Learning to meet language demands in multi-step mathematical argumentations: Design research on a subject-specific genre

Susanne Prediger & Kerstin Hein
TU Dortmund University, Germany


Abstract. Formal multi-step mathematical argumentations are a typical case of a highly specified subject-specific genre in the technical language with challenging demands in the academic language of schooling. The design research project presented here has the dual aim of (1) specifying the structural and language demands of formal mathematical argumentations and (2) designing a teaching-learning arrangement that uses structural scaffolding to foster students to successively meet these demands. These dual aims are pursued by an in-depth empirical analysis of students’ processes on the micro-level. For this purpose, 15 design experiments were conducted with 10 pairs of students in three design experiment cycles. The first two cycles served to develop the structural scaffolding and the third cycle served to investigate the initiated learning processes and the language demands on the lexical and syntactical level. The qualitative in-depth analysis of the teaching-learning processes in the design experiments shows how students can successively learn to conduct multi-step argumentations when supported by structural scaffolding. Expressing the argumentations in an adequate way, however, is an additional challenge. The empirical analysis reveals deep insights into the complex interplay of conditional, causal, and consecutive phrases that are necessary to combine premises, arguments, and conclusions in a logical sound way. The design research study has theoretical consequences for conceptualizing subject-specific discursive demands as well as practical implications as one design outcome is a prototype of the learning arrangement to foster students in a mathematics- and language-integrated way.

Keywords. Specifying academic language demands, subject-specific genre, formal argumentation, formal deductive reasoning, mathematics education

Although the important role of academic language of schooling has repeatedly been acknowledged in many theoretical and empirical studies across all subjects (Schleppegrell 2004; Gibbons 2002; Thürmann, et al. 2010; Snow and Uccelli 2009), there is still a research gap in identifying academic language demands more concretely. Thus, the specification of academic language demands for different subjects and topics must be put on the research agenda (Bailey 2007). Starting from broadly defining academic language of schooling as including all language that students need in order to acquire new knowledge and meaningfully engage with academic content in school contexts (Schleppegrell 2004; Bailey 2007; Moschkovich 2015), the main question is how to identify specific language demands in a specific school subject and, even more concretely, with respect to one subject-specific genre (Gibbons 2002).

This article reports on a project that focused on the case of a subject-specific genre that is a crucial part of the technical language of upper secondary mathematics (Grades 9-12) namely formal multi-step mathematical argumentations. Beyond its specific characteristics in the logical structures and its specific monological form, the genre poses challenging language demands in terms of academic language of schooling. The design research project presented here pursues the research questions of specifying the structural-logical demands as well as lexical and syntactical demands of this typical mathematical genre and of designing a teaching-learning arrangement that supports students to successively meet these demands.

Beyond the concrete case of formal mathematical argumentation, the article also promotes a methodological message: Academic language demands for a certain subject-specific genre can best be specified in the methodological framework of design research (Gravemeijer and Cobb 2006) because this framework allows the investigation of the language demands during the learning processes and therefore specifically resonates with the epistemic role of language (Prediger and Krägeloh 2016; Schleppegrell 2004).

The article first describes the theoretical background on academic language demands and the chosen mathematical genre of formal argumentation in Section 1. Section 2 describes the teaching-learning
arrangement based on structural scaffolding that is a design outcome of the first design experiment cycles and the research context for the study from Cycle 3 presented here. The methodological background of the design research study is presented in Section 3. Section 4 offers empirical insights into the functioning of teaching-learning arrangements by outlining students’ pathways towards structural-logical demands (Section 4.1) and specifies lexical and syntactical demands by investigating students’ oral and written ways of expressing logical connections (Section 4.2). The conclusion (Section 5) discusses achievements, limitations, and perspectives for further research.

1 Theoretical background and the chosen mathematical genre of formal argumentation

1.1 Specifying academic language demands in a systemic functional linguistics perspective

Academic language of schooling has been shown to have different roles for learning subject matter such as mathematics: It is the basic important learning medium used in the “communicate to learn” mode (Lampert and Cobb 2003; Pimm 1987). With respect to equity concerns, it is widely discussed in its second role as an unequally distributed learning prerequisite (Snow and Uccelli 2009). One educational consequence can be drawn immediately: If this prerequisite cannot be taken for granted for all students, it is a matter of equity to treat it as a learning goal in the classrooms (from “communicating to learn” to “learning to communicate,” Lampert and Cobb 2003; also Schleppegrell 2004; Thürmann et al. 2010).

