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Martin Kneser's Work on Quadratic Forms and Algebraic Groups

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ABSTRACT. This article provides an overview of the research work of Martin Kneser on the arithmetic theory of quadratic forms and algebraic groups, focusing on the period 1955 – 1970. To put Kneser's work in proper historical context, a survey of the theory of quadratic forms prior to that period, and an outlook on some subsequent work initiated or influenced by him is given.

1. A Short History of Quadratic Forms 1884 – 1954

The theory of quadratic forms emerged as a part of (elementary) number theory, dealing with quadratic diophantine equations, initially over the rational integers. The main questions in modern language are:

(a) the equivalence problem: when are two quadratic modules ("lattices") (L_1, q_1) und (L_2, q_2) over $\mathbb{Z}$ isometric?
(b) the classification problem: determine a set of representatives or a set of easily computable invariants for all isometry classes of lattices subject to natural restrictions (i.e. with given dimension, determinant, genus).
(c) The representation problem: for which $t \in \mathbb{Z}$ does there exist an $x \in L$ with $q(x) = t$?
(d) The determination of the representation numbers $a(t, L) = |\{x \in L \mid q(x) = t\}|$.

Here, the L_i are free $\mathbb{Z}$-modules of finite rank, or lattices in rational vector spaces V_i, and q_i is a quadratic form on L_i (or V_i). For problem (d), q should be (positive) definite, but using the action of the orthogonal group $\mathrm{O}(L)$ in an appropriate way, the problem (and its solution) can be extended to the general case. Also, the theory carries over to quadratic lattices over the ring of integers $\mathfrak{o}_k$ of an algebraic number field k and to some extent to rings of integers in arbitrary global fields.

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There are good reasons to date the beginning of the modern arithmetic theory of quadratic forms to HERMANN MINKOWSKI in the 1880s. Between 1884 and 1890 Minkowski developed the foundations of a general theory of quadratic forms over the rationals and rational integers. He already proved major results on all questions above in a modern way.

A brilliant work of the very young Minkowski is the prize-winning paper *Foundations of a theory of quadratic forms with rational coefficients* (in German) [**Min84**]. In the main part of this paper, he develops the local classification of integral quadratic forms. In the context of the prize question on sums of five squares, this was preparatory, but clearly of independent, even greater importance.

These investigations were continued in his Königsberg dissertation from 1885 *Investigations on quadratic forms. Determination of the number of distinct forms which are contained in a given genus* (in German) [**Min85**]. Minkowski states and proves a version of the mass formula (in German: "Maßformel", literal translation: "measure formula") which is already very similar to the modern one. In contrast to the works of previous authors (Eisenstein in the ternary case, Henry John Stephen Smith), the "right hand side" is a product of local densities over all prime numbers. In this context, Minkowski also introduces for the first time (more or less) today's notion of a genus of quadratic forms (in any number of variables).

On the first few pages of the note *On positive quadratic forms* (in German) [**Min86**], Minkowski gives a very clear and readable summary of his dissertation and also describes the contributions of Henry J.S. Smith, the other prize winner. Even today, this is useful reading for everyone interested in sums of squares.

In those days, the rational theory (classification over $\mathbb{Q}$) still was a by-product of the integral theory. Nevertheless, the paper *On the conditions under which two quadratic forms with rational coefficients can be transformed into each other* (in German) [**Min90**] practically contains the main theorem over $\mathbb{Q}$. With every rational quadratic form, Minkowski associates a system of invariants $C_p = \pm 1$, one for each prime. He shows that these invariants, together with the discriminant (a rational square class), determine the rational equivalence class. This result contains the "weak" local-global principle (for equivalence, not for representations), but the term is not yet used.

The next major step in the theory of quadratic forms is achieved by HELMUT HASSE in 1921: He introduces Hensel's p-adic numbers $\mathbb{Q}_p$ into the theory of quadratic forms and proves in his dissertation, published as [**Has23a**], the local-global principle for representations of numbers by rational quadratic forms in today's form. This principle has later been called "strong Hasse principle". In the second part of this work, published as [**Has23b**], he extends this principle to the representation of forms by forms and to the equivalence of forms. The local-global principle for equivalence is today usually called "weak Hasse principle". It turned out only much later, in the context of Witt rings, that it is actually more elementary in the sense that its proof can avoid norm principles and the existence of primes in arithmetic progressions. In the 1920s, the local and global theory of algebraic number fields was sufficiently far developed to quickly generalize the results to quadratic forms over arbitrary algebraic number fields; Hasse did this in the papers [**Has24a**] and [**Has24b**].

Summarizing we can say that Hasse's work gives full solutions to the (analogues of the) questions (a), (b) and (c) over number fields (not their rings of integers).

The modern approaches (after Minkowski) to the actual questions (a), (b) and (c) make systematic uses of this foundation, in particular of the p-adic numbers, but a considerable extension of the methods is required. Beginning with the 1930s, two major branches in the theory of integral quadratic forms have developed, the "analytic" and the "arithmetic" theory.

For more than 20 years (1935 – 1955), almost all progress in the analytic theory of quadratic forms after Minkowski is due to essentially one person, CARL LUDWIG SIEGEL. In three long, fundamental papers *On the analytic theory of quadratic forms*, parts I, II and III, (in German), he gives a solution to problem (d), also (in III) over number fields, at least to the extent where this is possible without additional assumptions. Any kind of solution of problems (c) and (d) has to take into account the fundamental fact that the direct generalization of the Hasse principle to rings of integers (instead of fields) fails to be true. In order to pass from fields to their rings of integers, one has to replace an individual quadratic form, or lattice M, by its whole genus, that is the set of lattices M' which are locally everywhere isometric to M, meaning $\mathbb{Z}_pM \cong \mathbb{Z}_pM'$ for all primes (places) p including infinity.

THEOREM 1 (Minkowski, Siegel). *Let N be a positive definite quadratic lattice of dimension n and $M = M_1, \dots, M_h$ a system of representatives of a genus of positive lattices of dimension m. Then the representation numbers $a(N, M_k)$ and the local representation densities $\alpha_p(N, M)$ are related by the formula*

$$\frac{1}{\sum_k |\mathrm{O}(M_k)|^{-1}} \cdot \sum_k \frac{a(N, M_k)}{|\mathrm{O}(M_k)|} = \frac{\gamma(m-n)}{\gamma(m)} \prod \alpha_p(N, M).$$

Here, the $\gamma(n)$ are inductively defined by

$$\gamma(0) = 1,\ \gamma(1) = \frac{1}{2},\ \gamma(2) = \frac{1}{2\pi},\ \gamma(m) = \frac{\gamma(m-1)}{m \cdot \rho_m} \quad \text{for} \quad m \geq 3,$$

where ρ_m is the volume of the m-dimensional unit ball.

