Convergence of Petviashvili’s Iteration Method

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Numbering consistent with [PS]!
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Convergence of Petviashvili's Iteration Method
Scalar, 1-D Wave Equation with Power Nonlinearity

\[ u_t - (\mathcal{L} u)_x + pu^{p-1}u_x = 0, \quad (1.1) \]

- \( u : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}, \ p > 1 \)
- \( \mathcal{L} : \) linear, self-adjoint (\( \langle u, \mathcal{L} v \rangle = \langle \mathcal{L} u, v \rangle \)), positive (\( \langle u, \mathcal{L} u \rangle \geq 0 \)) pseudodifferential operator in \( x \) of order \( m \).
- \( \langle f, g \rangle = \int_{-\infty}^{\infty} \bar{f}(x)g(x)dx \)
- Fourier: \( u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k)e^{ikx}dk, \ \hat{u}(k) = \int_{-\infty}^{\infty} u(x)e^{-ikx}dx \)

Stationary bound state solution \( u(x, t) = \Phi(x - ct) \) leads to boundary value problem (\( \int \left[ -c\Phi_x - (\mathcal{L} \Phi)_x + p\Phi^{p-1}\Phi_x \right] dx \))

\[
(1.3) \begin{cases} 
  c\Phi + \mathcal{L} \Phi = \Phi^p \\
  \lim_{|x| \to \infty} \Phi(x) = 0
\end{cases}
\]

or \( (1.5) [c + v(k)] \hat{\Phi}(k) = \hat{\Phi}^p(k) \), \( v(k) \geq 0 \) an \( m \)th order polynomial in \( |k| \)
Assumption 1.1

\[ p > 1, \, \nu(k) \geq 0, \, c > 0. \exists \text{ real analytical solution to 1 in} \]
\[ X = L^2(\mathbb{R}) \cap L^{p+1}(\mathbb{R}) \cap H^{m/2}(\mathbb{R}) \]

Approximate \( \hat{\Phi} \) through \( \hat{u}_{n+1}(k) = \frac{u_p^n(k)}{c + \nu(k)} \longrightarrow \text{usually divergent} \)

\[
\hat{u}_{n+1}(k) = M_n^\gamma \frac{u_p^n(k)}{c + \nu(k)} \quad \text{(1.8)}
\]

\[
M_n = \frac{\int_{-\infty}^{\infty} [c + \nu(k)] \hat{u}_n(k)^2 dk}{\int_{-\infty}^{\infty} \hat{u}_n(k) u_p^n(k) dk} \quad \text{(1.9)}
\]

Lemma 1.2: Fix points for (1.8), (1.9) correspond to bound states \( \hat{\Phi}(k) \) of (1.5) for \( \gamma \neq 1 + 2n, \, n \in \mathbb{Z} \).
Spectrum, Assumption 2.1

Define Operator to (1.1): $\mathcal{H} = c + \mathcal{L} - p \Phi^{p-1}(x)$ \hspace{2cm} (1.10)

- selfadj. in $L^2(\mathbb{R}) \rightarrow$ real eigenval., orth. spectr. decomp.
- Null space contains at least $\Phi'(x)$.
- cont. spectrum positive, bounded away from zero (ass. 1.1)
- negative spectrum not empty

$$\mathcal{H} \Phi = (1 - p) \Phi^p$$

$$\langle \mathcal{H} \Phi, \Phi \rangle = -(p - 1) \langle \Phi^p, \Phi \rangle = -\frac{p - 1}{2\pi} \langle \hat{\Phi}, \hat{\Phi}^p \rangle$$

$$= -\frac{p - 1}{2\pi} \langle [c + \nu(.)] \hat{\Phi}, \hat{\Phi} \rangle < 0$$
Spectrum, Assumption 2.1

Define Operator to (1.1): \( \mathcal{H} = c + \mathcal{L} - p\Phi^{p-1}(x) \) (1.11)

- selfadj. in \( L^2(\mathbb{R}) \) \( \longrightarrow \) real eigenval., orth. spectr. decomp.
- Null space contains at least \( \Phi'(x) \).
- cont. spectrum positive, bounded away from zero (ass. 1.1)
- negative spectrum not empty

Assumption 2.1 on Spectrum of \( \mathcal{H} \):

- \( \sigma^{\text{discr}}_{L^2}(\mathcal{H}) \) for eigenvalues \(< c\)
- \( \sigma^{\text{cont}}_{L^2}(\mathcal{H}) \) for eigenvalues \( \geq c \)
- Nullspace is one-dimensional
- dim. neg. space \( n(\mathcal{H}) \geq 1 \)
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Convergence Theorem