In order to treat the school academic register as a learning goal in mathematics classrooms, its relevant features have to be specified in detail. Although there is a consensus on typical lexical, syntactical, and discursive features in general, there is still a research gap in specifying the specific school academic language demands that are most relevant for learning specific mathematical genres or topics (Council of Chief State School Officers 2012; Bailey 2007). This process of specifying academic language demands has so far mostly been restricted to the method of analyzing textbooks and other curriculum materials (e.g., Bailey et al. 2007; Thürmann et al. 2010). Although this approach is insightful, it focuses mainly on receptive demands, whereas productive demands require an empirical approach for investigating students’ mental and linguistic processes and the required language means in these processes (Prediger and Zindel 2017).

In order to investigate students’ processes, we refer to a systemic functional linguistic perspective, following Halliday and Matthiessen (2004) and Schleppegrell’s (2004) application to schools. In systemic functional linguistics, language is regarded as a social semiotic system in which language means are investigated with respect to their functions. In Halliday and Matthiessen’s (2004) approach of lexico-grammar, the connection between lexis and grammatical structures are considered empirically by investigating the configuration of grammatical structures that are expected in different kinds of socially relevant tasks. These language means link the linguistic choices with the social purposes and situations that the “texts (spoken or written) participate in” (Schleppegrell 2004: 45).

In this article, the specification of language demands is conducted for the genre of formal mathematical argumentations. For this genre of formal argumentations in which words from other social contexts are used with mathematics-specific meanings, it is therefore especially crucial to analyze the interplay between logical content, grammaticalization, and phrases applied. The systematic functional approach can support specification of the language demands that are required for participating actively in classrooms. While foregrounding the functions and not the forms, this approach also gives important orientations for designing the teaching-learning arrangement (Schleppegrell 2004).

1.2 Formal multi-step argumentations as a mathematics-specific genre

Argumentation is a typical genre in school. It is present in all subjects and requires the performance of academic language of schooling in general (Rapanta et al. 2013), yet is has different concrete discursive demands in different subjects (Bailey 2007: 15).
In order to understand the general structure of argumentations, we introduce Toulmin’s (1958) scheme, which is widely accepted (Figure 1, left): The structure of an argumentation can thus be characterized by the act of connecting premises with a conclusion (called data and claim by Toulmin) by means of a warrant and optionally a backing. The data is supposed to support the claim, while the warrant is the assumption or principle that justifies the step from data to claim. Sometimes, a backing is given to support the warrant itself.

![Figure 1](Toulmin’s (1958) argumentation structure scheme and its didactical adaptation for mathematical formal argumentations)

Explicitness and completeness are the main differences between argumentations in everyday contexts and school contexts: According to Grice’s maxim of quantity, everyday argumentation structures usually make explicit only the main elements of data and claim or those elements that are relevant in a conversation situation (Grice 1975); only if the dialogue raises further doubts is the warrant made explicit. In school contexts, by contrast, the completeness of logical elements is mostly expected, especially the warrant (Rapanta et al. 2013). Different school subjects vary in the kinds of accepted warrants and in the rigor of connecting them to the data and claims. Whereas a subject such as history can also address hermeneutic warrants, science prioritizes empirical evidence or, e.g., physical laws (Driver et al. 2000).

In mathematics, argumentations are also relevant if everybody is already convinced, because they reveal the reasons. With respect to the geometric angle constellation in the first box of Figure 2, an everyday argumentation might be:

“The angles $\alpha$ and $\beta$ have the same measures because I can see that.”

In contrast, a mathematical argumentation would have to refer to warrants such as the vertically opposite angle argument (called “vertical angle theorem” for short). The more explicit argumentation structure is outlined in Figure 2.

“We start from the premise of the given geometric constellation for $\alpha$ and $\beta$. In order to argue using the vertically opposite angle argument, we need to check whether the premise of the if-then statement is satisfied. Indeed, $\alpha$ and $\beta$ are opposite in the given geometric constellation, so the vertically opposite angle argument can be applied. The argument says that if two angles are opposite when two lines cross, then they have an equal measure. Hence, we can conclude that $\alpha$ and $\beta$ have an equal measure.”

As the example shows, formal mathematical argumentations can be distinguished from other arguments by their monological instead of dialogical form: Usually, no interaction partner is required. Furthermore, they can be distinguished from other mathematical argumentations by the specific logical rigor in applying warrants: no semantic connection is accepted for connecting the data to the claim, but only logical derivations by implications in if-then structures (Duval 1991). This logical rigor specifically belongs to another function, not for persuading somebody, but for developing a logical deductive theory (see de Vil-
liers 1990 for different functions of proofs). That is why the applied arguments must be backed up them-
selves by already proven theorems. The example presented later will show how to prove the vertically opos-
site angle argument itself by applying already existing theorems as arguments. Formal mathematical argu-
mentation can comprise multiple steps in which several warrants and (intermediate) claims are com-
bined in order to connect the data and the (target) claim (Knipping and Reid 2015). For multiple steps,
Toulmin’s scheme is used several times within one argumentation and the claims of previous steps are used
as data within further steps (Duval 1991).

