“x-arbitrary means any number, but you do not know which one”

The epistemic role of languages while constructing meaning for the variable as generalizers

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Abstract. Conceptual understanding of variables is crucial for school success in algebra. This chapter presents a case study from a larger design research project in which multilingual low-achieving students are fostered to gain access to this topic in a content- and language-integrated learning arrangement. The empirical analysis of the videotaped teaching–learning processes especially shows the epistemic role of the language of schooling, a register to which underprivileged students have limited natural access in either of their languages.

0. Introduction

Language challenges in mathematics classrooms do not only appear for multilingual students (including bilinguals, and in Europe mostly immigrant students), but also for monolingual learners with underprivileged socio-economic background. As only 4% of students have immigrated to Germany themselves, almost all monolingual and multilingual students grew up in Germany and have developed good basic interpersonal communication skills (BICS, Cummins, 1979). In spite of these skills, large-scale studies show that many multilingual students and monolingual underprivileged students experience substantial language barriers resulting in limited school success and in particular limited achievement in mathematics (OECD, 2007). This discrepancy has been explained by the difference between BICS and cognitive academic language proficiency (CALP, Cummins, 1979) since the acquisition of CALP seems to necessitate access to learning opportunities that are not equally provided by all families. More recently, the construct of CALP has been linguistically elaborated by the constructs ‘language of schooling’ (Schleppegrell, 2004; Thürmann, Vollmer & Pieper, 2010) or, in German, ‘Bildungssprache’ (Gogolin, 2009; Feilke, 2012; Morek & Heller, 2012). These two constructs explain why even many German native speakers experience multilingual challenges in mathematics classrooms: all students have to mediate between three or six registers: their everyday language, the language of schooling, and the technical language of mathematics; and possibly each of them in first and second languages (Prediger, Clarkson & Bose, in press).

Although this distinction of different registers is now omnipresent in the academic discourse on multilingual classrooms, and even in the European policy discourse (Thürmann et al., 2010), substantial further research is needed for investigating the mechanisms on the micro-level of mathematics learning processes. Topic-specific empirical insights into these questions are necessary for supporting students in overcoming these language barriers, as Schleppegrell (2010, p. 107) has claimed.

This chapter contributes to these research needs with respect to an algebraic concept. The importance of algebra is clear since it counts as a gatekeeper especially for (language) minority students’ middle school success (Moses & Cobb, 2001). In particular, within algebra, the meanings of variables are crucial, which is why the exemplary mathematical topic ‘meaning of variables as generalizers’ has been chosen for investigating the following research questions with underprivileged low-achieving multilingual eighth graders (age 13/14 years):
Q1 Specification of topic-specific linguistic means: Which kind of linguistic means are crucial for the algebraic topic ‘meaning of variables as generalizers’?

Q2 Impact of language on learning processes: How does students’ proficiency in the mediating language influence the individual learning pathways to constructing the meaning of variables as generalizers?

Q3 Designs for fostering topic-specific language learning: How can the reconstructed language-determined limits be overcome by suitable language- and content-integrated learning arrangements?

These questions combine two general aims, foundational empirical insights into complex processes and developing concrete learning arrangements. The combined aims are treated within the research program of Didactical Design Research (Gravemeijer & Cobb, 2006; Prediger & Zwetzschler, 2013).

After presenting the theoretical background on the algebraic topic and the transitions between languages in Section 1, the methodological background of Didactical Design Research will briefly be sketched in Section 2. Section 3 offers some empirical snapshots from the design experiments that show the epistemic role of languages for constructing meanings.

1. Theoretical background

1.1 The algebraic topic: constructing meaning for variables as generalizers

Variables are among the most important concepts in algebra. Their conceptual understanding comprises two essential meanings:

- The variable as unknown is dominant for solving equations, more generally in the conception of algebra as “a study of procedures for solving certain kinds of problems” (Usiskin, 1988, p. 12). The unknown stands for a fixed number that has to be found by using the given relations.

- The variable as generalizer is needed in contexts where algebraic expressions or equations are formulated or interpreted in algebraic and non-algebraic contexts, mainly in the conception of “algebra as generalized arithmetic” (ibid, p. 11) or “algebra as the study of relationships of quantities” (ibid, p. 13). Unlike the unknown, the generalizer stands for many numbers at the same time or for successively changing numbers.

