Classification of p-Elementary and Extremal Lattices

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Foundations – Lattices

Definition 1.1

A (full) **lattice** on a quadratic vector space V over $\mathbb{Q}$ is a subset of the shape

$$\begin{aligned} L &= \{x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \mid x_1, \ldots, x_n \in \mathbb{Z}\} \\ &= \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \ldots \oplus \mathbb{Z}v_n \end{aligned}$$

for some basis $v_1, v_2, \ldots, v_n$ of V.

Isomorphism, also called **isometry**, of quadratic spaces and quadratic lattices is defined in the obvious way.

The theory can be extended to algebraic number fields K instead of the rationals. In that case, a lattice is a finitely generated $\mathfrak{o}_K$-module, where $\mathfrak{o}_K$ is the ring of integers in K.
In general, a lattice is not a free module. But it can always be represented as

$$L = \mathfrak{o}_K v_1 \oplus \mathfrak{o}_K v_2 \oplus \ldots \oplus \mathfrak{o}_K v_{n-1} \oplus \mathfrak{a} v_n,$$

for some (then non-principal) ideal $\mathfrak{a}$ in $\mathfrak{o}_K$.
The ideal class of $\mathfrak{a}$ depends only on L and is called the **Steinitz class** of L.

The most basic invariant of a lattice (after its dimension or rank) is its determinant:

Definition 1.2

The **Gram matrix** of a lattice L w.r.t. a basis $v_1, \dots, v_n$ is the symmetric $n \times n$-matrix $(b(v_i, v_j))$.
The **determinant** $\det L$ of L is the determinant of any Gram matrix of L.

Base change for lattices is given by matrices in $\mathrm{GL}_n(\mathbb{Z})$, which have determinant ± 1. Therefore, the determinant of the Gram matrix does not change.

Definition 1.3

A quadratic lattice is called an **integral lattice** if $b(L, L) \subseteq \mathbb{Z}$. Equivalently, $L \subseteq L^{\#}$, where

$$L^{\#} := \{x \in V \mid \forall y \in L : b(x, y) \in \mathbb{Z}\}$$

is the **dual lattice** of L.
It is called **even** if $b(x, x) \in 2\mathbb{Z}$ for all $x \in L$.
"Even" implies "integral".

$L^{\#}$ is generated by the dual basis (w.r.t. the form b) of some basis of L (and thus is indeed a lattice).

Definition 1.4

Let L be an integral lattice. The **discriminant group** of L is the factor group $L^{\#}/L$. Its order is $|L^{\#}/L| = |\det L|$.

The **exponent of** L is the exponent of its discriminant group, that is, the smallest natural number m such that $mL^{\#} \subseteq L$.

The **level** of an even lattice L is the smallest natural number m such that the rescaled dual lattice ${}^{m}L^{\#}$ (see below) is again even.

The level is equal to the exponent or 2 times the exponent of L.

Scaling and similarity

Notation: For $c \in \mathbb{Q}^*$ and a quadratic space V or a lattice L, we denote by ${}^{c}V$, ${}^{c}L$ the quadratic space (V, cb), respectively the quadratic lattice (L, cb), with bilinear form $(x, y) \mapsto c \cdot b(x, y)$.

Definition 1.5

A **similarity with norm** c is a linear map $\sigma : V \to W$ with $b_W(\sigma x, \sigma y) = c \cdot b_V(x, y)$.

L is **similar to** M if ${}^{c}L \cong M$ for some $c \in \mathbb{Q}^*$.

Later we shall treat the case where $L = M^{\#}$.

Remark:

For a lattice L on a real vector space one has ${}^cL \cong \sqrt{c}L$ (assuming $c > 0$, in particular for positive definite bilinear forms). But ${}^cL$ and $\sqrt{c}L$ are not identical, scaling is not the same as homothety. Typically $\sqrt{c}L$ does not live in V (a vector space over $\mathbb{Q}$).

