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Seminar for Applied Mathematics
ETH Zurich
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Korteweg-de Vries equation $u_t + uu_x + u_{xxx} = 0$

kdv_1soliton.mpg
http://www.ma.hw.ac.uk/solitons

kdv-2soliton.mpg
http://people.uncw.edu/hermanr/research/solitons.htm

But before KdV was J.S. Russell and his “Wave of Translation”
- quotation from John Scott Russell, 1834
Russell’s wave

http://www.ma.hw.ac.uk/solitons/soliton1b.html
Tidal Bore

http://www.severn-bore.co.uk
Tidal Bore and Internal Waves

Strait of Gibraltar - Camarinal Sill

- discussion of internal waves: Ablowitz and Segur, 1981, Sec. 4.1.b
Morning Glory

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Morning Glory

Freak Waves / Rogue Waves / Killer Waves / Extreme Waves

http://www.photolib.noaa.gov/historic/nws/wea00800.htm
Bay of Biscay, France

http://en.wikipedia.org/wiki/Freak_wave
Freak Waves / Rogue Waves / Killer Waves / Extreme Waves

ERS-2 SAR  Detected Extreme Wave

Aug 20, 1996, 22:51:17 UTC, 44.6° S, 7.1°

H_{max} 29.8 m

H_{max} / H = 2.9

REFERENCE FOR THE FIGURE ??
Brief history of solitons and completely integrable systems

- 1843 - J.S. Russell, Edinburg-Glasgow canal
  - witnessed a shallow water soliton
  - experimentally deduced $c^2 = g(h + a)$, where $h$ is the water depth and $a$ is the soliton amplitude

- 1895 Korteweg - de Vries equation (KdV) proposed
  \[ u_t + uu_x + u_{xxx} = 0 \]

- 1955 - Fermi, Pasta and Ulam (LANL)
  \[ \ddot{x}_n = x_{n+1} - 2x_n + x_{n-1} + \alpha \left[ (x_{n+1} - x_n)^p - (x_n - x_{n-1})^p \right], \quad p = 2, 3 \]
  - numerical study of weakly nonlinearly (quadratically and cubically) coupled oscillator chains
  - expected equipartition of energy (among all linear modes) caused by the nonlinearity
  - obtained recurrence after some time, e.g. $T \approx 1000 T_1, T_1 = \frac{2\pi}{\omega_1}, \omega_1 \ldots$
    - frequency of the fundamental linear mode
1965 - Zabusky and Kruskal
- numerical study of KdV with periodic BC with \(u(x, 0) = \cos(\pi x)\)
- obtained steepening and evolution into a pulse train (taller pulses faster)
- elastic interactions of the pulses, branded the name **soliton**

1967 - Gardner, Greene, Kruskal and Miura
- Inverse Scattering Transform (IST) for KdV

1968 - P. Lax
- generalization of the IST \(\rightarrow\) Lax pair

1972 - Zakharov and Shabat
- IST for the Nonlinear Schrödinger equation

1973 - Ablowitz, Kaup, Newell and Segur
- further generalization of the IST

...
Solitary waves:
- waves of permanent form
- localised (approach zero or another constant at infinity)

**Solitons** = solitary waves which
- interact strongly with other solitons and emerge from the collision unchanged apart from a phase and position shift
- no necessary and sufficient condition for soliton existence
- solitary waves arise due to perfect balance between nonlinearity - steepening (water), focusing (light), ... - and dispersion
- often (not always) PDEs with soliton solutions are “completely integrable”
- complete integrability:
  - no necessary and sufficient condition (unlike in Hamiltonian ODEs)
  - necessary cond.: infinite number of conserved quantities
  - usual method: inverse scattering transformation (IST)
  - solution of a system integrable via IST is a soliton $\Leftrightarrow$ its IC generates only point spectrum in the direct scattering problem
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Soliton equation examples:

1. Korteweg de Vries (KdV) equation (shallow water waves, internal waves, ...)
   \[ u_t + uu_x + u_{xxx} = 0, \quad x \in \mathbb{R}, \ t \geq 0 \]

2. Sine Gordon equation (Josephson junctions, self-induced transparency, ...)
   \[ u_{tt} - u_{xx} + \sin(u) = 0 \]

3. Cubic Nonlinear Schrödinger (NLS) equation (pulses in optical fiber, deep water waves, ...)
   \[ iu_t + u_{xx} + |u|^2 u = 0 \]

4. Massive Thirring model
   \[ i(u_t + u_x) + v + |v|^2 u = 0 \]
   \[ i(v_t - v_x) + u + |u|^2 v = 0 \]
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\[ i(u_t + u_x) + \nu + |\nu|^2u = 0 \]

\[ i(\nu_t - \nu_x) + u + |u|^2\nu = 0 \]
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Solitary wave (but not soliton) equation examples:

1. Cubic-quintic NLS (plasma waves, fiber lasers)

\[ iu_t + u_{xx} + |u|^2 u - \gamma |u|^4 u = 0 \]

2. NLS with saturable nonlinearity (pulses in optical fibers with saturable nonlinearity)

\[ iu_t + u_{xx} + \frac{|u|^2}{1 + \gamma |u|^2} = 0 \]

3. Generalized KdV (has solitary waves of compact support = “compactons”)

\[ u_t + (u^n)_x + (u^n)_{xxx} = 0 \]

4. Coupled Mode Equations (light pulses in fiber Bragg gratings)

\[ i(u_t + u_z) + \kappa v + (|u|^2 + 2|v|^2)u = 0 \]

\[ i(v_t - v_z) + \kappa u + (|v|^2 + 2|u|^2)v = 0 \]
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