To sum up, the mathematics-specific genre of formal mathematical multi-step argumentations shares the
basic structures with everyday argumentations, but is specific in

- its function (constructing deductive theories instead of convincing),
- the nature of the warrants (no semantic warrants but syntactical derivations by if-then statements
  according to formal logic),
- the explicitness of the accepted repertoire of warrants (only those backed up by already proven theo-
  rems; see Douek 1999), and, hence,
- having fewer ambiguities than in other types of argumentations, even if absolute rigor never exists in
  reality.

Figure 1 (right) provides an advance organizer for the didactical adaptation of Toulmin’s argumentation
structure scheme for the purposes of this study. In order to adapt to the typical mathematical language, we
use the word argumentation for the complete structure (Toulmin refers to it as argument) and argument to
refer to what Toulmin’s scheme refers to as warrant. Arguments or general logical implications in an if-
then structure relate the premise to the conclusion (data to claim in Toulmin’s scheme). In managing these
kinds of arguments, the general implication must be related to the concrete premise in the task. This requi-
res checking the validity of the premise in a concrete case.

1.3 Structural-logical and language demands when learning formal argumentations: State of research

The mathematics-specific genre of formal argumentations offers the specialist who has acquired it a
reliable tool for creating proofs. For students, however, the genre is difficult to learn. Some logical as well
as language demands have already been identified in several empirical studies in mathematics educa-
tion research (see Hanna and De Villiers 2012 for surveys on mathematics education research on proofs and
proving).

Structural-logical demands

Logical demands are related to nearly all features of mathematics-specific genre formal argumentation:

- Students tend to mix up premises, arguments, and conclusions in argumentations, which can lead to cir-
cular argumentation structures (Harel and Sowder 1998; Ufer et al. 2009). Hence, understanding the
logical structures is a central demand that is challenging for many students (Epp 2003; Selden and Seld-
en 1995).
- With respect to the nature of the arguments, Chazan (1993) has shown students’ difficulty in realizing
that empirical or semantic arguments are not accepted as warrants in formal argumentations.
- With respect to the possible arguments and theorems as backings, students must realize the theoretical
status and generality of theorems once the teacher has declared them as valid or proven. In school
mathematics, an additional challenge is that the set of accepted backings is often only made partially ex-
PLICIT by the teachers or textbooks (Douek 1999).
- Applying an argument to the concrete data requires students to understand the if-then structure of a logi-
cal implication. If-then statements are parts of syllogisms: Understanding their structure is crucial for
avoiding circularity (Miyazaki et al. 2017). Thus, they differ from everyday argumentations with respect
to the status of premises: In most everyday argumentations, if-then statements are only formulated when
the premises are satisfied (Nunes et al. 1993:130). In mathematics, in contrast, if-then statements are hypothetical, so the validity of the premises always has to be checked before applying an if-then statement as warrant. However, this is rarely made explicit (Durand-Guerrier et al. 2011). In the didactical adaptation of the Toulmin scheme, we hence added the premise check (Figure 1, right). This might also help both to distinguish the general if-then statement from its concrete application to a concrete premise and to identify the correspondences between premises in the data and the premises that have to be satisfied for the if-then statement.

The empirical findings that these kinds of multiple structural-logical demands are only met by a minority of students resonate with classroom observations about missing learning opportunities for formal argumentations: Most mathematics classes restrict themselves to reasoning semantically (Duval 1991). If formal argumentations are presented, they are presented mostly in a ready-made form (Harel and Sowder 1998), which results in students not becoming aware of how to compose an argumentation in the logical structure. Hence, more learning opportunities should be provided for linearly ordering all elements of a proof and expressing their logical relations in written form.

Language demands

For the process of understanding logical structures, their explication (by the teacher and the students) is of crucial importance (Durand-Guerrier et al. 2011). Learning to express logical connections in formal argumentations is hence not only necessary with respect to the communicative role of language, but also to the epistemic role of language (Schleppegrell 2004), as Duval (1991) has emphasized. This is specifically difficult because the logical connections are often not made explicit in school.

Although the logical structures of mathematical formal argumentations are specific, the language means for expressing the logical connections are mainly not technical terms but stem from the repertoire of the (academic) language of schooling. Causal (“because”) or consecutive (“so that”) connectives are used as well as conditional clauses (“if…then”) (Halliday and Matthiessen 2004; Clarkson 2004). As the later empirical analysis will show, it is not the isolated technical terms that are difficult, but combining the different connectives within and between sentences in a discursively adequate form.

