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20. Topic-specific design research with a focus on learning processes: The case of understanding algebraic equivalence in grade 8

Susanne Prediger & Larissa Zwetschler

Abstract
How can students acquire conceptual understanding for algebraic equivalence? What are the students’ conceptions and how can they be further developed? What hinders this process, and what can facilitate it? We present a case study addressing these questions in mathematics education that is conducted within the framework of topic-specific Didactical Design Research with a focus on learning processes. The framework is briefly presented, and the topic-specificity is explained. The illustration by the case study on understanding algebraic equivalence shows how a Didactical Design Re-search project is conducted in five cycles between the four working areas specifying and structuring learning goals and contents, developing the design, conducting and analyzing design experiments in laboratory and classroom settings, and developing local theories on teaching and learning processes.

1. Framework of topic-specific didactical design research with a focus on learning processes
Like other approaches to Educational Design Research (cf. Plomp & Nieveen, 2009; Van den Akker, Gravemeijer, McKenney, & Nieveen, 2006 for overviews), our framework of Didactical Design Research relies on the iterative interplay between designing teaching-learning arrangements, conducting design experiments and empirically analyzing the processes (following Cobb & Gravemeijer, 2006). It has two equally important aims, namely to generate 1. Research results (local theories on content specific teaching and learning processes with typical learning pathways and obstacles, and typical conditions and effects of design elements) and 2. Design results (evaluated teaching-learning arrangements and design principles and strategies) (see figure 1).

As we work in mathematics didactics, the problems treated are didactical and not generally educational, which means that we focus on learning specific contents and ask not only how to learn, but also what to learn and how these are intertwined (van den Heuvel-Panhuizen, 2005; see also Biehler, Scholz, Sträßer, & Winkelmann, 1994, for didactics as a research discipline). For example, if we treat the well-known problem, “How can mathematics classrooms be designed in a way that students really have the chance to understand the mathematics contents instead of simply learning calculation routines?”, a major didactical question is to clarify which knowledge elements, conceptions, experiences, and skills are exactly necessary for the conceptual understanding of a specific mathematical topic.

Whereas Plomp and Nieveen (2009) emphasize the need for Educational Design Research especially for new educational problems, most of the practical problems that we treat are well-known and to a certain degree “solved” on a very general level.
For example, there is a certain consensus within the community of mathematics didacticians and teachers regarding design principles for the aforementioned understanding-problem: for giving students access to conceptual understanding of mathematical topics, teaching-learning arrangements should generally start from meaningful (often out-of-school) contexts, provide adequate representations, and allow students to build adequate models of the mathematical topics (Freudenthal, 1991). Notwithstanding this general “solution” (or design principle), there is a further need to clarify how these general principles can be concretely realized for specific mathematical topics, such as dealing with algebraic expressions: Which meanings (mental models) are necessary for dealing with algebraic expressions?; Which contexts and which representations can be provided to support the individual construction of meanings for this specific topic?; How do they have to be connected in the learning process for allowing understanding?; What initial conceptions do students have prior to the instruction?; How can we build on them?; What typical obstacles appear for the concrete topic?; For answering these kinds of questions, the general, topic-independent principles must be enriched by very concrete, topic-specific design research. The design research aims at finding concrete ways of realization (design results) as well as at specialized knowledge regarding typical, topic-specific learning and teaching processes (research results). Although some elements of the local theories are of course transferable to the next topic (e.g., from algebraic expressions to fractions), the topic-specific research is needed in each case and characterizes our Didactical Design Research. Figure 1 shows the four working areas that are iteratively connected in the design and research processes. The areas are intertwined in the sense that each cycle builds upon the results of the previous cycle across the different working areas. Notice that the models deliberately have no explicit starting point, since it varies from project to project. Usually, projects start from didactical problems in context, but other projects also start from theoretical concerns or empirical results (e.g., Prediger & Schnell, 2014; for an example from stochastics).