The two constituents L and b of a quadratic lattice should be treated as objects of equal rights.
This is also the philosophy of Magma, the most advanced computer algebra system for lattices.

Theorem 1.1 (Finiteness of Class Number)

For a given determinant d, the number of isometry classes of (positive definite) integral lattices with determinant d is finite.

This is a consequence of reduction theory, which gives a lattice basis with $b(v_i, v_i) \leq C\, d^{1/n}$ for some constant C.
The constant C depends on the dimension of the lattice, and on the reduction theory chosen. Important notions of "reduced basis" have been introduced by Hermite, Minkowski, Voronoi, Ryskov, and LLL (A. Lenstra, H. Lenstra and L. Lovasz).

Foundations – Genera of lattices

Let p be a prime number. Every quadratic vector space (V, b) over $\mathbb{Q}$ embeds into a quadratic vector space (V_p, b) over $\mathbb{Q}_p$, where $V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and the natural extension $b_p : V_p \times V_p \to \mathbb{Q}_p$ is simply denoted by b again.
(V_p, b) is called the **completion** of (V, p) at the prime p.
This definition extends to $p = \infty$ with $\mathbb{Q}_\infty := \mathbb{R}$.
Similarly, a quadratic lattice L embeds into its **completion** $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$.
One also sets $L_\infty = V_\infty$.

A lattice L is integral if and only if each L_p is integral in the sense that $b(L_p, L_p) \subseteq \mathbb{Z}_p$.

$\mathbb{Z}_p$-lattices in quadratic vector spaces over $\mathbb{Q}_p$ can be defined as objects of their own right, with the usual notion of isometry.

The **determinant** of a quadratic $\mathbb{Z}_p$-lattice M is defined as the class modulo squares of units in $\mathbb{Z}_p$ of the determinant of a Gram matrix of M.

Every lattice over $\mathbb{Z}_p$ posseses a **modular decomposition**, also called **Jordan decomposition**

$$ {}^{p^{-r}}L_{-r} \perp {}^{p^{-r+1}}L_{-r+1} \perp \ldots \perp L_0 \perp {}^{p}L_1 \perp \ldots \perp {}^{p^s}L_s $$

where all L_i are self-dual $\mathbb{Z}_p$-lattices: $L_i = L_i^{\#}$.

If $p \neq 2$, a modular decomposition is unique up to isometry.

If $p = 2$, this only holds if all L_i are even. The lattice is called **totally even** in this case.

The local-global principle of Minkowski and Hasse for quadratic spaces does not hold for quadratic lattices. Therefore, the following notion is introduced.

Definition 1.6

Two lattices L and M are in the same **genus** if $L_p \cong M_p$ for all $p \in \mathbb{P} \cup \{\infty\}$.

To phrase it differently: isometry of all localisations of two lattices defines an equivalence relation, and genera are by definition the equivalence classes of this relation.

To specify or to distinguish genera, we use the **genus symbol** as introduced in [CoSl93], Chapter 15. This symbol is a string of local symbols, one for each prime $p = 2, 3, 5, \ldots$ dividing $2 \cdot \det L$. The local symbol at the prime $p \neq 2$ of a lattice L with modular decomposition as above is the formal product

$$ \prod_{j=-s}^{t} (p^j)^{\varepsilon_{j,p} n_{j,p}} \text{ with } \varepsilon_{j,p} = \left(\frac{\det L_j}{p} \right) \text{ and } n_{j,p} = \dim L_j \, . $$

We do not describe the dyadic symbol in full generality here. Among other things, the parity "even/odd" of the modular component belonging to $q = 2^t$ is recorded by a subscribed q_{II} respectively q_I.

Example: The lattice $E_8 \perp {}^2E_8$ and the Barnes-Wall lattice BW_{16} both have the genus symbol $1_{II}^{+8} 2_{II}^{+8}$, also written as $II_{16}(2_{II}^{+8})$.