Empirical studies have repeatedly shown that these well-specified target meanings are not attained by all students. Many students conceptualize variables simply as meaningless symbols that can be transformed according to formal rules or as a “symbol for an element of an replacement set” (Usiskin, 1988, p. 9; similarly Malle, 1993) or provide other deviant interpretations of variables (e.g., Küchemann, 1981; Kieran, 2007).

So far, the role of languages in this limited success has rarely been addressed in research: although some empirical written assessments show a strong statistical connection between language proficiency and algebra skills (MacGregor & Price, 1999), little is known on the role of different languages in students’ pathways to the meanings of variables.

A textbook analysis gives first hints on possible obstacles: many German textbooks try to support the construction of the second meaning, the variable as generalizer, by referring to typical linguistic expressions that are used outside algebra classrooms; for example the textbook task in Fig. 1 with “x-beliebig” and “x-mal” (literally meaning “x-arbitrary”1 and “x-times”).

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1 In German, “arbitrary” is often used for generalizing, as well as “x-arbitrary.” The translation of the typical German term could have also been “x-any.”
Assuming these expressions are known by the students, the authors’ intention is to remind students of out-of-school-language resources to help their individual construction of meaning. However, the empirical section will show that these linguistic resources cannot be taken for granted for all students since they are part of the language of schooling, not necessarily of students’ everyday register.

Hence, substantial experiences are needed for constructing the meaning of these generalizing expressions. These experiences can be gained by the well-established shapes and pattern approach (e.g., Mason et al., 1985; Stacey & MacGregor, 2001; Kaput, Blanton & Moreno, 2008), in which the investigation of growing patterns for shapes and/or number sequences leads to variables and algebraic expressions as means to express general patterns. For example in Fig. 2, the algebraic expression $2 + x\cdot 6$ gives the general rule for calculating the numbers of dots in the shape at the $x$-th position. (Note that $\cdot$ is the German sign for multiplication $\times$).

In our design research project, the shapes and patterns approach was used for a remediating course on variables and algebraic expressions for low-achieving multilingual grade 8 students. Although having some potential for learning mathematics, the students had difficulties during their first encounter with the variables (following other approaches) in their regular classrooms. Hence, the intended learning pathway for this second encounter with the variable comprised the following three stages: The students:

1. find growing patterns for different sequences of shapes and numbers,
2. express these growing patterns informally (with pictures, tables, own verbal descriptions, or with quasi-variable numbers such as 42), and
3. remember or re-discover (instead of invent) the variable as a means to express a symbolic rule for the general case.

As this approach relies heavily on informal expressions that the students invent (e.g., Akinwunmi, 2012), we hypothesized that language plays a crucial role in their constructed pathway. This would give an explanation for the strong connection of learning outcomes in algebra and language proficiency (as shown by MacGregor & Price, 1999). As little is known about the processes so far, this chapter investigates how the interplay between different languages contributes or hinders the intended learning pathways.
1.2 Communicative and epistemic role of languages

As sociolinguists have pointed out, multilingual challenges in mathematics classrooms cannot only be linked to different minority languages, but also to different registers. In sociolinguistics, a register is defined as a “set of meanings, the configuration of semantic patterns, that are typically drawn upon under the specific conditions, along with the words and structures that are used in the realization of these meanings” (Halliday & Hasan, 1976, p. 23). The social embeddedness of the communication situation is often emphasized: “A register can be defined as the configuration of semantic resources that a member of a culture typically associates with the situation type. It is the meaning potential that is accessible in a given social context” (Halliday, 1978, p. 111). Hence, registers are characterized by the types of communication situations, their field of language use, and the discourse styles and modes of discourse. In this sense, the language of schooling can be linguistically conceptualized as a register that is situated between, but overlapping with, both the everyday register and the technical register (Prediger et al., 2014; see Fig. 3).

The didactical relevance of the three registers has been outlined by Pimm (1987): learning mathematics always involves the transition between these registers, and this means moving consequently forward and backward, not only moving in the direction of the technical register (Prediger et al., 2014). For the multilingual learners, the three registers appear in their first and second language (L1 and L2, or even more; see Fig. 3).

