“GROWTH GOES DOWN, BUT OF WHAT?” A CASE STUDY ON LANGUAGE DEMANDS IN QUALITATIVE CALCULUS

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The instructional approach of qualitative calculus aims at developing conceptual understanding for the relationship between amount and change, e.g., by connecting multiple representations of complex context phenomena. This article presents a design experiment with two Grade 11 students’ pathways towards the mathematical concepts of amount, change, and change of change. Qualitative analysis is used to show how deeply concept and language development are intertwined and to unpack the language demands occurring on the students’ conceptual pathways.

THEORETICAL BACKGROUND: LANGUAGE DEMANDS IN QUALITATIVE CALCULUS: AMOUNT AND CHANGE OF CHANGE

Amount and change as core concepts in qualitative calculus

Calculus has often been shown to be conceptually challenging for many students. That is why approaches of qualitative calculus have been suggested in order to promote conceptual understanding of the relationship between quantities of amount and change (Thompson & Thompson, 1994) long before change is mathematized as average and instantaneous rate of change and the derivatives and their procedural rules (see Stroup, 2002, for steps even before the rate of change). “Understanding qualitative calculus is cognitively significant and ‘structural’ in its own right” (Stroup, 2002, p. 170). The “own right” is justified, e.g., by the relevance of qualitative concepts for out-of-school contexts such as newspaper headlines:

> “Fewer child births. In recent years, the population growth has decreased.”

Empirical evidence has been provided that many students and adults misinterpret this statement as reporting about declining populations. But it is the *population growth* function *f’* that decreases, not the *population amount* function *f*, and *f* can still grow even if *f’* decreases, then the growth becomes just slower. As Hahn and Prediger (2008) have shown, this misunderstanding occurs especially in a phenomenon they called *counter-directional covariation*, i.e., when the covariation of *f* and *f’* have different directions: One increases and the other one decreases. They explained the specific difficulties in understanding counter-directional covariation phenomena using their level model (see Fig. 1 from Hahn & Prediger, 2008). It builds upon Confrey’s and Smith’s (1994) distinction of two approaches for functions, the *correspondence* approach (asking what the value of *f* at x_1 and x_2 is) and the *covariation* approach (asking how *f* changes with *x*).

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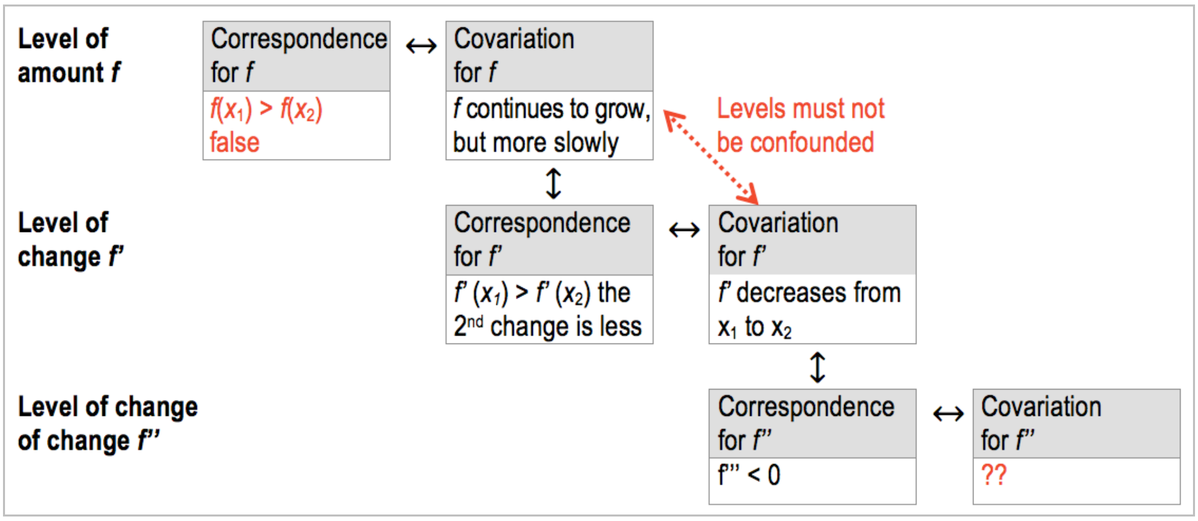


Fig. 1 Complex relationships of amount and change: Shifts of approaches and levels for the example of the headline "population growth decreased"

When shifting the levels (see Fig. 1), the covariation approach for *f* turns into a correspondence approach for *f'*, and on the next level, the covariation approach for *f'* turns into a correspondence for *f''*. In contrast, statements about covariation of *f'* must not be identified with statements about covariation of *f*, as they can have opposite directions.