This theorem is proved in [**Sie35-37**] I or [**Kne73**]. The local representation numbers, or rather "densities" $\alpha_p(N, M)$ are obtained as follows: for every power p^r, the (ordinary) representation number of N modulo p^r by M modulo p^r is directly defined; a natural normalization of this number is obtained by dividing through $p^{r\left(m-\frac{n+1}{2}\right)n}$. This "relative representation number" turns out to be independent of r for large r, by an appropriate version of Hensel's Lemma. It is the desired density. Notice that the left hand side is a weighted average of ordinary representation numbers. In particular, at least one summand has to be non-zero if the right hand side is non-zero. In this way, the original Hasse principle is contained in Siegel's theorem, which can be considered as a quantitative version of that principle.

We have attributed this theorem to Minkowski and Siegel, because the systematic use of the appropriate notion of genus was introduced by Minkowski, and also he had already stated and proved the (easier) case $n = m$ of the theorem.

Although this is a slight detour from our proper subject, we cannot survey the history of quadratic forms in the 20th century without naming ERNST WITT. His habilitation thesis, published as [**Witt37**], marks the beginning of the "algebraic" theory of quadratic forms, that is, the theory over arbitrary fields. There is no need to repeat here his well known cancellation theorem and the related extension theorem for isometries. The resulting uniqueness (up to isometry) of the maximal anisotropic part of a quadratic space is a paradigm for the structure theory of algebraic groups (to be developed more than 20 years later). An important aspect of Witt's work, that turned out to be relevant also for the integral theory, is the introduction of the geometric language: one deals with spaces, subspaces, lattices, sublattices, vectors and maps instead of polynomial equations, matrices and substitutions.

Witt was a very original mathematician. Despite the fundamental nature of his contributions to quadratic forms, his work can by no means be reduced to that. For instance, also in the theory of Lie algebras, in modular forms and in algebraic combinatorics he is still cited for some standard results. Concerning a special topic and a particular result, I want to mention Witt's paper *An identity between modular forms of degree two* (in German) [**Witt41**]. He shows that for $\widetilde{D}_{16}$ and $E_8 \perp E_8$ (the two 16-dimensional even unimodular lattices), not only the ordinary theta series coincide, but also the second degree Siegel theta series. This observation opened a new direction of research on "lattices and modular forms", which is still active today. We shall come back to this subject later.

The second, arithmetic, direction in the investigation of integral quadratic forms has been shaped to a large extent by the work of Martin Eichler. He was the first to bring systematically into play the role of the orthogonal group. More specifically, he introduced the notions of spinor norm and spinor genus which split up the failure of the Hasse principle for integral quadratic forms into two steps. This approach eventually led to a solution of problems (a) to (c) for indefinite forms which is similarly complete as Hasse's work over fields. In this context, Eichler also proved some initial results on "strong approximation" for the orthogonal group. Practically all of Eichler's work on these matters is contained in the monograph [**Eic52b**]. This book probably was not so much used as a text book, since other monographs like [**Jo50**] or [**O'Me63**] were easier to read and more accurate. Nevertheless, Eichler's book was an influential source of inspiration (and of open problems) for subsequent researchers. In particular, it was M. Kneser who brought the subject of approximation to maturity and thus arrived at complete results on the classification of indefinite integral quadratic forms. It should be mentioned that the arithmetic theory of quadratic forms makes up only a part of Eichler's number theoretical work. His later contributions are primarily devoted to modular forms and their relation to algebraic geometry.

Martin Kneser very clearly acknowledges the influence of Eichler on his own work in the introduction of the 2001 book version of his lectures on quadratic forms: "Für all dies vergleiche man das einflußreiche Werk *Quadratische Formen und orthogonale Gruppen*" [1]. Continuing, Kneser makes precise in what sense the book was influential on himself: "Schließlich ein persönliches Wort. Es ist ziemlich genau 50 Jahre her, daß ich als junger Assistent nach Münster kam, bald an Eichlers Seminar teilnahm, wo gerade die neuesten Ergebnisse aus seinem Buch *Quadratische Formen und orthogonale Gruppen* besprochen wurden. Da ich im Institut mein Arbeitszimmer mit Eichler teilte, hatte ich die besten Möglichkeiten,

[1] For all this, compare the influential work *Quadratic Forms and Orthogonal Groups.*

von einer Seminarsitzung zur nächsten die offen gebliebenen Fragen zu klären und so die quadratischen Formen an der Quelle zu studieren." [2]

There is little doubt that Kneser describes his own part in the interaction (or rather cooperation?) with Eichler too modestly.

2. Martin Kneser: Quadratic forms and the arithmetic of algebraic groups 1955 – 1970

In the mid 1950s, the theory of algebraic groups and the (arithmetic) theory of quadratic forms were still rather unrelated areas of research. On the side of groups, the classification of (semi)simple algebraic groups over algebraically closed fields was known by work of Claude Chevalley. Parallel to these achievements, Jacques Tits had (essentially) introduced the structures later called buildings which give a uniform geometrical interpretaion of all these groups, including the exceptional ones.

Already by the end of the 1950s, a completely new area of research had emerged, after Armand Borel had proved his fundamental theorem on the existence and conjugacy of maximal connected solvable subgroups. This made the classification of semisimple groups over arbitrary fields accessible, which was then rather quickly carried out mainly by Borel and Tits. They used k-split tori and the relative root system to reduce the question essentially to the anisotropic kernel, in analogy with the Witt decomposition of quadratic forms.

Over number fields, this approach embedded the earlier studies of algebras with involution, hermitian forms, Cayley octaves and Jordan algebras into a uniform theory. In this situation it was perfectly natural (after work of Lang and Tate) to introduce non-abelian Galois cohomology (H^0, H^1, abelian H^2) to treat such classification questions. Jean-Pierre Serre's course at the Collège de France 1962-63, leading to the famous Lecture Notes No. 5 *Cohomologie Galoisienne*, demonstrates how quickly the new method had been established.

The theory of semisimple groups over number fields in turn laid the foundations for a general treatment of arithmetic subgroups of algebraic groups, the fundamentals for which were developed by Borel and Harish-Chandra. Clearly, many substantial results had been obtained (much) earlier, mainly by Siegel, but the framework had dramatically changed, already by work of Weil in the late 1950s.