Fig. 3 Three registers relevant for mathematical learning (Prediger et al., 2014)

The socio-educational relevance of distinguishing the three registers has been explained by Gogolin (2009). Most teachers are aware that the technical register needs to be acquired in school, and hence give explicit learning opportunities for the technical language. In contrast, the school register (to which only students of privileged socio-economic background are already acquainted) is sometimes treated as a learning condition, instead of a learning goal, such as the expression “x-arbitrary” in Fig. 1. Empirical studies show that limited proficiency in the language of schooling is an important challenge for many (monolingual or multilingual) students in mathematics (Schleppegrell, 2004; Thürmann et al., 2010). This is immediately evident for language difficulties in test situations as shown in many American studies (Abedi, 2006), but also for the German language context where the proficiency in the language of schooling could be reconstructed as the background factor with the highest impact on mathematics achievement (Prediger et al., 2013).

However, the communicative role of the language of schooling is not only relevant in test situations. Every classroom interaction requires language as a medium for the transfer of knowledge. Many researchers have stressed that students with limited academic language proficiency are often hindered in showing their mathematical competences in classroom interactions (e.g., Schleppegrell, 2004). In contrast to these accounts of the communicative role of language, this contribution focuses on the epistemic role of the involved languages in the (individual or/and social) processes of knowledge construction and enculturation into mathematical practices.
Authors who emphasize the epistemic role of the language of schooling (e.g., Schleppegrell, 2010; Thürmann et al., 2010; see also Morek & Heller, 2012) especially point to its importance for higher order thinking practices such as abstracting, generalizing, or specifying causal connections (Morek & Heller, 2012, p. 75; Feilke, 2012). These sociolinguistic, didactical and socio-educational considerations show that the distinction of registers and their characteristics offer an insightful theoretical background for explaining possible language difficulties on a macro- and meso-level. However, the methodological potential of the distinction of registers for the empirical micro-analysis of concrete learning situations is limited by the situatedness of registers and their large overlap. Both features can hinder a unique assignment of utterance one of the registers.

For this reason, the methodological approach for empirical data analysis operationalizes the distinction of languages by situated repertoires with a higher potential on the micro-level of concrete learning situations. For analytical purposes, we do not distinguish the sociolinguistic registers but three situational activated linguistic repertoires: the technical repertoire (usually being a part of the technical register, see Fig. 4); the individual linguistic repertoire (that students bring into the situation, and which can comprise linguistic means from different registers); and the mediating repertoire (by which the teacher intends to mediate between the others, usually comprising different registers). These repertoires will be operationalized in Section 2.3.

![Fig. 4 Individual (I), mediating (M), and technical (T) linguistic repertoires within and between the registers](image)

2. Methodological framework

The specific epistemic role of different languages in student’s pathway to constructing meanings of a variable was investigated within a design research framework.

2.1 Topic-specific Didactical Design Research as methodological frame

Our framework of Topic-Specific Didactical Design Research (Prediger et al., 2012; Prediger & Zwetzschler, 2013) relies on the iterative interplay between designing teaching–learning arrangements, conducting design experiments, and empirically analyzing the processes. It is shown diagrammatically in Fig. 5 (following Gravemeijer & Cobb, 2006).
2.2 Design experiments as method for data collection

The design research project deals with a learning arrangement (comprising 10 sessions of 45 minutes each) designed for low-achieving multilingual grade 8 students in a remediating course on variables and algebraic expressions (hence, it is their second encounter with variables). In the overarching project, we conducted six design experiment cycles with a total of 68 students. In sum, about $190 \times 45$ minutes of design experiments were completely video-recorded and partly transcribed (selections made as they pertained to the research questions).

The case study presented in this chapter uses data from cycle 4 in which the design experiments were conducted in a laboratory setting (see Prediger & Zwetzschler, 2013) by the second author. The four multilingual girls involved in the case study, Ayla and Gözden, Meliha and Gülner, were 14/15 years old. Their parents immigrated from four Middle East countries before their birth or one year after.