1.4 Research and design interests for the current project

Empirical investigations of language demands for formal argumentations should take place in a research context where all students – notwithstanding their language proficiencies – are engaged in formal argumentations. As this is not usual in most classrooms, the research needs to take place in teaching-learning arrangements that are specifically designed for this purpose.

(Q1) How can a teaching-learning arrangement be designed that engages students in formal argumentations?

(Q2) Which typical learning pathways do students take towards successively meeting the structural-logical demands of formal argumentations in such a teaching-learning arrangement?

(Q3) Which lexical and syntactical demands occur for students when approaching formal argumentations?
2 Underlying design: Teaching-learning arrangement based on structural scaffolding

2.1 Design principle of making logical structures explicit

Due to the indicated differences between everyday arguments and formal deductive arguments (see Section 1.2), many mathematics education researchers have suggested making the logical structures of formal reasoning more explicit in the learning process (Douek 1999; Durand-Guerrier et al. 2011), for example, by Toulmin’s (1958) argumentation scheme (e.g., by Cho and Jonassen 2002). As Toulmin’s scheme is the most widely used analytical tool for grasping argumentation structures within and outside mathematics education (Rapanta et al. 2013), it is a promising starting point for making structures explicit to students.

This approach was adapted here for students of Grades 9 and 10. The chosen geometric context deals with theorems of angle sets (see Figure 3) because the possible arguments are well limited in this locally deductive theory (see Figure 4).

To visualize the adapted version of Toulmin’s (1958) argumentation structure (Figure 1, right) for students in Grades 9 and 10, the materialized argumentation structure form was introduced (see Figure 5 for an example). In this structure form, students can write down the initial situation in the task and apply all theorems (in if-then clauses) already proven as possible arguments for the next step of formal reasoning. The premise check is visualized by the premise-check box as shown in the second box.

2.2 Design principle of structural scaffolding

For enabling students to become acquainted with the logical structure of formal deductions, the design principle of scaffolding was applied in the variant of structural scaffolding (Hein and Prediger 2017). Originally, scaffolding was developed in one-to-one interactions for language learning, with the core idea being to build scaffolds so that learners can first engage in activities with support before they can conduct them independently without support (Wood et al. 1976). In the last decades, scaffolding has increasingly also been elaborated into a design principle for materials and computer tools, whole-class contexts, open geometrical problems (Miyazaki et al. 2015), other multiple learning contents (e.g., Lajoie 2005), and, in our case, for the structural-logical demands of formal argumentations with explicit given or self-derived if-then statements.

In enacting structural scaffolding, the materialized argumentation structure (Figure 5), with its
premise box, premise check box, argument box, and conclusion box (see also Section 2.1), not only makes the structure of (existing) formal argumentations explicit, but also, once introduced, empty *materialized argumentation structure forms* serve as scaffolds for students’ own construction of argumentations (such as depicted in Figure 5) or chains of argumentations. Working with this materialized structural scaffold for each step enables students to make explicit their often implicit ideas and to monitor completeness of their explicit reasoning. To sum up, the structural scaffold has three roles to play in the learning process:

(I) as visualizer for the extended structure,

(II) as working tool for the students to check the completeness of their explicit reasoning, and

(III) as red line for expressing the formal argumentation in oral or written form.

3 Methodological background of the design research study for specifying language demands in students’ learning pathways to formal argumentations

3.1 Methodological framework of design research and design experiments as method for data collection

For the dual aim of designing a teaching-learning arrangement and specifying structural-logical as well as lexical and syntactical demands by investigating the initiated learning processes, the methodological framework of design research with a focus on learning processes (Gravemeijer and Cobb 2006) provides a powerful approach. The concrete model of Topic-Specific Didactical Design Research (Prediger and Zwetzschler 2013) relies on the iterative and intertwined interplay of four working areas:

(a) specifying and structuring learning contents,

(b) developing the design of the teaching-learning arrangement,

(c) conducting and analyzing design experiments, and

(d) (further) developing local theories on teaching and learning processes.

Design experiments are considered to be the methodological core of design research studies (Gravemeijer and Cobb 2006). For this project, three design experiment cycles were conducted. The teaching-learning arrangement based on structural scaffolding that was presented in Section 2 is a design outcome of the first two design experiment cycles (conducted with 10 ninth and tenth graders) and now serves as the research context for the case study from Design Experiment Cycle 3 presented here.