If two lattices are in the same genus, all their localisations have the same determinant (a square class in $\mathbb{Z}_p$), and their determinants have the same sign, so they coincide in $\mathbb{Q}$.

Corollary and Definition 1.7

The number $h(\mathcal{G})$ of isometry classes in a genus $\mathcal{G}$ is finite. It is called the **class number** *of the genus.*

Basic task: Determine a set of representatives, and in particular the class number of any given genus of positive definite lattices.

Theorem 1.2 (Minkowski and Siegel)

Let L be a positive definite lattice of dimension ℓ and $M = M_1, \ldots, M_h$ a set system of representatives for a genus of positive definite lattices of dimension m. Then the weighted average of the number of representations $a(L, M_j)$ of L by the different M_j is the product of certain **representation densities** *$\alpha_p(L, M)$, p prime, with a certain factor "at infinity":*

$$\frac{1}{\sum_{j=1}^{h} |\mathrm{O}(M_j)|^{-1}} \cdot \sum_{j=1}^{h} \frac{a(L, M_j)}{|\mathrm{O}(M_j)|} = \frac{\gamma(m-\ell)}{\gamma(m)} \prod \alpha_p(L, M).$$

In this formula, the values $\gamma(n)$ are inductively defined by

$$\gamma(0) = 1,\ \gamma(1) = \frac{1}{2},\ \gamma(2) = \frac{1}{2\pi},\ \gamma(n) = \frac{\gamma(n-1)}{n \cdot \rho_n} \quad \text{für} \quad m \geq 3,$$

where ρ_n is the volume if the n-dimensional unit ball.

For $L = M_1$, this formula contains as a special case a representation of the **mass**

$$\sum_{j=1}^{h} \frac{1}{|\mathrm{O}(M_j)|}$$

of the genus as a product of local densities. This **mass formula** allows to test whether a set of inequivalent lattices is already a set of representatives for the genus in question.

Foundations – Theta series

If L is an even lattice of even dimension $n = 2k$ and level ℓ, we denote by

$$\Theta_L(q) = \sum_{m \geq 0} r_L(m) q^m\,, \quad r_L(m) := |\{x \in L \mid (x, x) = 2m\}|$$

its theta series, where as usual $q = e^{2\pi i z}$ and z is a variable in the upper half plane. This is a modular form of weight k for the group $\Gamma_0(\ell)$ and a certain quadratic character $\varepsilon : \Gamma_0(\ell) \to \{\pm 1\}$. Using standard notation for the action of $PSL_2(\mathbb{R})$ on modular forms of weight k, this means that

$$\Theta_L|_k \gamma = \varepsilon(\gamma) \Theta_L\,.$$

The character ε only depends on $(-1)^k \det L$, and is trivial if this number, also called the (signed) discriminant of L, is a square. So in that case we have modular forms of weight k and level ℓ in the strict sense, i.e. invariant under $\Gamma_0(\ell)$.

We denote by $\mathcal{M}_k(\ell, \varepsilon)$ the finite-dimensional complex vector space of these modular forms, and by $\mathcal{S}_k(\ell, \varepsilon)$ the subspace of cusp forms.

If L and M are lattices in the same genus, then the difference $\Theta_L - \Theta_M$ is a cusp form.

Extremal lattices

Atkin-Lehner involutions and strong modularity

Definition 2.1

Let L be an integral lattice, ℓ the exponent of its discriminant group $L^\#/L$, and q an exact divisor of ℓ. The **partial dual** $D_q L$ of L is defined as

$$D_q L := {}^q\!\left(\tfrac{1}{q} L \cap L^\#\right).$$

The lattice $D_q L$ is integral again, and $D_q(D_q L) \cong L$ (canonically).