**Content-focused**

Due to the importance of the learning content, “specifying and structuring learning goals and contents” is specified as an own working area in the FUNKEN-model for Didactical Design Research (Prediger et al., 2012; FUNKEN is the interdisciplinary graduate school in which the model was developed). This comprises not only the optimal order of pre-defined topics, but also their restructuring with respect to central meanings and relations to everyday thinking, for which the mediation between students’ perspectives and scientific perspectives is a major methodological step (Prediger, 2008; Duit, Gropengießer, & Kattmann, 2005). **Process-focused:** The empirical analysis is focused on individual (and socially contextualized) learning processes as generated by the developed teaching-learning arrangements.
2. Snapshots from the concrete case “Understanding algebraic expressions”

The following case study offers concrete examples for explaining the working areas in more detail. In the next sections we present it by specifying empirical and theoretical starting points, an overview on all design experiment cycles, and then insights into the different cycles.

**Empirical and theoretical starting points**

We can only sketch the theoretical background and the empirical state of research by illustrating the process of problem specification, the specification of learning content, and topic-specific research questions.

**Problem specification**

Starting point for the project was a typical error while transforming algebraic expressions and the subsequent dialogue between the teacher researcher (first author) and her student Lea, 15 years old.

44 Lea  [has written $10n + 3 = 13n$]
45 Teacher  Have a look, $10n + 3 = 13n$, is that really correct?
46 Lea  Yes, why?
47 Teacher  Just insert a number for $n$ and test it.
48 Lea  What do you mean by that?
49 Teacher  Well, just take the 4 instead of $n$ and write it down.
50 Lea  [writes $10 \cdot 4 + 3$, hesitates] No, it would be times. [writes $10 \cdot 4 + 3 = 43$]
51 Teacher  And the $13n$?
52 Lea  They would be, uh, 52.
53 Teacher  And, do you see something?
54 Lea  No, what? [looks at the numbers, hesitates] Is it wrong, though?
55 Teacher  The result isn’t equal?
56 Lea  No, but ... that is with 4, not with $n$.

(Prediger, 2009, p. 228)
Lea makes a typical error while transforming algebraic expressions that is well known from empirical investigations as syntactical error pattern “similar to similars” (Tietze, 1988). The student seems to assume that the two algebraic expressions $10n + 3$ and $13n$ are equivalent, since she does not realize her error when prompted to it (in line 45/46). Drawing on the theoretical distinction between syntax (routine transformation) and semantics (meanings of mathematics signs) (Malle, 1993), the teacher adopts a typical didactical strategy for dealing with syntactical errors, namely “go back to the meaning of mathematical signs” (Prediger, 2009). She hopes that Lea can identify and correct her error when she realizes that the assumed algebraic equivalence does not correspond to its meaning for concrete numbers (line 47, 49, 55). For this purpose, the teacher refers to the meaning “insertion equivalence” of algebraic equivalence (see below). However, Lea does not seem to connect the transformation equivalence of expressions (which she incorrectly applies) to the insertion equivalence, and without this conceptual understanding, she still cannot realize her error. This episode illustrates one facet of the central problem that is addressed in our design research project: Q1. How can students acquire conceptual understanding for algebraic equivalence?

Specification of learning content and topic specific research questions

Before the “how” can be investigated, the “what” must be specified (van den Heuvel-Panhuizen, 2005): Conceptual understanding of algebraic equivalence comprises three different meanings that have to be connected to each other: Two algebraic expressions are said to be equivalent:

(a) description equivalence: …, if they describe the same phenomenon (same geometric pattern, same situation, same function, …);

(b) insertion equivalence: …, if they have the same value for all inserted numbers;

(c) transformation equivalence: …, if they can be transformed into each other according to the transformation rules (commutativity, associativity, distributivity). (Malle, 1993, p. 46).

For the design of the teaching learning arrangement, we started with a core activity (illustrated in figure 2) that is often suggested in the literature (Wellstein, 1978; Mason, Graham, & Johnston-Wilder, 2005; Malle, 1993), namely compare expressions that describe the same geometric shapes: “While formulating and interpreting expressions in meaningful situations … the transformation rules result naturally, when a situation is described in two ways” (Malle, 1993, p. 239, italics added).

However, the first design experiments showed that this activity and the individual construction of intended meanings are not as “natural” as mentioned by Malle in the quotation. Weaker student, especially, seem to be hindered by alternative conceptions of the situations. These alternative individual conceptions might be the reason why the teaching/learning arrangement does not always facilitate the intended learning outcomes.

It is our social constructivist theoretical background (Smith, di Sessa, & Rochelle, 1993; but with a focus on social constitution of meanings, cf. Ernest, 1998) that suggests investigating students’ conceptions in more depth in order to develop the teaching/learning arrangement towards a more fruitful communicative mediation between student thinking and intended meanings. So we added two further leading questions for the research project design:

• Q2. What conceptions do students have and how can they be further refined?