To gather data in Cycle 3, 10 design experiments in laboratory settings were conducted and completely videotaped. The sample consisted of five pairs of students who had two sessions of 60 minutes each (in total about 10 hours of video material). The 14- to 16-year-old students already knew the geometry topic in view, angle theorems, and concentrated now on the new learning goal of “formal argumentation.” The five pairs of students worked with the second author of this paper as design experiment leader (referred to as “tutor” in the following). She introduced the scaffolds of materialized argumentation structure forms and their rationale, guided the students to work with them by building on their initial ideas and initiated reflection on the necessary logical structures. Each formal argumentation was jointly developed orally by the two students and then written down separately by each student.

3.2 Methods for qualitative data analysis of students’ written products and the videos of processes

For the qualitative data analysis, students’ written products and selected transcripts of the videos were analyzed with respect to their learning pathways towards successively meeting the structural-logical demands and the ways they express the logical connections in their proofs in oral and written form. For this article, 30 argumentations have been analyzed with respect to their logical structures: the analysis of five student pairs’ discussions for Proofs 2 and 3 as well as the written argumentations of these 10 students for the two proofs, which were written down after completing the argumentation structure form with the support of the tutor.
In analyzing students’ argumentation structures, Toulmin’s (1958) analytic scheme of identifying data, warrant and claim, has often been applied in mathematics education research (e.g., Krummheuer 1995). In our case, the adapted argumentation scheme (see Figure 1, right) serves as an adapted analytic tool by which every argumentation of a student can be analyzed with respect to the explicitness and correctness of the elements’ premise (P), argument (A), conclusion (C), and backings (B). In order to distinguish the name of the encapsulated argument from making its if-then structure explicit, the explicated argument (eA) was also coded, if existent. With this scheme, students’ pathways towards more complete argumentations can be traced efficiently for oral utterances as well as for written products.

In order to analyze students’ ways of expressing formal argumentations in their written products, a specific focus was put on the logical connections between the mentioned logical elements. The systematic comparison of linguistic means used allowed us to specify several lexical and syntactical demands in a code-developing procedure that was adapted from Jimenez-Aleixandre et al. (2000).

4 Empirical insights into students’ pathways towards conducting and expressing formal argumentations

4.1 Students’ pathways towards successively accomplishing structural-logical demands

4.1.1 Empirical insights into Katja’s and Emilia’s pathways to formal argumentation

The cases of Katja and Emilia, two ninth grade students, offer empirical insights into students’ typical pathways towards formal argumentations (analyzed in more detail in Hein and Prediger 2017).

Sequence 1: Approaching formal argumentations in Proof 1: Vertically opposite angle theorem

When first asked to prove that the vertically opposite angles α and β are equal (as in Figure 6), Katja and Emilia first give a typical one-expression answer:

338 Tutor So, here we have the new task [Gives them a sheet with the geometric constellation of Figure 6 (without γ) and the task: “Justify the following statement: The angles α and β have the same measure”]

339 Katja Well. [12-second break while students look at the sheet] Vertically opposite angles? [Students laugh]

This is a typical starting point found by many students in the design experiments: Initially, the students classify the type of relation between the two angles, and the preceding parts of the design experiment had already given them some idea that the tutor will not consider this as sufficient. However, they do not know what more to say. The tutor sensitizes them that the vertically opposite angle theorem is what they are supposed to prove based on existing arguments (not like in Figure 2). The arguments already available in this moment are the argument of supplementary angles (“If two lines intersect, then the measures of the two resulting supplementary angles add up to 180°.”) and the calculating argument (“If there are angle measures, then it is possible to calculate with them like with numbers.”).

In order to find the relevant connections, Katja and Emilia start by naming the angles γ and δ (Figure 6) and discussing calculation possibilities:

369 Emilia: Yes, so actually this can be – Yes, precisely – But we have no concrete numbers [points to the conclusion box from the previous task] – and then we can go on – so, I don’t know whether we can do this in such small steps, because we have no numbers at all, but then we could say, α plus γ equals 120, uh, 180 degrees. And β plus δ equals 120, uh, 180 degrees.

... 380 Emilia: And then, if, a system of equations could be created.
While searching for connections, the students offer details of calculation steps without making explicit the warrants of these relations (here the argument of supplementary angles). In their oral utterances, the steps are combined with temporal connectives (“and then” in Turn 380) but not yet by logical ones.