Theorem 2.8

Let $\hat{\Phi}(k)$ solution to (1.5), assumptions 1.1 and 2.1. Petviashvili Iteration (1.8), (1.9) converges to $\hat{\Phi}(k)$ in (small) neighbourhood of $\hat{\Phi}(k)$ if:

1. $1 < \gamma < \frac{p+1}{p-1}$
2. $n(\mathcal{H}) = 1$
3. Either $\Phi^{p-1}(x) \geq 0$ or $\lambda_{\text{max}}((c + \mathcal{L})^{-1}\mathcal{H}) < 2$ (ass. 2.7)

"If any of the conditions are not met, the Petviashvili iteration diverges from $\hat{\Phi}(k)$".
Fréchet Derivative, Contraction Principle

Fréchet Derivative

Let $\mathcal{B}, \mathcal{C}$ be Banach spaces, $D \subset \mathcal{B}$ open, mapping $A : \mathcal{B} \to \mathcal{C}$. $A$ is Fréchet differentiable in $g \in D$ if there exists a linear operator $L : \mathcal{B} \to \mathcal{C}$ such that

$$\lim_{\|h\| \to 0} \frac{\|A(g + h) - Ag - Lh\|}{\|h\|} = 0$$

Fixed Point Theorem ([HP], Lemma 4.4.8)

Let $\mathcal{B}$ be a Banach space, $D \subset \mathcal{B}$ open, assume that $A : D \to \mathcal{B}$ has a fixed point $\bar{f} \in D$, and let $A$ be Fréchet differentiable in $\bar{f}$ ($A'(\bar{f})$).

For all $0 < \varepsilon < 1 - \|A'(\bar{f})\|$, there exists an open set $S(\bar{f}, \delta)$ such that if $f_0 \in S(\bar{f}, \delta)$:

- The iterates $f_n := A f_{n-1} \in S(\bar{f}, \delta)$
- $\lim f_n = f$
- $\|f_n - \bar{f}\| \leq (\|A'(f)\| + \varepsilon)^n \|f_0 - f\|$

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Convergence of Petviashvili’s Iteration Method
Proof of Convergence

Let $A$ the iteration operator (1.8), (1.9): $\hat{u}_{n+1} = A(\hat{u}_n)$ in $X(\mathbb{R})$.

1. $A'(\hat{u}_n)$ continuous in $S(\Phi, \delta_c)$ (proof: [PS], Proposition 3.4 and additional calculation)

2. $\|A'(\Phi)\| < 1$, i.e. spectral radius of $A'(\Phi)$ is $< 1$.

By Continuity of $A'(\Phi)$, we have $\forall \ 0 < \varepsilon < 1 - \|A'(\Phi)\|$

$\exists S(\Phi, \delta(\varepsilon)) \subset X(\mathbb{R})$ such that $q = \sup_{\hat{u}_n \in S} \|A'(\hat{u}_n)\| < 1.$

By [HP], Lemma 4.4.7: $\forall \ \hat{f}, \hat{g} \in S: \|A(\hat{f}) - A(\hat{g})\| \leq q\|\hat{f} - \hat{g}\|.$

The contraction mapping theorem ([HP] theorem 4.3.4) assures that $A(\hat{u}_n)$ has unique, asymptotically stable fixed point in $S(\Phi, \delta)$. By the fixed Point theorem we get that

$$\|\hat{u}_n - \Phi\| \leq \left(\|A'(\Phi)\| + \varepsilon\right)^n \|\hat{u}_0 - \Phi\|.$$

q.e.d. theorem 2.8
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Proposition 3.1

Proposition 3.1 \( A'(\hat{\Phi}) \) (i.e. Operator (1.8), (1.9) linearized at \( \hat{\Phi}(k) \)) has spectral radius smaller than one (\( \|A'(\hat{\Phi})\| < 1 \)), if

- \( 1 < \gamma < \frac{p+1}{p-1} \)
- \( n(\mathcal{H}) = 1 \)
- assumptions 2.1 and 2.7 are met.