We now want to look at some of Martin Kneser's work as part of this general picture. We shall consider the following five fundamental contributions:

- (1) Class numbers of indefinite quadratic forms [**Kne56**]
- (2) Class numbers of definite quadratic forms [**Kne57**]
- (3) Representation measures of indefinite quadratic forms [**Kne61**]
- (4a) Strong approximation (Boulder Proceedings) [**Kne65a**]
- (4b) Strong approximation in algebraic groups I [**Kne65c**]
- (5) Galois cohomology of semisimple algebraic groups over p-adic fields, I and II [**Kne65d**].

[2] Finally a personal remark. Almost exactly 50 years have passed since I came as a young assistant to Münster, soon participated in Eichler's seminar, where at that time the latest results of his book *Quadratic Forms and Orthogonal Groups* were discussed. Since I shared my office in the institute with Eichler, I had the best opportunity to clarify from one seminar meeting to the next the remaining open questions and thus to study quadratic forms at the source.

All these papers except for (4a) are written in German, which was still the standard for German authors until the early or mid 1980s (at least in number theory). In the following three subsections, we shall sketch the contents of those papers and complement this by a quick report on the general developments at that time. The last subsection deals with smaller, scattered contributions by Kneser, which partly turned out to anticipate later developments in the constructive theory of integral quadratic forms of the 1980s and 90s.

2.1. Strong approximation and class numbers. In the paper (1), the strong approximation theorem for representations and for the orthogonal group is proved. In this context, the adelic orthogonal group is introduced for the first time. Later, in the 1960s, this developed into a new chapter in the theory of algebraic groups over number fields. In the present paper, the emphasis lies on the application to spinor genera and class numbers. It is shown that the number of spinor genera in a genus, which is a power of 2, can be interpreted as a group index. For this purpose, one has to compute local spinor norms which in turn leads to the question of generation of local orthogonal groups by reflections. Considerations of this kind go partly back to Eichler and were later continued and refined by various authors, including Kneser himself.

In the paper (2), Kneser introduces the method of *neighbouring lattices* as a new tool for the classification of positive definite lattices. The key observation on which this method is based had already been made by Eichler: if (V, q) is isotropic at p, then $\mathbb{Z}[1/p]$-lattices on (V, q) behave like indefinite lattices. More precisely, taking $\mathbb{Z}[1/p]$ as the ground ring, every spinor genus consists of only one class. In [**Eic52a**], the term "arithmetically indefinite" had been introduced to describe this setup. Kneser obtains this result as an immediate consequence of the strong approximation theorem from the previous paper (1), applied to the set of places $S = \{\infty, p\}$. As a consequence, for any two classes in the same spinor genus, there are representatives L, M s.t. $\mathbb{Z}[1/p]L = \mathbb{Z}[1/p]M$. It is elementary to see that such L and M can be connected by a chain $L = L_0, L_1, \ldots, L_i, \ldots, L_s = M$ of lattices such that $(L_{i-1} : L_{i-1} \cap L_i) = (L_i : L_{i-1} \cap L_i) = p$, for all i (i.e., L_{i-1} and L_i are p-neighbours). The resulting "neighbour method" is used in (2) to calculate the class number of the unit lattice I_n (sums of squares) up to dimension 16. It has been widely applied since then, also over number fields, for hermitian forms, and in computer implementations.

The paper (3) to my knowledge is the first publication on quadratic forms which makes full use of the adelic method. It demonstrates very well the elegance and effectiveness of this technique for the classical problem of representations by integral quadratic forms. Kneser considers classes, spinor genera and genera of representations of a number a by lattices M on a quadratic vector space of dimension n over a number field k. Extending earlier work of Siegel, who used analytic methods, and Eichler, he deals with the representation measure of a by M (or by the class of M) which generalizes the finite representation number of a by M for definite M. By definition, this number is a sum of measures of "representations", that is, of (classes of) pairs (x, M), where $x \in M$ with $f(x) = a$. Analogously (just by summing up) one defines the representation measure of a by a spinor genus or a genus of lattices. Alternatively, these representation measures can be seen as measures of certain subsets of the adelic coset space $\mathrm{O}_{\mathbb{A}}(V, x)/\mathrm{O}(V, x)$. The first theorem of (3) says that for $n \geq 5$ or $n = 4$ and $a \neq 0$, each genus of representations contributes the same amount to the representation measures of the different spinor genera in some genus of lattices, which therefore coincide. This is essentially already Siegel's result (who had given very long and complicated proofs), and Eichler in [**Eic52a**] had already done the crucial step of translating the question into arithmetic terms. The adelic setting simplifies Eichler's proof considerably and allows for the removal of some unnessecary restricitions. Kneser's paper sheds new light on Siegel's Main Theorem by viewing it as a result about an average over a genus of representations, not just a genus of lattices. In the second theorem of (3), Kneser uses this approach to prove a new result, not accessible by analytic methods, in the case $n = 3$. It involves the new concept of half-genera of ternary lattices. Concerning the paper (3), see also the remarks below at the end of subsection 2.3.

The papers (4a) and (4b) extend the investigation of strong approximation and class numbers from the theory of quadratic (and hermitian) forms to arbitrary (reductive) algebraic groups, whose structure theory over number fields then had just become available, by work of Borel and Tits. We need more notation and some definitions to describe the results precisely.

Notation

k	an algebraic number field
$\mathfrak{o}$	the ring of integers of k
$p, \ell, v, \ldots$	places (equivalence classes of valutations of k)
$k_p, \mathfrak{o}_p$	the completion of k, resp. $\mathfrak{o}$ at p
$p \in \mathfrak{o}_p$	a prime element for p, if p is finite
S_k	the set of all places of k
S	a finite set of places of k
$\mathbb{A}_k \subset \prod_{p\in S_k} k_p$	the ring of adeles of k
$(a_\ell)_{\ell\in S_k}$	a typical element of $\mathbb{A}$, so $a_\ell \in \mathfrak{o}_l$ f.a.a. ℓ
$\mathbb{A}(S) \subset \mathbb{A}$	the S-integral ideles, so $a_\ell \in \mathfrak{o}_\ell$ for $\ell \notin S$
V	a finite-dimensional vector space over k
L	a lattice in V
G	a linear algebraic group defined over $\mathfrak{o}$
$G(R)$	for any over-ring $R \supseteq \mathfrak{o}$ the group of R-points in G in particular
$G(\mathbb{A}_k)$	the adele-group of G over k
$G(k) \subset G(\mathbb{A}_k)$	diagonally embedded

The adele group of the general linear group $\mathrm{GL}(V)$ acts on the set of all lattices on V, since any of its elements $g = (g_v)$ stabilizes almost all localizations $L_p := \mathfrak{o}_p K$ of L and thus gL is well-defined by $(gL)_p := g_p L_p$.