2.3 Methods for data analysis

The methodological background of the analysis starts from the assumption that mental and interactional processes are linked, but should be carefully distinguished in order to reconstruct the trains of thought and the evolution of linguistic means in the interaction. For our sequential interpretative analysis of the transcripts, we reconstructed, for each speaker’s utterance, (a) the speaker’s individual construction of meaning (shortly called individual mental model), (b) the speaker’s linguistic means to express the mental model (shortly called linguistic realization), and (c) the listener’s interpretation of the linguistic means (shortly called interpretation).

Points (a) and (c) give hints to the mental processes of both students and researcher/teacher, and (b) refers to the interactional processes and the development of linguistic means. Point (c) is only mentioned if necessary.

For classifying the linguistic means and their mutual transition in (b) according to the used languages, the distinction of three situationally activated linguistic repertoires is used (see Fig. 4). The word level, sentence level, text level, and discourse level in different semiotic representations (words, signs, graphs, gestures), together with their mutual meanings are all used in this classification process. They are operationalized as follows:

- **Technical linguistic repertoire**: Linguistic means and their intended meanings are assigned to the technical linguistic repertoire in a specific learning situation when they belong to the general technical register and are part of the target language. In our concrete learning situation, the variable $x$ and its meaning as generalizer are the most prominent targeted linguistic means.
• **Mediating linguistic repertoire**: Linguistic means are assigned to the mediating linguistic register in a specific learning situation when it is used by the teacher or the material to mediate between the technical repertoire and learners’ language for communicating mathematical contents, meanings, or tasks. This includes especially numerical and graphical representations or artifacts. Hence the criterion for assigning this repertoire draws on didactical or interactional intentions and on an *a priori* specification of the target language that is to be mediated, not *a priori* on sociolinguistic categories.

• **Individual linguistic repertoire**: The reconstruction of the individual linguistic repertoire needs the strongest methodological control. For each linguistic means used by the learners in a learning situation, we check whether its activation can be traced back to an external model by the teacher or material, or whether it was initially activated by the student without external model. In the second case, we assign it to the initial individual repertoire; in the first case, we reconstruct an act of integrating a linguistic means from the technical or mediating repertoire into the student’s individual repertoire. In this way, we can reconstruct the micro-process of individual language development.

Hence, in contrast to the sociolinguistic registers, the linguistic repertoires are operationalized with respect to observable characteristics in the analyzed learning situation (see the analytic table in the next section for an example). The analysis focuses on the transitions between repertoires, for example marked by $[T \rightarrow M]$ if an idea or content is expressed first in the technical and then transformed to the mediating linguistic repertoire.

The complete sequential analysis and classification with respect to the transition of linguistic repertoires was carefully discussed between at least two researchers in order to achieve a communicative validation.

3. **Empirical snapshots from the design experiments**

3.1. **We do not know x-arbitrary**

*Episode 1: Ayla, Gözden, and unknown mediating terms*

For embedding the variable x (an important element of the intended technical repertoire) in the students’ individual linguistic repertoires, the course for low-achievers intended to use the mediating expression “x-arbitrary.” For investigating whether students are familiar with the expression x-arbitrary, it is briefly mentioned early in Stage (1) of the course (see Section 1.1) for diagnostic purposes: after having specified the number of dots for several positions (see Fig. 2), the girls Ayla and Gözden are confronted with the question on the worksheet: “What is the number for an x-arbitrary position?” The researcher/teacher (RT) of the design experiment diagnostically explores the girls’ thinking in the following transcript where 2/97 stands for the second transcript, line 97):

2/97 RT: What does x-arbitrary mean, do you have an idea?
2/98 Ayla: No, our teacher also says that, but we don’t know what it means.

... 2/102 Ayla: She never explains it, not really.
2/103 Gözden: [whispers to Ayla] I only know that in algebraic expressions, there … there is x.
2/104 Ayla: [to Gözden] Yes, but THIS is a position of the numbers [points to the worksheet]

... 2/111 Ayla: That’s how it is said in German. [laughs]
2/112 RT: [laughs] Right, that’s how it is said in German. But, ehm, what does arbitrary mean, though?
2/113 Ayla: any position of the numbers
2/114 RT: Yeah [nods for acknowledging]
2/115 Ayla: But you don’t know which one.
2/116 RT: Yeah, exactly … And that is how you hit the concept. That is the concept x-arbitrary.
That means any position, nothing more.

2/117 Ayla: Then, we can choose a position and then ...