Proof: Define \( \hat{u}_0(k) := \hat{\Phi}(k) + \hat{w}_0(k) \), \( \hat{w}_0(k) \) small and \( \langle \Phi', w_0 \rangle = 0 \). Generate \( \hat{w}_n(k) = \hat{u}_n(k) - \Phi(k) \) by linearized operators:

\[
\hat{w}_{n+1}(k) = \gamma m_n \hat{\Phi}(k) + p \frac{\Phi^{p-1} \ast \hat{w}_n(k)}{c + \nu(k)} \tag{3.1}
\]

\[
m_n = (1 - p) \frac{\int_{-\infty}^{\infty} \hat{\Phi}^p(k) \hat{w}_n(k) dk}{\int_{-\infty}^{\infty} \hat{\Phi}^p(k) \hat{\Phi}(k) dk} = M_n - 1 \tag{3.2}
\]

Proof: calculation, done in handout.
I Proof of Proposition 3.1

Define space $X_p := \{ U \in L^2 : \langle \Phi^p, U \rangle = 0 \}$.
We decompose $\hat{w}_n(k) = \hat{u}_n(k) - \hat{\Phi}(k)$ into

$$w_n = a_n \Phi(x) + q_n(x), \quad q_n(x) \in X_p \quad (3.3)$$

Immediately (3.2, 3.3): $m_n = (1 - p)a_n$ and by short calculations (see handout):

$$m_{n+1} = [p - \gamma(p - 1)]m_n \quad (3.4)$$
$$q_{n+1}(x) = q_n(x) - (c + L)^{-1}H q_n(x) \quad (3.5)$$

Want to prove that $w_n \xrightarrow{n \to \infty} 0$ to conclude that spectral radius of (3.1), (3.2) less than 1.

(1) $m_n \to 0$ if $1 < \gamma < \frac{p+1}{p-1}$. Superlinear: $\gamma = \frac{p}{p-1}$. 
II Proof of Proposition 3.1

(2) \( q_n \to 0 \):
Decompose \( q_n \) into EF of \((c + L)^{-1} \mathcal{H}\) (see later) in \( X_p \). We need two Lemmata (proven later):

**Lemma 2.4**
\[ \sigma \left((c + L)^{-1} \mathcal{H}\right) \text{ in } X_p(\mathbb{R}) \text{ has } n(\mathcal{H}) - 1 \text{ negative EV.} \]

**Lemma 2.5**
Positive spectrum of \((c + L)^{-1} \mathcal{H}\) in \( X_p(\mathbb{R})\):

1. Infinitely many discrete EV. \( 0 < \lambda < 1 \) (accumulating to \( 1^- \)).
2. If \( \forall x \in \mathbb{R}: \Phi_p^{-1}(x) \geq 0 \): no EV. > 1.
3. If \( \exists x_0 \in \mathbb{R}: \Phi_p^{-1}(x_0) < 0 \), we also have infinitely many discrete EV. in \( 1 < \lambda < \lambda_{\text{max}} \) (accumulating to \( 1^+ \)), and \( \lambda_{\text{max}} < 1 + \frac{p}{c} \left| \min_{x \in \mathbb{R}} \Phi_p^{-1}(x) \right| < \infty \).
We had \( q_{n+1}(x) = q_n(x) - (c + \mathcal{L})^{-1} \mathcal{H} q_n(x) \) (3.5)

\( \Phi' \) is EF of \((c + \mathcal{L})^{-1} \mathcal{H}\) to EV 0, but \( \langle \Phi', q_0 \rangle = 0 \) (\( \langle w_0, \Phi' \rangle = 0 \), use \( \langle \Phi, \Phi' \rangle = 0 \)) implies \( \langle \Phi', q_n \rangle = 0 \) by induction (use 3.5).

\[
q_n(x) = \sum_{k=1}^{n(\mathcal{H})-1} \alpha_k^{(n)} U_k(x) + \sum_{0<\lambda_k<1} \beta_k^{(n)} U_k(x) + \sum_{1<\lambda_k \leq \lambda_{\text{max}}} \gamma_k^{(n)} U_k(x) \tag{3.6}
\]

\[
\alpha_k^{(n+1)} = (1 + |\lambda_k|) \alpha_k^{(n)} \quad \lambda_k < 0 \tag{3.7}
\]
\[
\beta_k^{(n+1)} = (1 - \lambda_k) \beta_k^{(n)} \quad 0 < \lambda_k < 1 \tag{3.8}
\]
\[
\gamma_k^{(n+1)} = (1 - \lambda_k) \gamma_k^{(n)} \quad 1 < \lambda_k \leq \lambda_{\text{max}} \tag{3.9}
\]

For (max. linear !) convergence to 0 we need \( n(\mathcal{H}) = 1 \) and assumption 2.7.
IV Proof of Proposition 3.1

Remark: Add $\sum \lambda_j \cdot \delta^{(n)}_j \cdot U_j(x)$ to $q_n$:

- If $w_0$ not orthogonal to $\Phi'$ $\rightarrow$ Iteration of $w_n$ converges to $c_0 \Phi'$. translation in $x$ of $\Phi(x)$ to $\Phi(x + c_0)$, since we have linearized operator (first order correction!).