DEFINITION 1. *Suppose that G is represented as a subgroup of* $\mathrm{GL}(V)$. *The* G-class *of the lattice L is the $G(k)$-orbit of L. The* G-genus *of L is the $G(\mathbb{A}_k)$-orbit of L. The* G-class number *of L is the number of G-classes in the G-genus of L.*

DEFINITION 2 (Strong Approximation). *Let G be an algebraic group over k and S be a finite set of places of k. We say that* strong approximation holds for the pair (G, S) *if $G(k)G(\mathbb{A}(S))$ is dense in $G(\mathbb{A}_k)$.*

Partial results on strong approximation (not exactly in this language) for the various types of classical groups were already known by work of Eichler and Kneser from the 50s (see above for orthogonal groups). In the beginning 60s, Kneser started to relate the question to the structure theory of algebraic groups, and in particular found out that strong approximation in the above simple sense can hold only for simply connected groups. It took considerable effort to prove that this assumption is actually also sufficient (in the anisotropic case):

THEOREM 2 (Kneser 1965, Platonov 1969).
Strong approximation holds for all pairs (G, S)*, where* G *is simply connected almost* k*-simple and* $G(k_v)$ *is not compact for at least one* $v \in S$*.*

Actually, Kneser proves for classical groups in [**Kne65c**] that strong approximation holds for every simply connected almost k-simple group G for which the Hasse principle is true. After the announcement of the general case (under the same hypothesis) it turned out by work of Platonov that there is a different, eventually simpler proof of the strong approximation theorem, based on the Kneser-Tits hypothesis on generation by rational unipotent elements. This proof is independent of the Hasse principle; see [**PlRa94**].

2.2. Local-global principles and Galois cohomology. We now come to the last entry (5) of our above list. It is a contribution to various questions put forward by Serre in his course *Cohomologie Galoisienne* mentioned above.

DEFINITION 3. *The Hasse principle holds for an algebraic group* G *over* k *if the canonical map*

$$H^1(k, G) \to \prod_{v \in S_k} H^1(k_v, G)$$

is injective.

The following result is fundamental for the application of this principle to classification problems since it allows one to replace the right hand side in Definition 3 by the finite product over all real places of k; see also the comment below after Theorem 4.

THEOREM 3 (Kneser 1965). *If* G *is a semisimple simply connected group over a local field* k_p *of characteristic* 0*, then* $H^1(k_p, G) = 0$*.*

The proof of Theorem 3 uses the classification and structure theory of semisimple groups, but also a lot of case-by-case investigations. The desire for a uniform proof without case distinctions could only be satisfied about 20 years later; apparently, one has to pay the price of using some of the advanced parts of the Bruhat-Tits theory of group schemes over local fields [**BrTi87**]. The result holds if the residue field of k has cohomological dimension ≤ 1; in particular, the hypothesis on the characteristic of k is not needed.

It had been generally conjectured after Serre's course *Cohomologie Galoisienne* (see [**Ser65**]) that the Hasse principle should hold for large classes of semisimple groups over number fields, including all simply connected ones. Because of long-standing technical difficulties with the exceptional groups of type E_8, it took about 25 years until the following theorem was eventually proved completely.

THEOREM 4 (Kneser 1965, Harder 1965/66, Chernousov 1989). *The Hasse principle holds for all semisimple simply connected algebraic groups.*

The isomorphism classes of k-forms of an object defined over an algebraic number field k are in one-to-one correspondence with the Galois cohomology H^1 of its automorphism group, whose connected component often is not simply connected.

But if $\widetilde{G}$ denotes the universal cover of a connected semisimple group G, the exact cohomology sequence

$$1 \to Z(k) \to \widetilde{G}(k) \to G(k) \to H^1(k,Z) \to H^1(k,\widetilde{G}) \to H^1(k,G) \to H^2(k,Z)$$

allows for the derivation from Theorems 3 and 4 of immediate consequences also about $H^1(k,G)$ and the $H^1(k_v,G)$. See [**Kne65b**], [**Ser65**] or Springer's article in [**Boulder65**].

The proof of Theorem 4 has been given by Kneser for the classical groups in [**Kne65b**], see [**Kne69**] for details, and by Günter Harder in [**Har65**] and [**Har66**] for the exceptional forms of type D_4 and for the types E_6 and E_7, partly also for E_8 (the cases G_2 and F_4 are easy). The proof for groups of type E_8 could eventually be completed by Chernousov in his paper [**Che89**].

Here are some rough remarks about the strategy of the proof of Theorem 4 (and also of Theorem 3). Properties, and possibly vanishing of cohomology classes $\xi \in H^1(k,\bar{G})$, where $\bar{G} \hookrightarrow \operatorname{Aut} G$ is the adjoint group, are studied via properties of the twisted group G^ξ. Cohomology classes can have at most the order $2 \cdot 3 \cdot 5$, correspondingly the groups split over extensions of degree at most 30 (depending on the type, at most 6 for type $\neq E_8$). The proof also uses induction on the dimension of G. It is shown that every cohomology class is in the image of $H^1(k,T)$ or $H^1(k,H)$ for some k-split torus T, respectively an appropriate reductive subgroup H of G (see [**Har65**]). As Harder points out, certain simplifications can be obtained by making use of his later paper [**Har75**] (in which he treats primarily the case of function fields). A complete proof of Theorem 4 is contained in the book [**PlRa94**].

2.3. Siegel's theorem and Tamagawa numbers. We have mentioned earlier that the theorem of (Minkowski and) Siegel can be viewed as a quantitative version of the Hasse principle for quadratic forms. This raises the question for a quantitative or numerical variant of the Hasse principle for semisimple groups. An answer is given by the notion of Tamagawa number of an algebraic group and its computation, which we shall briefly sketch now.

The Tamagawa measure on the adelic points of a semisimple algebraic group G defined over a number field is a certain, canonically normalized product measure. It induces an invariant measure on the coset space $G(\mathbb{A}_k)/G(k)$, whose volume is actually finite. The *Tamagawa number* of G is defined as

$$\tau(G) := |\operatorname{disc} k|^{-\dim G/2} \operatorname{vol} G(\mathbb{A}_k)/G(k).$$

The first detailed treatment of these notions was given by André Weil in a course at Princeton in 1961; see [**Weil61**]; he calculated the Tamagawa number for various groups and conjectured the following:

THEOREM 5. *The Tamagawa number of any semisimple simply connected algebraic group is equal to one:* $\tau(G) = 1$.