2/128 Gözden: [writes down their working definition] “x-arbitrary is a position of number. You can chose which one. x stands for a number and arbitrary for a position.”

The girls’ relation to the mediating linguistic expression x-arbitrary is multifaceted: they know that it is used in German (#2/111) and that their teacher uses it (#2/98), but they say they do not know its meaning (#2/98). By formulating “how it is said in German” (#2/112) they signal that they do not feel part of this language community; this is one of the rare explicit indications for students’ awareness that the mediating repertoire belongs to the language of schooling in the described sociolinguistic sense. However, the girls realize that the mediating expression is connected to the variable x, the corresponding sign of the technical repertoire (Gözden in #2/103). The researcher believes that Ayla constructs the intended meaning when she translates “x-arbitrary” into her individual linguistic repertoire by “any position” (#2/113), since “any” can refer to the intended meaning “all or arbitrary.” However, the researcher does not realize that Ayla and Gözden stick to the meaning as one number instead of all numbers (#2/115 and #2/128). Although both girls activate the word “any” by which generalization usually can be expressed, they do not yet have access to the underlying practice of generalization. Due to space restriction, we show only one of the analytic tables, that of the dialogue above (see Table 1).

Table 1 Analyzing the interaction of Episode 1

<table>
<thead>
<tr>
<th>Line in transcript</th>
<th>Students’ mental processes</th>
<th>Interactional processes: linguistic realization [and change of register]</th>
<th>Researcher/teacher’s mental processes</th>
</tr>
</thead>
<tbody>
<tr>
<td>#2/97/2/98 A,G: no meaning, but identified as part of teacher’s German language</td>
<td>RT: “x-arbitrary” [T→M] A: “our teacher also says that, but we don’t know what it means” [M = alien register]</td>
<td>Intention: give an explanation for technical symbol x as generalizer by a mediating term x-arbitrary</td>
<td></td>
</tr>
<tr>
<td>#2/103 G: x-arbitrary contains symbol x</td>
<td>G: “in algebraic expressions, there is x” [M → T]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#2/104 A: x in x-arbitrary does not correspond to symbolic variable x</td>
<td>A: “Yes, but THIS is a position of the numbers” [x-arbitrary] [T ≠ M]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#2/113-2/115 A: constructs meaning for x-arbitrary as one unknown</td>
<td>A: “any position of the numbers” “but you don’t know which one” [M → T]</td>
<td>Interpretation of Ayla’s “any”: constructed meaning as generalizer</td>
<td></td>
</tr>
</tbody>
</table>

The diverging meanings becomes evident two sessions later: after having written down many arithmetic expressions (see scan in the transcript), the teacher gives them the algebraic expression $2+3\cdot x$ for another sequence. However, they are still unsure what x and x-arbitrary mean. The excerpt of transcript from the fourth session starts when they discuss again what values to insert for x:

4/102 Ayla: […] But how can you know what number you should calculate there?
4/103 RT: Well, first, we do not need to calculate. We first want to …
4/104 Ayla: … or is it for all?
4/105 RT: Exactly. It is for all positions.
4/106 Ayla: Ah! Okay!

4/110 Ayla: [to Gözden] Look, that stays always, doesn’t it? [points to the arithmetic expressions in the table] And then, [points to the algebraic expression $2+3\cdot x$] you can insert every number you want,
and that is, then every calculation becomes the same.

Only after having written down the arithmetic expressions for many positions (hence after a hands-on experience of generalizing; see Fig. 9 for the activity) can the students extend their individual meaning of \( x \) from the unknown (“know which number to calculate there,” #4/102) to the variable as generalizer. Now the linguistic means \( x \) and \( x \)-arbitrary are successfully integrated into the individual repertoire, and Alya (#4/110) can explain its meaning to Gözden (#4/110).

The scene shows that constructing the meaning of a variable as generalizer does not only depend on isolated (technical or mediating) words, but is deeply connected to the practice of generalizing itself. That is why Lee (1996) talks about an *initiation into the culture of generalizing*.

**Episode 2: Meliha and Gülner and the repdigit**

The episode of Meliha and Gülner starts in a similar way when they are asked to search the 42nd position of three sequences. The scan of their work in Fig. 6 shows that the process of finding shortcuts (and hence a generalizable pattern) starts quite slowly and only begins in the third sequence: they stop writing down all numbers until 42, but the video shows there repeated additions of 6 on the calculator. So far, no multiplicative shortcut has been found.