If q and r are as above and coprime, the operators D_q and D_r (on isometry classes of lattices of fixed exponent ℓ) commute:

$$D_q D_r L = D_r D_q L.$$

Clearly, D_q is also defined for lattices over $\mathbb{Z}_p$ (of appropriate exponent), and the operator D_q commutes with localization and thus maps genera onto genera. For a $\mathbb{Z}_p$-lattice L with modular decomposition as above, and $q = p^s$, we have

$$D_q L \cong L_s \perp {}^p L_{s-1} \perp {}^{p^2} L_{s-2} \perp \ldots \perp {}^{p^s} L_0 \,,$$

so D_q acts by reversing the sequence of p-modular components. The operator D_q does not affect the other localizations $L \otimes \mathbb{Z}_{p'}$, $p' \neq p$ as modules, but because of rescaling by q, it in general does change the genus of the quadratic form.

Definition 2.2 (H.-G. Quebbemann)

Let L be an integral lattice and denote by ℓ the exponent of its discriminant group $L^\#/L$. If $D_m L \cong L$ for all exact divisors m of ℓ then L is called **strongly modular**. It is called **modular** if $D_\ell L \cong L$.

So L is modular if it is similar to its dual lattice, the scaling factor being implied by its level.

Example: The lattice $E_8 \perp {}^2E_8$ and the Barnes-Wall lattice BW_{16} are modular of level 2 (and thus strongly modular).

In the definition of strong modularity, one could restrict m to prime powers since

$$D_m L \cong \prod_{q|m} D_q L,$$

where q runs over the prime powers exactly dividing m. Except for the self-dual case, $\ell = 1$, a modular lattice must have even dimension, $n = 2k$, say. Its determinant is ℓ^k. If L has even determinant and is totally even and modular, then its dimension is divisible by 4.

In the following we shall assume that ℓ is square free. This includes the assumption that L is totally even, and ℓ is equal to the exponent of $L^\#/L$. Denote by $\Gamma_0^*(\ell)$ the normalizer of $\Gamma_0(\ell)$ in $PSL_2(\mathbb{R})$. The factor group $\Gamma_0^*(\ell)/\Gamma_0(\ell)$ is 2-elementary abelian, generated by certain cosets $W_m\Gamma_0(\ell)$, $m|\ell$, which are independent mod $\Gamma_0(\ell)$; for $\ell = m$ one obtains the Fricke involution W_ℓ represented by $\left(\begin{smallmatrix} 0 & -1 \\ \ell & 0 \end{smallmatrix}\right)$. Since the character ε defining the genus of L is quadratic, W_ℓ acts on $\mathcal{M}_k(\ell,\varepsilon)$, and if ε is trivial, all the W_m act as commuting involutions on $\mathcal{M}_k(\ell,\varepsilon)$ (the **Atkin-Lehner involutions**). Hence, if ℓ is prime or ε is trivial, $\mathcal{M}_k(\ell,\varepsilon)$ splits into eigenspaces $\mathcal{M}_k(\ell,\chi)$ with respect to the characters $\chi : \Gamma_0^*(\ell) \to \mu_4 = \{\pm 1, \pm i\}$ extending ε.

Proposition 2.1 (Atkin-Lehner-identity)

For any lattice L of level ℓ, and any divisor m of ℓ, the theta series of the partial dual, $\Theta_{D_m L}$ is proportional to the Atkin-Lehner-transform $\Theta_L|_k W_m$ of the original theta series, with a certain numerical factor depending only on the genus.

See [Que97] for a precise statement and formula. In particular, if L is strongly modular, then Θ_L is an *Atkin-Lehner-eigenform*:

$$\Theta_L \in \mathcal{M}_k(\ell,\chi)\,,\ \chi = \chi_{\mathrm{gen}\,L}$$

for a certain character χ depending only on the genus of L.