• Q3. Which elements of the teaching learning arrangement can hinder this process, and which can facilitate it?
Overview on five design (experiment) cycles and different types of results

Table 1 gives a very brief overview of the complex process of the design research study with its five cycles in different stages. Although each cycle covered all four working areas (see Figure 1), they were conducted by different agents, prioritize different aims, and, therefore, had different methods and results (cf. Nieveen, 2009, p. 96). The research questions were always given by Q1-Q3, but with different priorities concerning the different specified aims.

<table>
<thead>
<tr>
<th>Table 1: Five cycles of design and design experiments</th>
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<tbody>
<tr>
<td>Preliminary Analysis 0</td>
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<tr>
<td>Didactician as teacher/researcher (=1st author)</td>
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<tr>
<td>(2005-2009)</td>
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<td></td>
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<tr>
<td>Design Cycle 1</td>
</tr>
<tr>
<td>Designer team (didactician, expert teacher, PhD student = 2nd author) (Sep 10 - Jan 11)</td>
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<td></td>
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<tr>
<td>Design Experiment Cycle 2a</td>
</tr>
<tr>
<td>PhD student &amp; didactician as designer &amp; researcher (Jan 11 - Apr 11)</td>
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<td></td>
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<tr>
<td>Design Experiment Cycle 2b</td>
</tr>
<tr>
<td>PhD student &amp; didactician as designer &amp; researcher (Apr 11 - Nov 12)</td>
</tr>
<tr>
<td>i) First analysis of videos with respect to further challenges and resources</td>
</tr>
<tr>
<td>ii) Deep qualitative analysis of transcripts (of Cycles 2a and 2b) with respect to typical challenges, resources and critical moments in the processes, with Vergnaud’s (1996) analytical framework</td>
</tr>
<tr>
<td>Results</td>
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<td></td>
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</tbody>
</table>
Table 1: Five cycles of design and design experiments (continued)

<table>
<thead>
<tr>
<th>Design Experiment</th>
<th>Aim</th>
<th>Methods</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cycle 3</strong></td>
<td></td>
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</tr>
<tr>
<td>Regular teachers as testers; PhD student &amp; didactician as designer &amp; researcher (Sep 11 - Nov 12)</td>
<td>Testing the complete prototype teaching learning arrangement in regular classrooms; Deepening empirical insights into learning pathways and teaching conditions</td>
<td>Design exp. in field setting in 2 classes (20-32 x 45 min.); triangulated by design exp. in laboratory setting with 6 pairs of students on selected tasks (3-5 x 45 min.)&lt;br&gt;i) Evaluation with respect to learning outcomes and practicality by analyzing classroom videos, teachers’ diaries, students’ written protocols, and tests on students’ conceptions&lt;br&gt;ii) Deep analysis of transcripts with respect to learning pathways, effects, conditions</td>
<td>i) Enhanced teaching and learning arrangement with redesign of selected tasks&lt;br&gt;ii) Further contributions to the local theory on learning and teaching algebraic equivalence (especially on learning pathways) (→ PhD thesis, Zwetzschler, 2013)</td>
</tr>
<tr>
<td><strong>Cycle 4</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular teacher as tester; PhD student &amp; didactician as designer &amp; researcher (Dec 11 - Mar 12)</td>
<td>Selectively testing the new tasks in the teaching learning arrangement; first observation on the succeeding teaching learning arrangement (transition to transformation equivalence)</td>
<td>Design experiments in field setting in one class (16 x 45 min.); focused by design exp. in laboratory setting with 2 x 2 students on new tasks (3 x 45 min.)&lt;br&gt;Evaluation of the classroom tests and videos with respect to cognitive accessibility and success of developing conceptual understanding in the new tasks</td>
<td>Completely (formatively) evaluated teaching learning arrangement that facilitates the development of conceptual understanding in the intended way; insights into conditions in the teaching arrangement</td>
</tr>
<tr>
<td><strong>Cycle 5</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Designer team &amp; editors &amp; didactical experts from publisher as designers (Jan 13 - Dec 14 )</td>
<td>Finalizing the teaching learning arrangement for regular use</td>
<td>Detailed editing with respect to theoretical and empirical results; but also linguistic issues, layout, etc.;&lt;br&gt;Collecting illustrating examples and episodes for the hand book for teachers</td>
<td>Chapter of a textbook for students in middle schools and handbook for teachers on typical pathways, obstacles and outcomes (→ Mathewerkstatt 8, Leuders, Prediger, Hußmann, &amp; Barzel, 2012)</td>
</tr>
</tbody>
</table>