- $\text{Ker}(H) > 1$, non-orthogonal $w_0$: Not necessarily convergence to $\Phi'$, bifurcation. We need assumption 2.1.

q.e.d Proposition 3.1
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Orthogonal Basis

\((c + \mathcal{L})^{-1} \mathcal{H}\) in \(L^2\): \(\mathcal{H}\) selfadjoint, \((c + \mathcal{L})\) positive \(\rightarrow\) EF of generalized EVP (2.4) \(\mathcal{H} U = \lambda (c + \mathcal{L}) U\) form an orthogonal basis of \(L^2\).

Lagrange Multipliers

Analysis II: Extremum of function \(f(x, y)\) under constraint \(\phi(x, y) = 0\) computed through 3 equations:

\[
\phi(x, y) = 0 \quad \nabla [f(x, y) + \lambda \phi(x, y)] = 0
\]

Generalize to infinite dimensions: looking for extremum of \(F[\psi]\) under constraint \(C[\psi] = 0\):

\[
C[\psi] = 0 \quad \frac{\delta}{\delta \psi} (F[\psi] + \nu C[\psi]) = 0
\]
Lemma 2.3

The negative space of $\mathcal{H}$ in $X_p(\mathbb{R})$ has dimension $n(\mathcal{H}) - 1$.

Proof:

Need to find solutions $(\mu, \psi)$ to $(\mathcal{H} - \mu)\psi = 0$ under constraint that $\langle \Phi^p, \psi \rangle = 0$. Use Lagrange Multiplier $\nu$ to get

$$\langle \Phi^p, \psi \rangle = 0 \quad \frac{\delta}{\delta \psi} \left( \frac{1}{2} \langle (\mathcal{H} - \mu)\psi, \psi \rangle + \nu \langle \Phi^p, \psi \rangle \right) = 0$$

in other words: $\langle \Phi^p, \psi \rangle = 0 \quad \mathcal{H}\psi = \mu \psi - \nu \Phi^p(x)$ \hspace{1cm} (2.7)

Decompose $\psi$ with $L^2$ EV-EF pairs $(\mu_k, u_k)$, $\mu \not\in \sigma_X(\mathcal{H})$:

$$\psi(x) = \nu \left[ \sum_{\mu_k<0} \frac{\langle u_k, \Phi^p \rangle}{\mu - \mu_k} u_k(x) + \sum_{\mu_k>0} \frac{\langle u_k, \Phi^p \rangle}{\mu - \mu_k} u_k(x) \right]$$ \hspace{1cm} (2.8)
\[ \psi(x) = \nu \left[ \sum_{\mu_k < 0} \frac{\langle u_k, \Phi^p \rangle}{\mu - \mu_k} u_k(x) + \sum_{\mu_k > 0} \frac{\langle u_k, \Phi^p \rangle}{\mu - \mu_k} u_k(x) \right] \]

1. \( u_k \in X_p \): \( \mu_k \) is eigenvalue of \( \mathcal{H} \) over \( X_p \).
2. \( u_k \not\in X_p \): Still need to fulfill constraint equation:

\[ F(\mu) = \frac{1}{\nu} \langle \Phi^p, \psi \rangle = \sum_{\mu_k < 0} \frac{\langle \Phi^p, u_k \rangle^2}{\mu - \mu_k} + \sum_{\mu_k > 0} \frac{\langle \Phi^p, u_k \rangle^2}{\mu - \mu_k} \overset{!}{=} 0 \quad (2.9) \]

Discussion of (2.9):
- Mon. decr. for \( \mu \leq 0 \) and \( \mu \neq \mu_k \), cont. in \( (\mu_k - 1, \mu_k) \).
- Eigenvalues \( \mu_k \) of (1): \( F \) continuous at \( \mu = \mu_k \).
- \( F \overset{\mu \to -\infty}{\longrightarrow} 0^- \)
- \( F(0) = -\langle \Phi^p, \mathcal{H}^{-1} \Phi^p \rangle = \frac{1}{p-1} \langle \Phi^p, \Phi \rangle > 0 \)
- \( \pm \infty \) at \( \mu = \mu_k \) for \( u_k \not\in X_p \).