Research focus on specifying language demands in conceptual development

In general, developing conceptual understanding has been shown to depend much more on language than procedural knowledge, which can be traced back to the epistemic function of language (Moschkovich, 2010; Schleppegrell, 2007): Constructing meanings of abstract mathematical concepts requires the use of concise and powerful language as a thinking tool for disentangling complex thoughts. For these purposes, everyday language must be enriched by the *school academic language register*, as its language features are optimized for expressing reified and abstract relationships in concise and explicit ways (Schleppegrell, 2007; for functions, Prediger & Zindel, 2017).

Although the general affordances and challenges of school academic language have been profoundly analyzed (Moschkovich, 2001; Schleppegrell, 2007), this general knowledge is not yet concrete enough to support language learners in their mathematics learning. Thus, more topic-specific research is required to specify the concrete language demands for specific mathematical topics (Prediger & Zindel, 2017). This paper intends to contribute to this research agenda by unpacking two students' learning pathways in a design experiment towards the distinctions of amount, change, and change of change with the following research questions:

(RQ1) Which language demands occur when students engage in productive struggle with the qualitative concepts of amount, change, and change of change?

(RQ2) How can students' learning be supported?

Design of the investigated teaching learning arrangement

Following Duval (2006) and many other mathematics education researchers, developing conceptual understanding can be fostered by the design principle of *connecting multiple representations*. This design principle has also proved to be powerful for language learning, as relating different registers and representations of the multiple semiotic system supports the language learners' construction of meanings (e.g., Prediger & Zindel, 2017). Furthermore, a key language challenge in mathematics teaching "is to help students move from everyday, informal ways of construing knowledge into the technical and academic ways that are necessary for disciplinary learning [and connect with subject-specific] ways of using language to construct knowledge" (Schleppegrell, 2007, p. 140). In this quotation, the sequencing for language registers is directly combined with the sequencing of epistemic practices; hence it calls for integrating conceptual and language-related learning opportunities.

For sequencing learning opportunities along a concept- and language-related continuum, a good starting point is students' intuitive distinctions of amount and change as documented by Stroup (2002): "Learners are observed to be able to move between rate and amount renderings — most often in graphical forms" (p. 170). Thompson and Thompson (1994) have shown in a case study that "computational language" is not enough to construct meanings that allow for distinguishing amount and change, but they leave open how "conceptual language" should look.

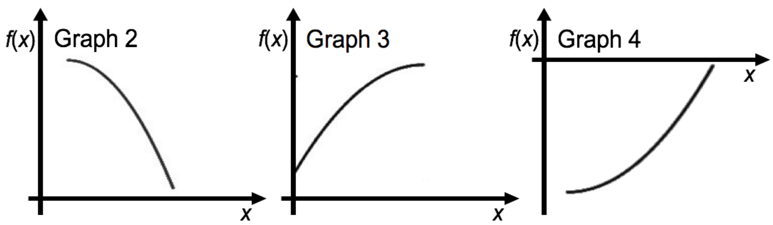
The design is already a first answer to RQ2: With activities as printed in Fig. 2, the learning arrangement combines all three design principles: (1) relating representations

Task 1. Match the newspaper headlines with Graph 1-6 shown below. Add fitting axes labels.

Fewer child births
In recent years, the population growth has decreased.

Volcanic island arises
The fast growth will soon bring the island above sea level.

Orangutans threatened
Palm-oil plantations entail dramatic decrease of population.



[Fourth headline and three other graphs not printed]

Task 2. Match the formal conditions A-H from the cards with graphs and headlines from Task 1.
Justify your decisions.