Extremality of modular forms and lattices

Definition 2.3

a) Let $\mathcal{M}$ be a subspace of $\mathcal{M}_k(\ell)$. We say that **extremality is definable** with respect to $\mathcal{M}$ if the projection $\mathcal{M} \to \mathbb{C}^d$ to the first $d = \dim \mathcal{M}$ coefficients of the q-expansion

$$f = \sum_{m \geq 0} a_m q^m \mapsto (a_0, a_1, \ldots, a_{d-1})$$

is injective. The unique element $F_\mathcal{M} \in \mathcal{M}$ with q-expansion

$$F_\mathcal{M} = 1 + \sum_{m \geq d} a_m q^m$$

is then called the **extremal modular form** in $\mathcal{M}$.

b) Let L be an even lattice of dimension $2k$ and level ℓ and $\mathcal{M}$ be a subspace of $\mathcal{M}_k(\ell, \varepsilon)$ with $\Theta_L \in \mathcal{M}$ (where as above ε denotes the character defined by the determinant of L). We say that L is **extremal with respect to** $\mathcal{M}$ if extremality is definable with respect to $\mathcal{M}$ and $\Theta_L = F_{\mathcal{M}}$.

Informally, a lattice is extremal if its minimum is as large as the space of modular forms where its theta series lives allows.

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Definition 2.4 (Extremal lattice)

Consider a genus $\mathcal{G}$ of level ℓ, determinant ℓ^k, and containing (strongly) modular lattices (e.g. ℓ prime). Let δ be the character on $\Gamma_0(\ell)$ and let χ be the character on the group of involutions W_m describing $\mathcal{G}$:

$$\chi(W_p) = g_p(L) \text{ for } L \in \mathcal{G}$$

with $g_p(L)$ the Gaussian sum from [Que97].

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A (strongly) modular lattice in $\mathcal{G}$ is called **(modular) extremal** if it is extremal with respect to the subspace

$$\{f \in \mathcal{M}_k(\ell, \delta) \mid f|_k W_\ell = \chi(W_\ell) f\}$$

and **strongly (modular) extremal** if it is extremal with respect to the subspace

$$\mathcal{M}_k(\ell, \chi) = \{f \in \mathcal{M}_k(\ell, \delta) \mid f|_k W_m = \chi(W_m) f \ \forall\, m \| \ell\}.$$

The minima of (hypothetical) extremal lattices are as follows:

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n \ ℓ	1	2	3	5	6	7	11	14	15	23
4	–	2	2	2	2	2	4	4	4	6
6	–	–	2	–	–	4	4	–	–	(8)
8	2	2	2	4	4	4	6	6	6	(10)
10	–	–	2	–	–	4	6	–	–	(12)
12	–	2	4	4	4	6	8	8	8	
14	–	–	4	–	–	6	8	–	–	
16	2	4	4	6	6	6	(10)	10	10	
18	–	–	4	–	–	8	10	–	–	
20	–	4	4	6	6	8	(12)	12	12	
22	–	–	4	–	–	8				
24	4	4	6	8	8	10				

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n \ ℓ	1	2	3	5	6	7	11	14	15	23
28	–	4	6	8	8	10				
32	4	6	6	10	10	12				
36	–	6	**8**	10	10					
40	4	6	8	12	12					
44	–	6	8	12	12					
48	6	8	10							
52	–	8	10							
56	6	8	10							
60	–	8	12							
64	6	10	12							
68	–	10	12							
72	**8**	10								

Existence and uniqueness of extremal lattices

See [SchaSchu99] for more detailed information about the tables below.

Example: the Barnes lattice in dimension 6

There is a well-known self-dual hermitian $\mathbb{Z}[\alpha]$-lattice, $\alpha = \frac{1+\sqrt{-7}}{2}$, of dimension 3 and minimum 2, which we denote be J_3 (following [Coh76]). Its unitary group is $U(J_3) = 2 \times G_{168}$, where $G_{168} \cong L_3(2) \cong L_2(7)$ is a famous simple group of order 168.