The eventual full proof of this theorem required the effort of several people over a long period. For most of the classical groups, a case-by-case verification had been given by Weil around 1960; see the notes quoted above. For split groups, the theorem was proved by Langlands [**Lan65**], using his notion of Eisenstein series for adele groups. This work was extended to quasi-split groups by Lai only in 1980 [**Lai80**]. The general case was finally proved by Kottwitz in 1988; as an essential step he showed a certain invariance property under inner twists.

Here are a few more remarks on the history of Siegel's theorem and Tamagawa numbers. It seems that the theory of adelic algebraic groups starts with work of Takashi Ono, who treated first the commutative case, using Chevalley's idele theoretic approach to class field theory (see his contribution in the Boulder Proceedings [**Boulder65**] for a survey). In the late 1950s, Tamagawa introduced adelic algebraic varieties, the Tamagawa measure and thus (implicitly) the Tamagawa number of an algebraic group over a number field. He himself did not publish much about it, but apparently he knew that Siegel's theorem is equivalent to $\tau(\mathrm{SO}) = 2$. To my knowledge, the first published exposition of these concepts is a talk by André Weil in the Séminaire Bourbaki in May 1959. Of course, Weil was not only surveying Tamagawa's work; a considerable part of his own research at that time dealt with the relations between discrete groups and number theory, and more specifically with putting Siegel's work on arithmetic groups into an algebraic-geometric framework. We have already mentioned Weil's Princeton lectures from 1961, where he presents his own results on the calculation of $\tau(G)$ for all classical groups.

I could not figure out to what extent Kneser contributed to the question of Tamagawa numbers. However, to my best knowledge he was the first who had realized that one can conveniently use the adelic orthogonal group for a proof of the Minkowski-Siegel formula. This remark is contained as a footnote already in his paper (1) = [**Kne56**] on p. 326. There Kneser comments on his definition of "Spaltvektoren" and "Spaltautomorphismen" (these are ideles without infinite components; the terminology did not come into later use) as follows: "Die unendlichen Primstellen haben wir außer Betracht gelassen, da wir sie nicht brauchen; nimmt man sie mit hinzu, so erhält man das genaue Analogon zu den CHEVALLEYschen Idelen, das man mit Vorteil beim Beweis des SIEGELschen Hauptsatzes über quadratische Formen verwenden kann." [3]

In the paper (3) = [**Kne61**], this remark is made precise: in formula (2) of that paper, which roughly reads $\mu = \mu_\infty \cdot \mu_0$, Kneser considers the representation of a vector x (or the form value $f(x)$) in a quadratic space over a number field by a lattice uM in the genus of the lattice M in V, where u is an element of the adelic orthogonal group. The number μ is the Haar measure of (the image of) a double coset containing u in the adelic homogeneous space $\mathrm{O}_A(V,x)/\mathrm{O}(V,x)$ (stabilizer of x), and μ_∞ is the measure of representation of x by M on which we have reported above. The factor μ_0 is the Haar measure of $\mathrm{O}_0(V,x) \cap \mathrm{O}_A(M,x)$, where O_0 denotes the finite part of the adele group; in particular, μ_0 depends only on the genus of M. In the first footnote on p. 191 Kneser gives further explanations on μ_0: "Dieser Faktor stellt sich als das Inverse des Produkts von $\mathfrak{p}$-adischen Darstellungsdichten heraus. Summiert man über die verschiedenen Doppelnebenklassen, so erhält man (...) SIEGELS Satz, vorausgesetzt ..." [4] (he then refers to [**Weil61**] for some further details). In the definite case over the rationals, the details of this proof have been carried out by Kneser in his lectures at the university of Göttingen; see [**Kne73**]. In the continuation of this footnote, Kneser in addition explains how one has to modify the proof in order to obtain the representation numbers for

[3] We have not considered the infinite primes since we do not need them; if one takes them into account, one obtains the exact analogue of Chevalley's ideles which can be used advantageously in the proof of Siegel's main theorem on quadratic forms.

[4] This factor turns out to be the inverse of the product of $\mathfrak{p}$-adic representation densities. By summing over the different double cosets, one obtains (...) Siegel's theorem, provided ...

representations with congruence conditions: "Ersetzt man im Falle definiter Formen mit rationalen Koeffizienten $O_A(M, x)$ durch die Gruppe derjenigen $u \in O_A(V, x)$, die nicht nur M, sondern mit einer beliebigen aber festen Zahl v alle Restklassen von M mod vM festlassen, so erhält man die Verallgemeinerung von VAN DER BLIJ [**Blij49**]." [5]

2.4. Outlook I: Constructive theory of integral quadratic forms. With the results we have reported on so far, the arithmetic theory of quadratic forms and orthogonal groups had reached a certain degree of completion. This is certainly true for the construction of the general foundational theories. Almost surely, this is the main (but not the only) reason why Kneser had only few publications on quadratic forms after the mid 1970s (not counting a couple of papers of essentially historical nature).

The developments concerning integral quadratic forms after 1970 are not our theme here, but roughly summarizing one can say that the investigation of individual objects or restricted situations, often in interaction with other fields (finite group theory, modular forms, invariant theory, theory of singularities, topology) gained more attention. Like in other parts of mathematics, general theories, as opposed to concrete objects, were no more the only serious goal. These general changes of attitude have been particularly striking in the theory of finite (simple) groups, but can also be observed in quadratic forms. This development paralleled new developments in combinatorial mathematics and a renaissance of classical fields like extremal problems in geometry of numbers. Also, the dramatically improving facilities for performing concrete computations led to new activities and shed new light on theories where general finiteness results are important, like class numbers of quadratic forms.

With this general picture of some number theoretic and algebraic developments in mind, we will now have a brief look at three further papers by Kneser (still from the same period of the mid 50s to the early 70s), and also at some later work initiated by him.

In the note *On the theory of crystal lattices* (in German) [**Kne54**], Kneser gives a short and conceptual proof of the well known fact that every positive definite lattice uniquely decomposes into orthogonally indecomposable lattices. The proof is achieved by regarding an appropriate set of generating vectors as a graph, where by definition any two non-orthogonal vectors are connected by an edge. The desired components of the lattice now are found as the sublattices generated by the connected components of that graph. From today's point of view, such an approach can be considered as more or less straightforward. But in those days, the combinatorial or graph-theoretic way of thinking was not yet common. This short paper is a good example of Kneser's general ability of bringing matters to the point. His search for conceptual, if at all possible "perfect" proofs is characteristic for all his publications, no matter what the mathematical subject is.