A	B	C	D	E	F	G	H
$f(x) > 0$	$f(x) > 0$	$f(x) < 0$	$f(x) < 0$	$f(x) > 0$	$f(x) < 0$	$f(x) > 0$	$f(x) < 0$
$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) < 0$	$f'(x) > 0$

Task 3. Match the third conditions as well (on the back side of the cards).

$f''(x) < 0$	$f''(x) < 0$	$f''(x) < 0$	$f''(x) > 0$	$f''(x) > 0$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$

Fig. 2. Activity for connecting multiple representations of counter-directional covariation

and registers, (2) eliciting students' intuitive conceptual and language-related resources in order to construct meanings, and (3) providing challenges for productive struggle in order to elaborate students' concepts and language integratively in more concise and formal ways. Thereby, the phenomenon of counter-directional covariation is in the core and worked through by relating representations and contrasting similar texts, graphs and formal conditions. Task 1 of the activity was treated in a similar form in the beginning of calculus (as suggested by Stroup, 2002). In order to ensure a pertinent conceptual focus during all of the teaching units of calculus rather than only in the beginning, the activity sequence in Fig. 2 was given after introducing the formalization for the derivative and the second derivative in order to reconstruct meanings for it.

METHODOLOGY OF THE DESIGN RESEARCH STUDY

Design Research as methodological framework. The overarching Design Research project in which the presented case study is embedded has the dual aim of designing language- and mathematics-integrated arrangements (in this case, the construction of meanings of the function-derivative relationship using the presented design principles) and developing an empirically grounded local theory of students' learning processes and the occurring language demands. For empirically specifying language demands, the methodological framework of Design Research with a focus on learning processes has proven valuable (e.g., in Prediger & Zindel, 2017).

Design experiments for data collection. Design experiments are the major method of data gathering in design research studies (Gravemeijer & Cobb, 2006). In the overarching project, four design experiment cycles were conducted with 16 pairs of students in Grades 10 and 11 (14-16 years old). This paper uses data from Cycle 3 in which design experiments were conducted in laboratory settings with nine pairs of 11th graders. Two sessions of 45-60 minutes each were completely video-recorded for each pair of students (in total about 1000 minutes of video material). In this paper, the analysis focuses on a case study of two girls, Emily and Layla, and the first author as design experiment leader (in the following referred to as "tutor").

Methods for qualitative data analysis. The qualitative analysis of the transcripts was conducted with the aim of qualitatively tracing the students' conceptual learning pathway by locating the intermediate steps in the level model from Fig. 1. In a second step, the students' language resources and obstacles were extrapolated in the lexical, syntactical, and discursive dimension, following Schleppegrell (2007).

EMPIRICAL INSIGHTS INTO THE CASE OF EMILY AND LAYLA

Rediscovery of counter-directional covariation and language resources

Emily and Layla had worked on similar headlines some weeks before. Meanwhile, they had learned to formalize the derivative and calculate it procedurally. So they had to rediscover the phenomenon of counter-directional covariation: When Emily and Layla start to work on Task 1 (from Fig. 2), Emily falsely matches the headline with Graph 2:

65 Emily — Yes and here [*hints to the headline "Less child births" and to the decreasing Graph 2*] the population growth just decreases, that means it was so much [*hints to the beginning of Graph 2*] […] and then it is less and less, well that means probably, how many people live at which time, and then there live simply less people.

66/7 Tutor/Layla — Mhm.

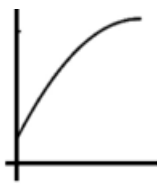
68 Emily — [*hints to the headline again*] I just remember, when we say, the growth becomes less, we had last time, that this can however become more, but not as much as before?

In Turn #65, Emily confounds population growth *f'* with the function *f*, which captures the population amount ("how many people live at which time") and transfers the increase to the wrong level. But in #68 she remembers that both levels can have different directions, even if she cannot yet express it. When the tutor invites her to be more precise about the axis labels, she formulates it more concisely:

76 Emily — There are fewer people born, and, therefore, the population grows not that fast anymore.

On this base, the students decide to choose Graph 3, but take much longer to discuss the axes' labels. They finally reject population growth and decide on population:

250 Emily — Ok, the number of people increases, but there are not so many additionally; thus, not so many in a year are added



251 Tutor — Mhm

252 Emily — After all, the growth becomes less, as it does not increase so steeply [*hints to the end of Graph 3*]

254 Layla — Thus, though, the population, however, increases, but simply not that much. And for this, the growth decreases, yes.