The group $U(J_3)$ is a primitive irreducible complex reflection group, occurring as no. 24 in the basic list of such groups given by Shephard and Todd [ShTo54].

A natural Gram matrix for J_3 is

$$J_3 \cong \begin{pmatrix} 2 & \overline{\alpha} & 1 \\ \alpha & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It also occurs in the work of Mimura [Mim82], where hermitian lattices over imaginary quadratic fields, generated by norm 2 vectors, are classified in a “direct” approach.
The desired extremal 7-modular lattice is obtained from J_3 by ordinary transfer. It is isometric to the Barnes lattice P_6 and to the Craig lattice $A_6^{(3)}$.

Extremal lattices with minimum 4

Result: For each pair (n, ℓ) such that ℓ is as before, and $n = n(\ell)$ is the minimal dimension for extremal lattices with minimum 4, there exists an extremal lattice, and is unique.

ℓ	1	2	3	5	6	7	11	14	15	23
$n(\ell, 4)$	24	16	12	8	8	6	4	4	4	2
h	24	24	10	5	8	3	3	3	3	2
h_{sm}	24	16	10	5	6	3	3	3	3	2

Table 3.2: Minimal dimensions for extremal lattices with minimum 4, and class numbers

ℓ	1		2			3				
n	32	40	20	24	28	14	16	18	20	22
h						29	163			
h_{ext}	≥ 22	≥ 3	**3**	≥ 1	≥ 1	1	6	≥ 1	≥ 1	≥ 1

ℓ	5	6+	6−	7		11
n	12	12	12	8	10	6
h	48	308	284	8	30	5
h_{ext}	4	6	4	1	4	1

Table 3.3: Extremal lattices of minimum 4 in non-minimal dimensions

Comments on Table 3.3:

For $\ell = 1$, much larger, but non-constructive lower bounds are known as a consequence of Minkowski-Siegel's Main Theorem: Already in 1983, Meinhard Peters gave the bound $8.45 \cdot 10^{51}$ for $n = 40$, whereas the bound $1.096 \cdot 10^{7}$ has only been proved in 2002 by Oliver King [King03], as a consequence of a refined mass formula.

The exact entry 3 for $(\ell, n) = (2, 20)$ has been proved by Christine Bachoc and Boris Venkov in [BaVeNe01], using spherical designs.

Extremal lattices with minimum 6

ℓ	1	2	3	5	6	7	11	14	15	23
$n(\ell, 6)$	48	32	24	16	16	12	8	8	8	4
h						395	31	80	91	6
h_{ext}	≥ 3	≥ 3	≥ 1	**1**	≥ 5	0	1	1	2	1

Table 3.4: Extremal lattices with minimum 6 in minimal dimensions

The remarkable uniqueness in the case $(16, 5^8)$ was proved by Bachoc, Nebe and Venkov in [BaVeNe01].

ℓ	1		2			3				
n	56	64	36	40	44	26	28	30	32	34
h_{ext}	≥ 1	≥ 1	≥ 3	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1

ℓ	5	6	7		11
n	20	20	14	16	10
h_{ext}	**?**	≥ 1	$\geq$ **1**	≥ 1	2

Table 3.5: Extremal lattices of minimum 6 in non-minimal dimensions

Extremal lattices with minimum 8

Here, we resrict to the unimodular case. The following recent result of G. Nebe solves a long-standing problem:

Theorem 2.1 (G. Nebe, 2010)

There exists an even unimodular lattice of dimension 72 with minimum 8.

This lattice is constructed as a tensor product over the ring $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ of the Barnes lattice J_3 described above with a complex form of the Leech lattice.

Dimension 80: The first two extremal lattices of dimension 80 for which the minimum 8 could actually be verified had been constructed by Christine Bachoc and Gabriele Nebe in 1997 [BaNe98].