A similar example is the paper *Two remarks on extreme forms* [**Kne55**], where Kneser gives a new proof of the classical theorem of Korkine, Zolotareff and Voronoi that an integral quadratic form is extreme if and only if it is perfect and eutactic. Almost at the same time and independently, also Barnes gave a new proof of

[5] In the case of definite forms with rational coefficients, if one replaces $O_A(M, x)$ by the group of those $u \in O_A(V, x)$ that preserve not only M, but also all cosets of M mod vM for some arbitrary but fixed v, one obtains the generalization of VAN DER BLIJ [**Blij49**].

Voronoi's theorem, which was the basis for later variations and extensions of the theory by various authors. But if one is just interested in the original theorem, Kneser seems to offer the shortest proof.

The paper *Linear relations between representation numbers of quadratic forms* (in German) [**Kne67**] was dedicated to Carl Ludwig Siegel on the occasion of his 70th birthday. It takes up the question treated by Witt in the above-mentioned paper [**Witt41**], but uses completely different methods. By direct calculation with (sub- and over-)lattices Kneser shows that not only for $n = 2$, but also for $n = 3$, there are as many n-dimensional sublattices of a given isometry class in the lattice $\widetilde{D}_{16}$ as there are in $E_8 \perp E_8$. That is, the represention numbers of ternary sublattices, in other words, the Siegel theta series of degree 3, coincide for $\widetilde{D}_{16}$ and for $E_8 \perp E_8$. That these computations are possible by hand, and in an understandable way, of course relies on the high symmetry of the two big, representing lattices, but also on the fact the the configurations of roots (norm 2 vectors) of all involved lattices suffice to control the situtation. The method of describing certain lattices by their root system and "glue vectors" (in the dual lattice) had come into general use only 25 years later (popularized by Conway and Sloane). The paper under consideration shows that Kneser was aware of these ideas and was able to apply them in a masterful way, long before more ambitious classification programs for lattices were initiated, and sophisticated techniques of various kinds were developed by B.B. Venkov, H.-G. Quebbemann, and later by many others.

It would be simplistic to reduce the recent (in the sense of this section) theory of integral quadratic forms to constructive aspects. Another feature is a renewed emphasis on analytic aspects, in particular on the study of theta series. Since the 1980s, one could even speak of a certain convergence of the arithmetic and the analytic theory of quadratic forms, including questions about Siegel modular forms and the study of related objects like weight enumerators of various kinds of codes. An important question which historically belonged to the realm of analytic methods is the problem of representability of numbers or forms by an individual positive definite form (not just a genus of forms). A classical theorem of Kloostermann and Tartakovskii says that a form of rank $m \geq 5$ represents every number which satisfies the necessary local conditions and in addition is larger than some appropriate constant C (depending on the form). The original proof uses the circle method of Hardy and Littlewood. The theorem can also be derived from Siegel's main theorem, phrased in terms of theta series, together with estimates for the Fourier coefficients of cusp forms. In his lectures at the University of Göttingen in 1973/74, Kneser gives a purely arithmetical proof of that theorem, which is related to his proof of the Minkowski-Siegel Theorem presented in the same course (cf. the end of subsection 2.3 above). Also, a variation of the theorem for primitive representations is given. These results are merged with an analogous result by J. Hsia over number fields and results by Y. Kitaoka about representations of forms of rank $n \geq 1$ (by forms of rank m, as above) into the influential paper [**H-K-K78**]. It is shown that the conclusion of the Kloostermann-Tartakovskii theorem remains true, provided $m \geq 2n + 3$. Using analytic methods, this result could be proved only for $n \leq 2$, by work of Y. Kitaoka; for larger n, the estimate for m is weaker. Also, the arithmetic method allows to treat representations with congruence conditions and primitive representations. Only recently, an alternative method, namely ergodic theory for the orthogonal group and its homogeneous spaces, became available, by work of J. Ellenberg and A. Venkatesh. It gives a better bound $m \geq n + 5$, but so far it guarantees merely the existence of the above constant C and does not give any method to produce an explicit C.

To finish this partial overview of Kneser's work, I want to report briefly on four of the in total 21 doctoral dissertations which were supervised by him. This is my personal choice. (As earlier, the titles are translated from German.)

Hans-Volker Niemeier: *Definite quadratic forms of discriminant* 1 *and dimension* 24 (1968)
Horst Pfeuffer: *One-class genera of totally positive quadratic forms in totally real algebraic number fields* (1969)
Jürgen Biermann: *Lattices with small automorphism group in genera of $\mathbb{Z}$-lattices with positive definite quadratic form* (1981)
Yuriko Suwa-Bier: *Positive definite quadratic forms with equal representation numbers* (1984)

In the dissertation of H.-V. Niemeier, the complete list of all positive definite even unimodular lattices in dimension 24 is derived. For many well known reasons, coming e.g. from modular forms, coding theory, finite group theory, this is the most natural of all classication problems for integral quadratic forms (excluding comparatively trivial cases like the corresponding problem in dimensions 8 and 16). At that time, the Leech lattice was known for a couple of years, and the neighbour method did exist, so it was quite natural, although tedious, to apply this method to enumerate the whole genus and to derive in this way the uniqueness of the Leech lattice as an even unimodular lattice in dimension 24 with minimal norm 4. On the way, Niemeier describes the gluing theory for the root lattices in a complete and fully explicit fashion. Later, great insight into the classification of this particular genus of lattices was gained by work of Boris Venkov, who gave an a priori determination of the root systems of the minimum 2 lattices, and John Conway, who proved the uniqueness of the Leech lattice more directly. But the list of all lattices is due to Niemeier, and since Venkov leaves the proof of uniqueness of the lattice, for each given root system, as a tedious case-by-case verification to the reader, in my opinion Niemeier's work was indispensable also after Venkov's work. Alternatively, one could (and did, Conway and Sloane) determine the orders of the orthogonal groups of all known lattices and check the completeness of the list with the mass formula.