Notice that a cycle was not necessarily completed before the next one started; this is especially true for the deep analysis of the Cycles 2b and 3. Furthermore, some cycles included micro-cycles, for example when an activity was slightly changed between the first and second design experiment of the same series.

In the following subsections, we will illustrate these steps by one thematic focus that played a role in all cycles, namely the challenge of generality of algebraic expressions and geometric shapes.
Although it is only one among others, this topic is now selected for illustrating typical stages and insights in the research design process. Despite the fact that the teaching learning arrangement also comprises social settings, computer sources and specific interactional arrangements, this article focuses mainly on the tasks as one design element of the comprehensive arrangement.

**Design Cycle 1: First design and redesign of the teaching learning arrangement**

Figure 2 shows two core activities in our teaching learning arrangement that were designed at the stage of Design Cycle 1.

(I) Different expressions for the same figure?
Which students calculate the same area?
And which of the expressions calculate the area of the given geometric shape correctly?

(II) Checking for different numbers
Insert different numbers for the variables.
Check which of the expressions are equivalent.

![Figure 2: Tasks (I) for experiencing description and (II) for insertion equivalence (first stage)](image)

Following the suggestions in the literature (Wellstein, 1978; Mason et al., 2005; Malle, 1993), we developed Task 1 (in Figure 2) for fostering students conceptual understanding of description equivalence: by considering four algebraic expressions of fictitious students that are to be compared, the learners are fostered to discover that different algebraic expressions can describe the same geometric shape (similarly for functions in Kieran & Sfard, 1999). Till’s and Ole’s expressions describe the area of the given shape, whereas Merve’s and Paul’s expressions do not. Task II (in Figure 2) suggests checking the equivalence by inserting numbers; this refers the algebraic expressions back to arithmetic expressions. Students who have already found out that some expressions are description equivalent can experience that they have the same value for each insertion. In this, they are supposed to construct the meaning of insertion equivalence. A later part of the teaching learning arrangement addresses the transformation equivalence, being constructed in a process of progressive schematization (Treffers, 1987). For this case study, we exclude this step in the learning pathway.

For the design within the larger design research project KOSIMA, the activities presented were embedded into a complete teaching learning arrangement (Prediger et al., 2011) that follows several central design principles.
The most important ones for this case study are:

- first construct meaning (here description and insertion equivalence), then schematize it into syntactic transformations (Freudenthal, 1991; Prediger, 2009);
- constantly relate different representational registers (Bruner, 1967; Duval, 2006) in order to foster the individual construction of meanings - here concretely switch between the symbolic register (algebraic expressions), the graphical register (geometric shapes), and the numerical register (arithmetical expressions after insertions);
- involve students in rich mathematical discussions for fostering interactive social constructions of meanings (Ernest, 1998).

The expert evaluation mainly considered the quality of implementation of these design principles with respect to conceptual coherence, cognitive accessibility, and suitability of contexts. It initiated partial designs before the next cycle started.

**Design Experiment Cycle 2a: Experimenting with selected parts of the design**

For gaining empirical insights into typical challenges on the learning pathways (specification of research question $Q_2$), we conducted two design experiments with core activities like those from figure 2, being prepared by less complex tasks in which students calculated the area of simpler shapes and wrote and interpreted algebraic expressions. The videos of students’ processes were roughly analyzed with respect to typical challenges and resources. For this, the videos were categorized with respect to students’ conceptions expressed in their interactions and actions.