After a long discussion, the students have unpacked what decreasing growth means (in #68, 76, and 250); for this, they shift the level *f'* back to level *f* (see Fig. 1). During that period, they speak about *processes* of change in covariation approaches without condensing them into nouns. Then, the labelling of axes is requested, and as this requires nouns, this request serves as a scaffold to conduct a condensation by nominalization. It is only in #254 that Layla has condensed all information in the covariation phrases "*f* increases, even if *f'* decreases" on two levels without going back to correspondence approaches (see Fig. 1).

With respect to occurring language demands, this analysis shows that the students have many *lexical language resources* to express processes of change, e.g., "decrease/increase" (#65), "get/become less/more" (#68, 252), "not so many are added" (#250). The greater challenge is, however, to apply them with successive conciseness. This challenge is related to *syntactical and discursive demands* on making explicit the levels to which the change processes refer and to express their mutual relationships. As linguistically explained by Schleppegrell (2007), making something explicit requires sentences with explicit references (instead of unspecific "this" and "it"), which requires a condensation of processes into nouns, which function as lexical markers for objectified concepts: The long sentence "How many people live at which time, and then there are simply fewer people living" is later condensed into the nominalization that allows combination with a verb "population increases/decreases" (#254).

Language demands while constructing meanings for the second derivative

During their pathway, Emily and Layla quickly succeed in assigning the formal conditions about f and f' to the graphs and headlines as requested in Task 2 from Fig. 2. Their struggle begins again with Task 3, which requests matching $f''(x) > 0$ or $f''(x) < 0$.

320 Emily — Oh, God [second derivative]. I know only that you can, eh, the turning points […] eh, always zero, something […]

355 Tutor — The second derivate stands for the growth of the increase. [[*the German word for slope has two meanings, "slope" and "increase"; "increase" is preferred in this case as it is the more common everyday usage*]]

360 Emily — The growth of the increase?

361 Layla — Yes perhaps, how much that grows again?

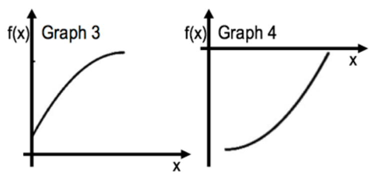
366 Emily — The growth, the growth of what? The function?

368 Emily — […] Well, we have a function and the derivative says always, what is the increase.

372 Emily — If it increases quickly or something like that?

With f'', Emily and Layla only associate procedural aspects of calculating turning points by using $f''(x) = 0$ in #320-354. When the tutor informs them about the standard interpretation as "growth of increase" (#355), they cannot make sense of it as they do not know how to refer the growth to something else ("the growth of what," #366). So far, they have not referred growth to any increase other than the increase of one level (amount, change, or change of change) they pointed out; they cannot cope with the double nominalization. In #368, Emily unpacks the nominalizations, which prepares her tentative interpretation of f' as "increases quickly" in #372. As this idea from #372 gets lost again, the tutor comes back to it:

390 Tutor — Ela, you just said that this is perhaps a statement about how strongly the increase increases. Look at Graphs 3 and 4. What are the commonalities of the increase?



391 Layla — **It** is positive.

402 Emily — Aha, because **this here** [*hints to Graph 4*] increases then, well, **that** goes more **That**, well, **here** is [*hints to Graph 3*], it grows but then **it** gets less […]

406 Emily — [And for Graph 4] it looks more as if **this** grows still steeper and not [like in Graph 3] **the increase** goes in direction 0, thus gets less.

414 Emily — Ok, both increase. But **here** [*hints to Graph 3*] The **increa**— well, **that** increases though, OK, that is what we assigned, because **this** gets less, thus, does not grow so steeply.