When being prompted by the materialized argumentation structure form, the students start to fill it in (Figure 5 is their product). The next excerpt from the transcript shows that the form supports them to focus on what they need for the argumentation:

403 Emilia: Well then – eh, I would say – I know, I think, that here [Points to the argument box], we first write that the supplementary argument is our argument. Then we think what has to be there [Points to the premise-check box]
404 Katja: [Writes “supplementary angle” in the argument box, 21 sec break] Yes, that here
405 Emilia: Ah, I wanted to write that
406 Katja: …that we γ here
407 Emilia: Yes, that α and β have the same supplementary angle.
408 Katja: [3 sec break] Where?
409 Emilia: Here [Points a finger at the premise check box]
410 Katja: [Writes in the premise-check box: α and β have the same supplementary angle γ]
411 [Discussion with the tutor about whether the second angle δ is necessary]
412 Emilia: Uh, then I would now write here, uh, - α plus γ equals 120 degrees and β plus γ equal 120, uh, 180 degrees. Why do I always say 120? Yes,
413 Katja: [Writes both equations in the conclusion box; see Figure 5]

While completing the argumentation structure form, the students succeed in identifying all relevant elements of the logical structure. They start by selecting the argument and checking whether its premise is satisfied (Turn 403). After filling in the form, they condense the proven theorem as a new argument to be used for further proofs (in non-printed Turns 452-471): “Argument of vertically opposite angles: If two lines cross each other, then the opposite angles are equal. (They are called vertically opposite angles.).”

This episode illustrates how the scaffold can support students to construct an explicit formal argumentation and to understand the logical structure, even if their language does not yet address logical connections explicitly but stays rather deictic (“here” and “there” in Turns 403, 404, 409, 417).

In order to give an expert model of how the connections could be expressed, the tutor finally resumes as follows:

474 Tutor: […] Also, this premise of intersecting straight lines was considered, so that we have two times two supplementary angles. Here, as a premise and because we have supplementary angles, we could apply the supplementary argument that says that two supplementary angles add up to 180 degrees. That is why it is satisfied for our supplementary angles and, uh, here two times two were considered, this means from this, we have two times this equation with, eh, our angles. […]

Sequence 2: Mastering formal argumentations in Proof 2

In Sequence 2, Emilia and Katja are asked to conduct Proof 2 on the alternate interior angle theorem (Figure 7) based on the vertically opposite angle argument (which was derived in Proof 1), the so-called equality argument (If \( \delta = \mu \) and \( \mu = \pi \), then \( \delta = \pi \rightarrow \) transitivity), and the corresponding angle argument (which can only be derived from

![Figure 7 Alternate Interior Angle Theorem](image-url)

![Figure 8 Katja’s and Emilia’s three steps of formal Proof 2 for the Alternate Interior Angle Theorem](image-url)
the parallel axiom and is hence left unproved for the students, see Figure 2).

Again, the structural scaffolding supports the students to construct the three-step argumentation structure (see Figure 8): In Step 1, they use the vertically opposite angle argument for deriving that $\gamma = \delta$. In Step 2, they use the corresponding angle argument for deriving that $\alpha = \beta$. For deriving that $\alpha = \gamma$ in Step 3, they use the equality argument and produce the last chain of reasoning.

Figure 9 shows the written text that Katja produced for the last step of argumentation after filling in the materialized argumentation structure form. It provides the first situational evidence that she has understood the logical structure of formal reasoning and can express some of the logical connections. In contrast to the beginning of their learning pathway, she makes the argument explicit (“the equality argument says that”) as well as premises of its application (“Now we know that $\gamma$ and $\beta$ have the same measure and $\alpha$ and $\beta$.”).

In expressing the logical connections, she adopts first elements of a language offered by the tutor in Turn 747 (“the supplementary argument that says”). She also expresses the deduction from the argument to the conclusion: “from this we can conclude.” This language means will be further analyzed in Section 4.

Figure 9 Katja’s written argumentation of Step 3 in Proof 2 on the alternate interior angle theorem

4.1.2 Analysis of 30 argumentations: Successive accomplishment of structural-logical demands

Katja’s and Emilia’s argumentations were compared to 30 other argumentations with respect to the mentioned elements of the logical structure (five pairs’ discussions of Proofs 2 and 3 as well as the written argumentations of the 10 students for the two proofs that were written down after completing the argumentation structure form with the support of the tutor). As the highly condensed results in Table 1 show, that Katja’s and Emilia’s pathways are typical could be extrapolated from other students’ pathways.

In conducting these comparisons and grasping the subtle differences, the codes of the analytic schemes had to be differentiated and completed as in Figure 10.

Figure 10 Extended analytic scheme for students’ individual argumentation structures
Table 1. Structural analysis of students’ argumentations on angle theorems (Correct references to elements of the logical structure are coded using I (initial situation), A (argument name), and C (conclusion); see Figure 9. Wrong references are coded using parentheses, e.g., (A). Each line signifies all elements of a step that are mentioned together (in order of appearance).