Similar to Ayla and Gözden, they do not know what \( x \)-arbitrary means and guess many different possibilities, such as repdigits (#3/110 in the original transcript, not shown here), prime numbers (#3/116), or letters that stay the same (#3/123). Then the teacher explicitly refers to the mathematics classroom (#3/126):

3/116  Gülner  [the numbers] that you can only divide by itself

…

3/123  Gülner  [the letter \( x \)] that it stays the same

…

3/126  RT  Perhaps you know the term \( x \)-arbitrary from math classrooms?

3/127  Gülner  We only know the variables. For example \( x \). That we insert, though.

3/128  RT  Yes exactly, there you can insert. But a variable, it can be even more. Or?

3/129  Gülner  Calculating formulas

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**Fig. 6** Meliha and Gülner’s pathway to generalizing and the variable as generalizer
The term x-arbitrary (that was meant to mediate the variables from the technical repertoire) again poses difficulties; as earlier, the shift $T \rightarrow M$ did not help to construct an adequate meaning. Instead, the shift back to the technical repertoire (suggested by the researcher/teacher’s explicit reference to math classrooms in #3/126) helps the students to connect the term “x-arbitrary” to the technical concept of variable. Gülner activates the meaning of the variable as a placeholder into which one can insert values (#3/127), although with a certain vagueness of imprecise grammar.

We conclude that in both case studies, the mediating register did not work in the intended way for students who do not feel acquainted with the term x-arbitrary. This empirical finding gives a first contribution to research question Q1: students’ individual meaning construction for variables as generalizers cannot be initiated simply by giving them a translation using the mediating repertoire. Instead of explaining the concept by one mediating term, the complete thinking practice of generalizing first has to be established.

This thinking practice, as well as the concrete term, is part of the language of schooling, marked as an alien sociolinguistic register by the girls. As the linguist Feilke (2012) has emphasized, the language of schooling and the thinking practice of generalizing are directly connected, since the language of schooling provides the linguistic (lexical, grammatical, and discursive) means for higher order thinking practices. This is no surprise since, historically, every register evolved for being most suitable for specific purposes. The register of schooling is optimized for making explicit concentrating, discussing, and generalizing. Feilke defines generalizing as “presenting circumstances as independent from personal, temporal and local situational references and assuming their general validity” (Feilke, 2012, p. 9, translated). He names some typical linguistic means for realizing this practice: grammatical means (such as generalized or generic forms ‘he instead of I or you,’ or cutting out the agent by passive forms, generic use of articles), lexical means (as in ‘all conditions,’ ‘always,’ …), and routines such as defining.
With Feilke, we assume that Meliha and Gülnur as well as Ayla and Gözden still need to develop their linguistic means for generic forms and for many numbers addressed at the same time (simultaneous aspect; Malle, 1993). This challenge goes much deeper and is more subtle then the simple term x-arbitrary. The following Episode 3 is aimed at further exploring and perhaps strengthening this assumption.

3.2. Experiencing generalizing and variables as generalizers

We continue to follow the learning pathway of Meliha and Gülnur and show how experiences with generalizing and generalizers can be established together with the necessary linguistic interchanges that support their learning.

For finding and expressing growing patterns informally, the remediating course offers (in stage (2); see Section 1.1) different ways of finding the higher positions in sequences (with pictures, tables, “Merve’s” verbal description, and expressions with the quasi-variable 35, as printed in Fig. 7; here with texts translated into English).

Interestingly, the students’ first attempt to adopt the strategies they have learnt in one context to the next sequence (in Fig. 8) shows many initial mistakes and a prevalent willingness to work with given linguistic models: the girls do not only transfer the phrasing (“I have the position, that I have to multiply by 3 and then add 1”), but also the “+1” in the verbal and symbolic description. The first arithmetic expression 1+5·60=181 is later corrected to 2+3·60=182.