415 Tutor — **What** gets less?

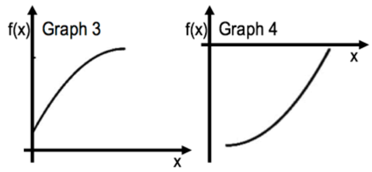
416 Emily — The increase

418 Emily — Ok, **it** increases **here** [*hints to beginning of Graph 3*] more steeply, that means, within the time, **it** becomes more. And **here** [*hints to middle of Graph 3*], in this section, less is added, even if the time interval is equal [*gestures a slope triangle*]. And, eh, **here** [*hints to the end of Graph 3*] the **increase** gets always less.

Again, the main language demands are not on the lexical level, but on the syntactical and discursive level: In these turns, all references to what increases or decreases (the subjects for the predicates) are marked in bold. The high amount of *underdetermined deictic expressions* (”it,” “this,” and “that”) for the addressed level shows the big need to become more explicit, because every underdetermined expression causes a risk to confound the levels f, f', and f''. Explicit navigation through the levels of Fig. 1 is only possible with explicitly articulated references to the levels. The tutors’ prompts in #390 and 415 are intended try to support the students to make the references explicit. And, indeed, Emily can work with these prompts in #418 when she refers to different parts of the graph: at first underdetermined in language, then very precise (“the increase in this time interval,” #418). However, they have still not cracked the meaning of the formal condition $f''(x) < 0$, with reference to what f and f' mean.

457 Tutor — OK, what can this mean for the second derivative. There is the option that $f''(x) < 0$ or $f''(x) > 0$. What can this mean, now?

458 Emily — Perhaps, **this** [*hints to Graph 3*] is bigger than zero, because **it** gets less, than … and then **it** becomes zero? I mean, that **it** is first more and then **it** gets less. And here [*hints to Graph 4*] **it** is first less and then it is more. That is perhaps, anyway, that **this** is less than zero, and when **it** is less than zero, that means that the **increase** becomes less?



459 Layla — Perhaps, the second [derivative] describes the **growth of the increase** and **here** [*hints to Graph 4*], **this** increases and there [*hints to Graph 3*] **it** decreases.

478 Emily — [...] OK, we have said that when the **increase** is zero and then increases [as in Graph 4], then **it** [*she means the second derivative*] is bigger than zero, and when **it** [*she means f''*] is less than zero, than **it** decreases [*she means increase f' of f*].

It takes 20 more turns for Emily to correct the misinterpretation in #458 and assign the formal condition appropriately. #458 and 459 shows that shifting to the level of f' (change of change) again requires other language means in which the increase must be treated as object and therefore be nominalized. In #478, Emily finally succeeds in making sense of the second derivative by reaching the level of f'' (see Fig. 1), even if still with many underdetermined references and a restriction to positive f''.

Summing up, language demands already visible in the easier Task 1 become crucial obstacles in dealing with formal conditions for the second derivative in Task 3. Lexical means are not missing (all students in our data had sufficient lexical resources), instead, the main challenge is their concise use in sentences without underdetermined references. In the process of making the relationships explicit, nominalizations and the necessary objectifications go hand in hand (Schleppegrell, 2007).

DISCUSSION AND OUTLOOK

What does it mean to talk about amount and change conceptually? The case study of Emily and Layla has provided insights into students’ pathways through productive struggle with the concepts of qualitative calculus: amount, change, and change of change. Requesting the connection of multiple representations is a main answer to

RQ2, and it is shown to help students to successively refine their thinking about the relationship of amount and change and express the phenomenon of counter-directional covariation in successively concise ways. The major finding on RQ1 is that language demands in Grade 11 do not occur on the lexical level but on the discursive and syntactical level for establishing clear references and nominalizations for objectifications. This can give important hints for the further design of this and other teaching learning arrangements: Teachers' prompts for making references explicit are crucial for developing conciseness of language (another major answer to RQ2).

Future research is planned so that the methodological limits of the case study can be overcome by (1) increasing the sample, (2) varying the tasks and activity settings, and (3) transferring the research framework to other mathematical topics. As almost no research has investigated higher grades, the differences from research results in earlier grades found here motivate a further focus on the upper secondary level.

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