It has recently been shown by Damien Stehlé and Mark Watkins [SteWa10] that a certain lattice constructed by Rainer Schulze-Pillot already in 1992 is actually extremal and not isometric to one of the Bachoc-Nebe lattices.

A fourth extremal lattice is described in [Wat10].

Algorithmic Classification

Some new results in dimensions 14 and 16

Proposition 3.1 (see also [GrLam10])

The genus $II_{16}(5^4)$ *has class number* 848. *It contains exactly one lattice with minimum* 4. *This lattice has* $2640 = 2^4 \cdot 3 \cdot 5 \cdot 11$ *minimal vectors and* $288,000 = 2^8 \cdot 3^2 \cdot 5^3$ *automorphisms.*

Further data for this genus: consider $o(L) := |\mathrm{Aut}(L)|$:

- 831 lattices for which $3 \mid o(L)$
- 529 lattices for which $5 \mid o(L)$
- 155 lattices for which $7 \mid o(L)$
- 8 lattices for which $11 \mid o(L)$
- 0 lattices for which $13 \mid o(L)$

Proposition 3.2

The genus $II_{14}(7^5)$ *has class number* 8664. *It contains* 8516 *lattices with minimum* 2 *and* 148 *lattices with minimum* 4.

Further data for this genus: consider $o(L) := |\mathrm{Aut}(L)|$:

- 5261 lattices for which $3 \mid o(L)$
- 631 lattices for which $5 \mid o(L)$
- 84 lattices for which $7 \mid o(L)$
- 0 lattices for which $11 \mid o(L)$ or $13 \mid o(L)$
- 24 lattices for which $o(L) = 2$

So, in this large genus, lattices with trivial automorphism group, which will we the majority when $\det \to \infty$, are still rare.

Theorem 3.1 (B. Hemkemeier, R.S., 2010)

There exists an extremal lattice in the genus $II_{14}(7^7)$. *It has* $560 = 2^5 \cdot 5 \cdot 7$ *minimal vectors (of norm* 6*) and* $1008 = 2^4 \cdot 3^2 \cdot 7$ *automorphisms.*

The integral representation of the cyclic group C_7 *given by its orthogonal group consists of two 7-dimensional indecomposable representations*

The free $\mathbb{Z}[\zeta_7]$*-sublattice of rank* 2 *of this* $\mathbb{Z}[C_7]$*-lattice has determinant* 7^8, *genus* $II_{12}(7^6 \cdot 49)$, *posseses* 56 *norm* 6 *vectors, and is glued with the unique binary lattice in* $II_2(7 \cdot 49)$, *with glue group* 7^2.

The following numbers come from a running computation which by now misses less than 10^{-5} of the mass of the genus. It might eventually lead to a brute force proof of the uniqueness of the above lattice (without any assumption on the automorphism group).

Proposition 3.3

The genus $II_{14}(7^7)$ *has class number* $\geq 82,929$. *It contains at least*

46, 515 *lattices with minimum* 2

36, 413 *lattices with minimum* 4

1 *lattice with minimum* 6.

Work in progress: Open Cases for extremal lattices

Members of my group in Dortmund currently work on filling some of the remaining gaps in the above tables:

- For the genus $(14, 11^7)$, show that no (modular) lattice of minimum 8 exists.
- Investigate the case $(18, 11^9)$, minimum 10.
- Prove the existence of an extremal lattice (with minimum 6) in the genus $(20, 5^{10})$
- Investigate the cases $(2k, 7^k)$ for $k \geq 11$ (minimum ≥ 8)

Open tasks and questions

- Analyse systematically hermitian structures, including hermitian tensor products, on known modular and extremal lattices.
- Decide whether an extremal lattice exists in the case $(36, 3^{18})$.
- Study the successive minima, in particular the "well-roundedness" of p-elementary lattices.
- Perform a numerical study of the behaviour of the depth parameter of B. Souvignier's programs for automorphisms and isometry [PlSo97] in large genera of lattices.

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