H. Pfeuffer's dissertation deals with the growth of the class number $h = h(\mathcal{G})$ of genera $\mathcal{G}$ of lattices in totally positive definite quadratic spaces over totally real number fields. Roughly speaking (suppressing certain problems coming from non-free lattices), the result is that h tends to infinity in any of the three parameters dimension of the space, norm of the determinant (volume) ideal and the field discriminant. The special case of the field $\mathbb{Q}$ had been treated thirty years earlier by W. Magnus. The result is derived from Minkowski's mass formula and its generalization by Siegel to number fields. Recall that the mass is defined as $\operatorname{mass} \mathcal{G} = \sum_{i=1}^{h} |\mathrm{O}(L_i)|^{-1}$, where the L_i run over a set of representatives for the isometry classes in $\mathcal{G}$; so it is the inverse of the quantity of Theorem 1 for $M = N \in \mathcal{G}$. The size of the mass depends on the three mentioned parameters. It is classical and relatively easy to see (cf. growth of Bernoulli numbers) that the mass tends to infinity rapidly, in fact exponentially, with the dimension. With more work, involving the estimation of local densities at primes dividing the determinant, one can also see that the mass grows with the determinant of the genus. It was Pfeuffer's contribution to extend these computations and estimates to arbitrary number fields, where a major technical difficulty is the determination of the local densities at dyadic (ramified) primes. For this, refined versions of Hensel's lemma due to Kneser are used; see eg. [**Kne73**]. The estimation of the mass from below in terms of the field discrimant is easy. The growth of the class number then follows from the trivial estimate $\mathrm{mass}(\mathcal{G}) \leq \frac{1}{2} \cdot h(\mathcal{G})$.

The dissertation by J. Biermann on small automorphism groups also belongs to the realm of consequences of the Minkowski-Siegel mass formula. With the result of Magnus and Pfeuffer on the growth of class numbers in the background, one asks for the orders of the orthogonal groups $\mathrm{O}(L)$ of the lattices L in one genus. The naive question of how good or bad the estimate $|\mathrm{O}(L)| \geq 2$, used to derive the growth of the class number $h(\mathcal{G})$ from the growth of $\mathrm{mass}(\mathcal{G})$, actually is, leads, after hard work, to a reasonable answer: for fixed dimension and large determinant, most lattices in $\mathcal{G}$ have trivial orthogonal group, i.e. $\mathrm{O}(L) = \{\pm\,\mathrm{id}\}$, and thus the estimate converges to the truth. A more precise statement is the following:

Theorem 6 (J. Biermann, 1981). *For a genus $\mathcal{G}$ of totally definite integral quadratic forms, let*

$$h_0(\mathcal{G}) := \mathrm{card}\{[L] \in \mathcal{G} \mid \mathrm{O}(L) = \{\pm\,\mathrm{id}\}\}.$$

Then, for fixed dimension $n \geq 3$ of the lattices,

$$\frac{h_0(\mathcal{G})}{h(\mathcal{G})} \to 1, \ \textit{if} \ \det \mathcal{G} \to \infty.$$

In my opinion, this thesis has been an important contribution to the literature on quadratic forms which would certainly have deserved publication in a journal. It looks plausible, but cannot be read off easily from Biermann's work, that the same holds for $\max\{n, \det\} \to \infty$. I am not aware of any serious hints that, over number fields, h_0/h tends to 1 also as a function of the field discriminant. But certainly this is a natural guess. The genera of even unimodular lattices of fixed dimension $n = 4$ over real quadratic fields of discriminant $d \to \infty$ could supply a first, manageable test case.

Unlike in the previous three cases, I have chosen the dissertation of Suwa-Bier not for its results, but for the problem itself. The question treated in this work is a very natural one and has been studied by various authors in quite different contexts: to what extent is an integral quadratic form determined by its representation numbers, or, more geometrically, a lattice L in Euclidean space $\mathbb{R}^n$ determined by the norms of its vectors, including multiplicities. In differential geometric terms, this problem had been studied as the problem of isospectral flat tori $\mathbb{R}^n/L$ already by John Milnor in the 1960s, and he had used the above 16-dimensional example put forward already by Witt. Later, examples of "isospectral" lattices of dimensions 12 and 8 were given by Kneser and Y. Kitaoka, respectively. The latter considered theta series and used the theory of modular forms. In 1988 the dimension of such examples was pushed down to 4 by Alexander Schiemann and independently by Ken-Ichi Shiota. Schiemann made exhaustive computer tabulations to approach systematically the smallest determinant 1729 for which such an example exists. Shiota found his examples in the course of the investigation of theta series as generators for certain spaces of modular forms of weight 2. See [**Schi90, Shi91**].

In dimension 2, our problem is elementary and has essentially nothing to do with modular forms: it is readily seen that the first three values of the spectrum and their multiplicities determine the three coefficients of a reduced Gram matrix and thus also the isometry class of a lattice. It was generally believed, after no counterexampes could be found, that a similar approach should work in dimension 3. Suwa-Bier could prove that the number of non-isometric lattices with the same spectrum is bounded by a constant only depending on the dimension, and is at most four in dimension 3. But a classical proof of the desired sharp result, with paper and pencil and based on hand-made case distinctions, eventually appeared to be impossible. The actual solution was given by Alexander Schiemann in his dissertation, supervised by F. Grunewald, in 1993 at the University of Bonn: a three-dimensional lattice is indeed determined up to isometry by its representation numbers [**Schi93**]. Roughly speaking, the strategy of the proof is as follows: a certain list of "good cases", distinguished by appropriate inequalities for pairs of reduced positive definite 3×3-matrices, is generated on a computer; this list turns out to cover eventually all cases. To formulate it a little bit more precisely, the 12-dimensional cone of pairs of reduced positive definite 3×3-matrices is covered by a (large) number of subcones, each defined by inequalities coming from vectors representing one of the successive minima, and such that the desired uniqueness is by construction true on each subcone.

2.5. Outlook II: Small class numbers. In the 1980s, the question of class numbers of lattices, or arithmetic groups, came up (again) in a geometric context: finite group theorists and geometers (Tits, Kantor, Timmesfeld, Stroth, Ronan, Meixner, Wester and others) worked on the classification of certain classes of (locally) finite incidence geometries belonging to a Coxeter diagram (and more general diagrams), together with an automorphism group acting transitively on the maximal flags of the geometry. The maximal flags are called "chambers"; the description of a geometry as a "chamber system" is also common. The Coxeter diagram says that the chamber systems in question locally look like finite buildings. There exists an appropriate covering theory for chamber systems (related to group amalgamations), and the universal 2-cover under rather general assumptions is a building. If the diagram belongs to the known list of affine Coxeter-Dynkin-diagrams and the rank is ≥ 4, then this building is known to be a Bruhat-Tits building. Also, a chamber transitive automorphism group lifts to a chamber transitive group on the universal cover, which is a discrete subgroup of the (known) full automorphism group of the affine building. This lifted group is in principle known: it is arithmetic. A general survey of these works is given in the paper [**Kan90**] by William M. Kantor.