**Episode 3: Again this x-arbitrary**

For initiating stage (3) of the course, the teacher now comes back to the practice of generalizing and asks the girls to find the expressions for many different positions (see Fig. 9). This repeated operative experience offers the fundamental idea of coming back to the variable and x-arbitrary:

5/  56  RT  [...] ok. Now, task b) and c) ask how to calculate for an x-arbitrary position? Describe like Merve. And how does Pia write an expression for x-arbitrary positions?
5/  57  Gülnur  Ah, again x-arbitrary.
5/  62  RT  [...] You can consider Merve’s way again and then think about what such an x-arbitrary means here now.
5/  63  a  Gülnur  [looks onto the sheet with the table from Fig. 9]
    b  Perhaps, it is [points to the table with the pen]
    c  the x-arbitrary position [moves the pen up and down]
    d  the position that always changes.
    e  So, not the sequence of numbers and the sequences of shapes but the position.
    f  Can be x-arbitrary, because it often changes. We have, we need not always make 
    g  the 43rd position, for example. We sometimes need to make the 120th position.

Gülnur immediately recognizes the difficult concept x-arbitrary (#5/57) and remembers that they did not understand it in Episode 2. The researcher/teacher encourages them (in the non-printed lines #5/58–5/61) and hints at the verbal description given by the fictitious Merve from Fig. 7 (#5/62). In line #5/63, Gülnur develops the idea that x-arbitrary addresses the changing positions that they have considered in the table of Fig. 9. Her utterance vividly shows how linguistic means can successively evolve: before verbalizing this idea explicitly, she uses the gesture of moving the pen up and down in the table to communicate her idea (#5/63c). Based on the operative experience with many arithmetic expressions and the pointing gesture, she finds the linguistic means to address the idea within her individual linguistic repertoire: “the position that always changes” (#5/63d). The generalizing activity is translated into a temporal consideration for which she can find words. The generic examples by which she intends to strengthen her argument (#5/63g) are accompanied by other temporal terms such as “always,” or “sometimes.” Hence, the transition [M →] now successfully takes place in several steps.

In the succeeding activities of the remediating course, the girls consolidate their constructed meanings and deepen their experiences with generalizing for sequences. The linguistic analysis
shows that they gain more and more confidence in the notion of variable as generalizer and integrate the mediating term x-arbitrary into their individual linguistic register. They accomplished this even for explaining huge algebraic expressions such as $x \cdot 3 + x \cdot 5 + x \cdot 17 + 2 \cdot 19 + 468$, which resulted from their mathematization of a complex word problem (with $x$ being the varying number of students in a calculation of costs), as noted in the following:

8/ 12 Meliha Because, x-arbitrary, that is, where the students. They change. Thus we have at x, always students.

Hence, they manage the mathematizing, that is, the transition between the word problem text in the mediating repertoire and the technical repertoire of an algebraic expression $[M \rightarrow T]$. They also manage the interpretation, that is, the transition back from the technical to the individual repertoire $[T \rightarrow I]$ that works fluently here [together $M \rightarrow T \rightarrow I$], showing the integration of the term x-arbitrary into the individual repertoire.

These observations and the careful analysis of the nature of linguistic means for expressing generality in the individual repertoire offer further contributions to research questions Q1 and Q2: Gülnur’s use of many temporal terms in her individual linguistic repertoire shows, on the one hand, that she has found a pathway to generalizing via the operative variation of expressions in Fig. 8. This operative activity makes the idea of generality accessible by offering a dynamic perspective (one position inserted after the other). On the other hand, Gülnur’s use of many temporal terms confirms Feilke’s emphasis that the simultaneous consideration of many possibilities needs the construction of new linguistic means that are not necessarily part of students’ individual repertoires. In contrast, Ayla in Episode 1 succeeded in tentatively expressing simultaneous generality by saying, “you can insert every number you want, and that is, then every calculation becomes the same” (#4/110 in Episode 1).

As the temporal linguistic means for expressing successive changes seem to be more accessible than linguistic means for expressing simultaneous generality, the mental activity of dynamically changing numbers seems a very appropriate pathway to the thinking practice of generalizing. In this sense, the learning arrangement has been optimized in preceding design experiment cycles so that it now allows Gülnur to bridge the gap by intensively offering operative experiences. This provides an important contribution to research question Q3.