<table>
<thead>
<tr>
<th>Pairs of students</th>
<th>Oral expressions while constructing the proof with structural scaffold (both students)</th>
<th>Written products after structural scaffolding (for each student)</th>
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<th>Written products after structural scaffolding (for each student)</th>
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General descriptions about the geometrical arrangements were coded as I (Initial situation) and the premises from the task or former argumentation steps as P. The name of the argument was coded as A, the conclusion as C, the explicated argument as eA, and when only the premise of argument was made explicated as eaP. Because premises and conclusions of the if-then statements have correspondences in the concrete data, there are also combinations with elements from both levels, such as the explicated argument applied, coded as eAa; the explicated argument applied premise, coded as eAaP; and the explicated argument applied conclusion, coded as eAaC. When the new argument that was proofed was named, it was coded as nA and as neA when it was made explicit.

The codes P, A, eA, eAaP, and C in Table 1 can be explained by their verbal representations for Katja’s text in Step 3 (in Figure 9). She starts by mentioning the premises from the step before P for Step 3: “Now we know that \(\gamma\) and \(\beta\) have the same measure and so do \(\alpha\) and \(\beta\).” In the next sentences, she mentions the argument name (A) and the explicated argument (eA). She applies the explicated argument applied to the premise (eAaP) by saying “Here the first is \(\gamma\), the second \(\beta\), and the third \(\alpha\)” and then derives the conclusion (C).

Besides the already addressed elements of the logical structure, the students sometimes merged the explication of the if-then statement with their application; this was done in three different ways and coded accordingly:

- The code explicated argument applied (eAa) was given, for example, in Leonard’s text for Proof 2, Step 3 (see Table 1): “The equality argument, which says that if \(\alpha = \gamma\) and \(\gamma = \beta\), then \(\alpha = \beta\).” In his text, he has changed the names for the variables so that they correspond to the given data.
- The code explicated argument applied to conclusion (eAaC) was given when the conclusion was addressed while explicating the argument, for example, by Lydia in Proof 2, Step 2: “The corresponding angle says, that \(\alpha\) and \(\alpha\)’ are equal.”
- The code explicated premise of argument (eaP) was given when only the premise of the argument was made explicit, for instance, by Jannis in Proof 2, Step 2: “The premise for the corresponding angle argument is that two parallel lines are crossed by another line.”

Coding all 30 argumentations in terms of the addressed elements of the logical structure allows both a comparison between students and an internal comparison for each student with respect to the difference between verbal and written argumentations within each proof. Without going into the details for every line in Table 1, we summarize the following main observations:

- In the oral communication about compiling the argumentation, students jump between steps, whereas in the written texts, the steps are mostly addressed in the logical sequence.
- As in Katja’s and Emilia’s cases, the oral communication about the argumentation often starts with addressing the arguments and only successively accomplishes the other elements. As the video data shows, this process is substantially scaffolded by the materialized argumentation structure. In completing the premise-check box, often the tutor needed to give some support. This explains that the order of finding all elements does not necessarily correspond to the order of reasoning.
- Once the materialized argumentation structure form is completed, the written texts (which are produced without the tutor’s influence) address the demanded elements in a more complete way.
- In the written texts, the order of writing corresponds more to the order of reasoning, even if some steps are repeated (e.g., Emilia in Proof 3).
- The if-then structure of the arguments are mostly made explicit and in many cases combined with the premises of the tasks.

These observations show that the structural scaffolding can substantially influence the students’ argumentations in their structure and explicitness. Furthermore, they show the crucial role of genre-specific writing that achieves explicitness much easier than oral communication. The comparison of the 30 argumentations analyzed shows additionally that the teaching-learning arrangement has already fulfilled its major goals, even if some students still need more support.
4.2 Specifying lexical and syntactical demands in expressing formal argumentations

The observation of students’ reluctance to use logical connectives was the starting point for investigating language demands not only on the structural-logical level but also on the lexical and syntactical level. The complexity of these demands on the different levels could already be seen in the tutor’s utterance after Proof 1 on the vertically opposite angle theorem (Turn 474 at the end of Sequence 1 in Section 4.1.1).

Table 2 shows five examples of students’ ways of expressing the logical connections after having completed the materialized argumentation structure forms. Although the students produce relatively complete argumentations with respect to the addressed logical elements, they differ in strategies for expressing the logical connections between the logical elements and argumentation steps.

It is worth restructuring the results of the text analysis (sketched in Table 2) according to the different demands in each step and the lexical or syntactical solutions the students found:

1. Connecting different steps of argumentation. Some students connect the different steps of argumentation by cohesive conjunctions, others do not connect them at all (as Florian does by simply writing one sentence after another) or they connect them by temporal adverbials (e.g., Cora’s use of “now”).