To describe the relation with class numbers precisely, we maintain the general notation introduced previously, and specifically we consider the following:

k	a totally real algebraic number field
$\mathfrak{o}, p, k_p, \mathfrak{o}_p$	as before
$G \subset \mathrm{GL}(V)$	simply connected semisimple, almost simple over k anisotropic at the infinite places
p	a fixed finite place of k s.t. $\mathrm{rk}_{k_p} G \geq 2$
$\bar{k}_p := \mathfrak{o}/p\mathfrak{o}$	the residue field at p
$\Delta := \Delta(G(k_p))$	the Bruhat-Tits building of $G(k_p)$.
L	a lattice in V s.t. $\mathfrak{o}_p L =: L_p$ defines a vertex of Δ
$\Delta_0 \cong \Delta(G(\bar{k}_p))$	the residue (star, link) of L in Δ.
$\Gamma := G(\mathfrak{o}[\frac{1}{p}])$	a $\{p\}$-arithmetic discrete subgroup of $G(k_p)$
$\Gamma_0 := G(\mathfrak{o})$	the finite stabilizer of L in $G(k)$.

The following proposition makes precise the relation between class numbers and chamber transitivity of discrete groups as indicated on p. 40 of Kantor's above-mentioned paper.

Proposition 1. *Under the above assumptions, the following properties of the lattice L (resp. the arithmetic groups Γ, Γ_0) are equivalent:*

(1) Γ *acts chamber transitively on* Δ.
(2) (i) Γ_0 *acts chamber transitively on* Δ_0,
   (ii) $h_G(L) = 1$.

Proof: "$\Longrightarrow$": (i) is obvious from the assumption, since the chambers of Δ_0 are exactly the chambers of Δ containing the vertex L. For (ii), we have to show that

$$G(\mathbb{A}_k) = G(k) \cdot G(\mathbb{A}(\infty)). \tag{1}$$

Since G is isotropic at p, we can use strong approximation for the set of places $\infty \cup \{p\}$:

$$G(\mathbb{A}_k) = G(k) \cdot G(\mathbb{A}(\infty \cup \{p\})). \tag{2}$$

Since Γ acts chamber transitively on Δ it also acts vertex transitively on the vertices of a given type, which for "type L" translates as

$$G(k_p) = \Gamma \cdot G(\mathfrak{o}_p) = G(\mathfrak{o}[\frac{1}{p}]) \cdot G(\mathfrak{o}_p). \tag{3}$$

Given an arbitrary adele $(\sigma_\ell) \in G(\mathbb{A}_k)$, first use (2) and write it as $\sigma \cdot (\tau_\ell)$ with $\sigma \in G(k)$ and $\tau_\ell \in G(\mathfrak{o}_\ell)$ for all $\ell \neq p$. Then use (3) and write $\tau_p = \gamma \cdot \delta$ with $\gamma \in G(\mathfrak{o}[\frac{1}{p}])$ and $\delta \in G(\mathfrak{o}_p)$. Now replace the original decomposition of (σ_ℓ) by

$$\sigma_\ell = (\sigma\gamma) \cdot (\gamma^{-1}\tau_\ell) \text{ for all } \ell.$$

Since p is a unit in all $\mathfrak{o}_\ell, \ell \neq p$, we have $\gamma \in G(\mathfrak{o}_\ell)$ for all $\ell \neq p$ and thus $\gamma^{-1}\tau_\ell$ is still in $G(\mathfrak{o}_\ell)$. Furthermore, $\gamma^{-1}\tau_p = \delta$ is in $G(\mathfrak{o}_p)$ by construction. Thus the second factor of the new decomposition is in $G(\mathfrak{o}_\ell)$ for all ℓ, and therefore the given adele is a member of the right hand side of (1).

"$\Longleftarrow$": Because of assumption (i), we only have to show the transitivity of Γ on the vertices of "type L", that is, the vertices in the orbit $G(k_p)L \subset \Delta$. But this transitivity is equivalent to (3), as has already been used. To prove (3), just apply assumption (1) to adeles which are 1 outside p: for any given $\sigma_p \in G(k_p)$, there exists $\sigma \in G(k)$ and an adele (τ_ℓ) with $\tau_\ell \in G(\mathfrak{o}_\ell)$ for all ℓ s.t. $\sigma_p = \sigma \cdot \tau_p$ and

$\sigma \cdot \tau_\ell = 1$ for all $\ell \neq p$. But this means $\sigma \in G(\mathfrak{o}_p)$ for all $\ell \neq p$, thus $\sigma \in G(\mathfrak{o}[\frac{1}{p}])$, as desired. □

This proof is completely analogous to the derivation of the neighbour method from strong approximation. See in particular equation (3) and compare Kneser's 1957 paper.

As a consequence of the above result, discrete chamber transitive groups on affine buildings are very rare. Examples had been found in the works of Kantor and Meixner/Wester mentioned above. A full classification has been announced by Kantor, Liebler and Tits in [**KLT87**]. The "generic case" of the proof deals with the non-existence of such a subgroup for almost all algebraic groups G. It is briefly sketched in that announcement (see also Kantor's survey quoted above). It uses only condition (i) (or rather the chamber transtivity on residues of all types). This condition alone is already very restrictive, by a theorem of Gary Seitz. A complete proof of the classification is not published.

Shortly after the appearance of [**KLT87**], a proof of the finiteness result based on the computation of covolumes of S-arithmetic groups has been given by Borel and Prasad in [**BoPr89**] and [**Pra89**]. Since covolumes and class numbers are related by the (known) Tamagawa numbers, a concrete application of these results for particular classes of groups would probably lead one back to Proposition 1. For instance, for orthogonal groups the finiteness of similarity classes of lattices satisfying condition (ii) follows from Pfeuffer's result described above. For spin groups and for unitary groups one could use the results on the growth of class numbers of the 1971 dissertation of Ulf Rehmann [**Reh71**], again supervised by M. Kneser. Over the rationals, the finite list of all positive definite lattices of rank ≥ 3 and class number 1 is known from the work of G.L. Watson. (To be precise, in his published work Watson restricts himself to lattices which are "square free". This is a local condition, for every prime p, and is irrelevant for our actual problem since it is fulfilled for all lattices representing a vertex of the Bruhat-Tits building.) This list is rather long (going up to dimension 10) but practically all lattices (and primes) are then immediately ruled out by condition (i). Over number fields, the list of lattices with class number 1 is probably shorter, but it is presently unknown. It is an open question whether one could use a clever combination of the transitivity condition (i) and the class number, or covolume condition (ii) effectively for a revision of the KLT-classification.

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