**Episode 4: From x-arbitrary position to x-arbitrary kilometer**

Once the meaning of x-arbitrary positions is consolidated for the students, a slow process of transfer starts. This can be exemplified with an episode from the tenth session with Meliha and Gülnur in which they were asked to find a general algebraic expression for the following word problem: “Every tour with the taxi costs a basic fee of 3€. For each kilometer, a price of 2€ is added.” To conduct the transition $[M \rightarrow T]$, Gülnur develops the expression $x \cdot 3 + x \cdot 2$ and explains:

10/ 30 Gülmar And for what stands x, I have, I have no question but I have written x times 3 plus x times 2. Ah, wait [she writes “How much is the price for an x-arbitrary position?”]

... 10/ 34 Gülmar [reads her question]

10/ 35 RT Ah, ok, for an x-arbitrary position. And what does x-arbitrary position mean here?

10/ 36 Gülmar Ehm, well, we do not know how many kilometer, well [writes on her sheet, corrects the first x into f]

The prompt to pose a question for interpreting the algebraic expression $[T \rightarrow M]$ urges her to think about the meaning of the variable. It is remarkable that, in her question, she does not activate her individual repertoire of speaking about changes in temporal terms but draws on the learned mediating term “x-arbitrary position,” so far without connecting it to the taxi situation in the word problem.
The researcher/teacher intervention (#10/35) draws her focus to the kilometers. Instead of changing the question, she first corrects her mistake in the algebraic expression (#10/36). Hence, the initiated transition \([T \rightarrow M \subseteq I \rightarrow T]\) enables her to evaluate her symbolic expression \([T \rightarrow M \subseteq I \rightarrow I \rightarrow T]\). She then shifts from “x-arbitrary positions” to “x-arbitrary kilometers.”

In the end, Gülnur succeeds in the connection of the variable to the general number of kilometers. This enables her to transfer her algebraic competencies of using the variables as generalizer to new everyday contexts. The trajectory of transitions between repertoires \([M \rightarrow T \rightarrow M \subseteq I \rightarrow I \rightarrow T]\) shows that she could integrate the linguistic means into her individual repertoire but still needed the connection to everyday contexts: using “x-arbitrary position” does not immediately imply that the connection to kilometers is drawn, but after a prompt, it could be.

This snapshot shows that, for these students, the language- and content-integrated learning arrangement could provide a pathway to generalizing as a mathematical practice and the variable as generalizer.

4. Conclusions

The case study on the meaning of variables as generalizers offer empirical insights into topic-specific language challenges for underprivileged multilingual students: teachers and teaching materials often use linguistic means in the mediating linguistic repertoire that go beyond isolated terms such as x-arbitrary. Beyond the word level, the sentence, text, and even discourse level is often addressed, as in our case study the thinking practice of generalizing. These linguistic means often belong to the sociolinguistic register that has been termed language of schooling and plays an important epistemic role in students’ pathway to conceptual understanding and higher order thinking practices. However, for socially underprivileged mono- or multilingual students, the register contains many linguistic means that are not part of their initial individual repertoires.

Successful learning arrangements for these disadvantaged students should therefore consider the mediating repertoire to be part of the learning goals, not the learning resources. If the learning arrangements provide opportunities to construct meanings for these mediating linguistic means with important epistemic roles, then the construction of meanings for mathematical concepts can be successfully supported. The thorough and deliberate shift between repertoires and registers can contribute to these processes (Prediger et al., 2014).

For the concrete exemplary topic “generalizing and variables as generalizers,” the well-established shapes and pattern approach has proved to be useful when being combined with operative variations of arithmetic expressions (cf. Fig. 9). These later experiences give the opportunity to get access to the “foreign” culture of generalizing (Lee, 1996), a thinking practice that is deeply connected to the language of schooling (Feilke, 2012) and should therefore be the issue of further empirical investigations.

To conclude, our design research project could contribute to four research demands (a and c being postulated by Schleppegrell, 2010, p. 107):

(a) topic-specific research, here on an algebraic concept that is central for middle school mathematics, namely variables as generalizers,

(b) a conceptual framework that helps to analyze the transition between languages not only on the sociolinguistic meso-level, but also on the micro-level of the concrete situation,

(c) practically and empirically approved instructional designs for developing students’ language of schooling together with the specific mathematical topic, and

(d) empirical insights into typical learning pathways initiated by these instructional designs.