2. Connecting the premise with the conclusion. The complexity of connecting the premise with the conclusion depends on the degree of explication: Florian, who has not explicated the argument, can simply relate the premise and the conclusion using “due to the…argument.” The other four students (Cora, Emily, Katja, and Jannis) choose to explicate the argument itself.

3. Explicating the argument (eA). When the students explicate the argument, they refer to the conditional structure (if-then statement) as expressed in the material, not like in predicate calculus, using, for example, every set of corresponding angles has the same measure (see Selden and Selden 1995). Cora, Katja and Emilia introduce this explication using the phrase “the argument says that.” This meta-language is necessary to address the argument and integrate its logical status and content in the sentence. Only Jannis chooses another form of expression, writing “the premise for the…argument is that,” thus another meta-language term. This shows how difficult it is to embed the explicated if-then statements properly in a sentence.

4. Connecting the premise check to the explicated argument (C→eA). When connecting the premise check to the explicated argument, Cora again uses the temporal connective “now.” In contrast, Emilia uses “as” and Katja “as…we can apply.”

5. Distinguishing the general if-then statement and its application to the given premises. Connecting the premise check to the explicated argument is difficult because it requires the mental act of distinguishing the general if-then clause from its application to the concrete condition. In the examples in Table 2, two of the five students make this act explicit: Cora uses the angle names α, α’, and γ instead of the δ, µ, and π of the argument. Jannis specifies “in that case.”

6. Deriving the conclusion. Only Florian, with his very short sentences, and Cora, who does not make the concrete conclusion explicit, need no logical connective for deriving the conclusion. Emilia and Jannis, on the other hand, use “this implies,” while other students in the sample (not in Table 2) use “so that,” “from this we conclude,” “therefore” and Katja writes “As we have all of it given, we can say.” Thus, students have a number of different options for causal or consecutive connectives.
This analysis shows an interesting tendency: More logical elements being explicated require more complex logical connections and, therefore, logical connectives, and more appropriate linguistic expressions are needed to connect them within a sentence or between sentences. Causal and consecutive connections are coordinated by diverse lexical and grammatical forms. These observations resonate with the lexicogrammatical analysis of Halliday and Matthiessen (2004: 43), who have shown how “one clause (or scomplex) enhances the meaning of another by qualifying it in one of a number of possible ways by reference to time, place, manner, cause, or condition” (Halliday and Matthiessen 2004: 410). Within the reconstructed category of “causal-conditional” enhancement, they distinguish “cause reason” (for giving reasons) from “cause result” (for deriving conclusions) and show the subtleties of differences that are not yet all mastered by the students in our design experiments. The detailed analysis on a lexicogrammatical level has offered deep insights into the complex interplay of conditional, causal, and consecutive phrases, which are all necessary to combine conditions, arguments, and conclusions in a logically sound way.
5 Discussion of results and limitations

Formal multi-step argumentation is a subject-specific genre that is typical in higher mathematics and functions especially as a gatekeeper for tertiary education mathematics in Germany. The investigation of the structural-logical demands by extending Toulmin’s (1958) argumentation scheme was a first important step for specifying the related language demands connected to this mathematics-specific genre.

In order to support students in acquiring this genre, the design principle of structural scaffolding has been applied in a teaching-learning arrangement based on the so-called materialized argumentation structure. The qualitative investigation into students’ learning processes in the teaching-learning arrangements has offered the first process-oriented qualitative evidence that (1) students can quickly understand the structural-logical demands and (2) the structural scaffold offers opportunities for discussing abstract connections that were formerly not easily accessible through words.

This study has methodological limitations that will still have to be overcome in later design experiment cycles or future research. Limitations include the sample size (10 students in the presented design experiment cycle, 20 in total in all three cycles) and the mathematical topic addressed. The teaching-learning arrangement focused only on angle theorems; future research should extend to other topics. Furthermore, an important limitation of the teaching-learning arrangement itself was that it did not extend to prepare the fading out that helps the students become more independent from the materialized scaffold.

In spite of these limitations, investigating students’ oral and written attempts in expressing the logical connections in their proofs has provided some first insights into the complexity of syntactical and lexical demands related to the logical connections required. Whereas some students did not yet provide any linguistic logical connectives, others unfolded a great deal of creativity in expressing the logical connections in their utterances and texts. In order to enable all students to participate successfully in the genre of formal argumentation, in the future we will need to continue to analyze these language demands and provide more systematic scaffolds for students with less developed language proficiency (Clarkson 2004).

On a general methodological level, the study gives an example of why it is sometimes necessary to first initiate learning processes before the language demands for higher order skills can be specified, supported, and then hopefully enhanced in students’ learning. For such a developmental program, design research with a focus on improving learning perspectives is a promising methodological framework within subject-specific research approaches.

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References