Have \( \#(2) = \# \text{poles} - 1 \). Get \( n(\mathcal{H}) - 1 \) negative EV over \( X_p \).

q.e.d. Lemma 2.3
Lemma 2.4

The spectrum of \((c + L)^{-1} \mathcal{H}\) in \(X_p(\mathbb{R})\) has \(n(\mathcal{H}) - 1\) negative eigenvalues \(\lambda\).

Proof:

\[ n(\mathcal{H}) = \text{dimension of negative space of quadratic form } \langle U, \mathcal{H} U \rangle \]
\[ \equiv n\left( \langle U, \mathcal{H} U \rangle \right), \quad U \in X_p(\mathbb{R}). \]

By generalized inertial theorem \(n\left( \langle U, \mathcal{H} U \rangle \right)\) is the same in any orth. basis of \(X_p\) diagonalizing \(\langle U, \mathcal{H} U \rangle\) wrt. positively weighted inner product:

- Orth. (wrt. \(\langle ., . \rangle\)) basis through \(\psi(x)\) as defined in (2.8).
- Orth. (wrt. \(\langle (c + L), ., . \rangle\)) basis out of generalized EVP (2.4).

q.e.d. Lemma 2.4
Lemma 2.5

Positive spectrum of \((c + \mathcal{L})^{-1} \mathcal{H}\) in \(X_p(\mathbb{R})\):

1. Infinitely many discrete EV. \(0 < \lambda < 1\) (accumulating to \(1^-\)).
2. If \(\forall x \in \mathbb{R}: \Phi^{-1}(x) \geq 0\): no EV. \(> 1\).
3. If \(\exists x_0 \in \mathbb{R}: \Phi^{-1}(x_0) < 0\), we also have infinitely many discrete EV. in \(1 < \lambda < \lambda_{\text{max}}\) (accumulating to \(1^+\)), and \(\lambda_{\text{max}} < 1 + \frac{p}{c} \left| \min_{x \in \mathbb{R}} \Phi^{-1}(x) \right| < \infty\).

Proof (bounds only):
Continuity / Discreteness of spectrum out of spectral theory.
Rewrite (2.4) as

\[
(c + \mathcal{L})U - \frac{p}{1 - \lambda} \Phi^{-1}(x) U = 0
\]  
(2.12)
\[(c + \mathcal{L})U - \frac{p}{1-\lambda} \Phi^{p-1}(x)U = 0 \quad (2.12)\]

Multiply (2.12) by \(U\) and integrate:

\[
\lambda = 1 - p \frac{\langle U, \Phi^{p-1}U \rangle}{\langle U, (c + \mathcal{L})U \rangle} \quad (2.13)
\]

1. \(\forall x \Phi^{p-1}(x) \geq 0 \rightarrow \lambda < 1.\)
2. \(\exists x_0 : \Phi^{p-1}(x_0) < 0: \)

\[
\lambda = 1 - p \frac{\langle U, \Phi^{p-1}U \rangle}{\langle U, (c + \mathcal{L})U \rangle} < 1 + p \frac{\min_{x \in \mathbb{R}} \Phi^{p-1}(x)}{\langle U, (c + \mathcal{L})U \rangle} \langle U, U \rangle
\]

\[
< 1 + p \frac{\min_{x \in \mathbb{R}} \Phi^{p-1}(x)}{c \langle U, U \rangle} = 1 + \frac{p}{c} \min_{x \in \mathbb{R}} \Phi^{p-1}(x)
\]

q.e.d. Lemma 2.5
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Overview

Main Theorem 2.8
Let $\Phi(k)$ solution to (1.5), assumptions 1.1 (solution space) and 2.1 (Nullspace, bifurcation).
Petviashvili Iteration (1.8), (1.9) converges to $\Phi(k)$ in (small) neighbourhood (Continuity of linearized operator, Fixed Point Theorem) of $\Phi(k)$ if:

1. $1 < \gamma < \frac{p+1}{p-1}$ (Proposition 3.1, convergence of $m_n$)
2. $n(\mathcal{H}) = 1$ (Proposition 3.1, convergence of $q_n$)
3. assumption 2.7 is met. (Proposition 3.1, convergence of $q_n$)

"If any of the conditions are not met, the Petviashvili iteration diverges from $\Phi(k)$". (Bifurcation)

Remark: Generalization to more dimensions possible!
References