The thematic focus “challenge of generality of expressions and geometric shapes” arose while observing Paula and Daniel (grade 9), who solved Task I in an unexpected but typical way. We briefly sketch some typical moments on their learning pathway on Task I and II (analyzed in more detail in Zwetzschler & Prediger, 2013): In Task I, Paula and Daniel succeeded in checking for some expression if they correctly described the area of the geometric shape or not. When struggling with the more complex expression suggested by the fictitious Till, they decided to calculate its value and compare it to other expressions’ values. As calculating the value is only possible after inserting numbers, they determined the length of the sides in the geometric drawing by counting units (a technique that might have been slightly suggested by the grid paper but also appeared without it). This technique of determining one specific number for the variables led them to a successful decision regarding Till’s expression. However, the technique is not compatible with the table of different insertions given in Task II. When starting with Task II, Paula and Daniel were astonished:

Paula: We filled in the right numbers and he took any number whatever?
Daniel: Huh? That’s not possible. (…)
You just can’t insert different numbers.

Daniel and Paula disagreed on inserting different numbers instead of the fixed lengths of the concrete drawing. With this restriction of possible insertions, they did not get any access to Task 2 and to general insertion equivalence.

In the first analysis of the video, we recognized their problem, on the one hand, as a typical misconception regarding variables as a specific hidden number (Küchemann, 1980) and, on the other hand, as a well-known challenge in geometry (Parzysz, 1988): understanding a geometric figure as general instead of seeing only the concrete drawing.
We rediscovered that general formulas for areas are not at all easy to understand for students in their generality; hence this insight has to be added to the specification of learning contents (see figure 1).

As a consequence for the redesign, we developed additional activities for dealing with this challenge of generality of geometric figures and variables. Task III in figure 3 was introduced in order to let the students experience that the same algebraic expression can describe the area of different (drawings of concrete) triangles, all of them belonging to the same geometric figure (the so-called *general* triangle) and being described by the same *general* expression.

**Task III** One expression for all triangles?

a) Draw three absolutely different triangles and find a way to calculate their area by splitting and adding the drawings. Write down expressions for the three areas.

b) Can you find a general expression to calculate all three areas?

c) Can you really calculate the areas of all triangles with your general expression? Draw and try.

**Possible student solution:**

Figure 3: Task (III) for discovering the generality of figures and expressions

**Design Experiment Cycle 2b: Deeping the analysis on selected parts of the design**

Although these first observations already informed the redesign of some elements of tasks and communicative activities, we decided to deepen the analysis with a more systematic analysis of higher methodological control in order to get deeper answers to question Q1-Q3. For this, we developed analytical tools based on Vergnaud’s (1996) constructs theorems- and concepts-in-action and conducted systematic qualitative analysis of transcripts (Zwetzschler & Prediger, 2013). Within the language of this analytic tool, the observation from above would be rephrased as follows: Paula’s and Daniel’s reaction is guided by their theorem-in-action <For comparing two expressions, I can compare the values of the expressions>, but limited to one insertion, the lengths in the drawing. Behind this, we reconstructed their concepts-in-action Variable as specific hidden number and Shape as concrete drawing. By rephrasing all observations in the same analytic language of theorems-in-actions (see Vergnaud 1996, here marked by <…>) and concepts-in-action (marked by), we could compare the conceptions of different students.

More importantly, the analytical tools allowed us to see more systematic connections between different individual conceptions and better understand the relations between different obstacles on the pathway to a general insertion equivalence.
Rather than identifying only well-known misconceptions, we were also able to recognize Paula’s and Daniel’s concept-in-action Value equivalence of expressions as specific insertion equivalence as still too limited, but nevertheless an important resource on the pathway to a general insertion equivalence (cf. Zwetzschler & Prediger, 2013).

In parallel to this deep analysis of data from Cycle 2a, we conducted new design experiments in laboratory settings with the enhanced learning arrangement. The aim was to definitely sharpen the research focus and to complete the prototype of the teaching and learning arrangement. We were especially interested regarding how the students could transfer their experiences from the new Task III to the later treated Task I and II.

We found interesting moments for this question in the learning pathway of Jan and Niclas (grade 7). The boys worked on Task II and later on Task I and tried first to find their own ways of calculating the area by splitting the shape. After different, but not completely correct, ideas, Niclas suggested:

58 Niclas: Now I would just do that [points to the bottom angle of the left triangle] times that [points to side h]

59 Teacher: Huh?

60 Niclas: Uh, calculate and then that [points to b] times that [points to a] then that [points to b] times that [points to a] and then add.

At this point, Niclas described a way to calculate the area without referring to its concrete measure but to characteristics of the figure. He was guided by the theorem-in-action <For calculating the area of a shape, I can split it into areas that I can calculate>. Due to his concept-in-action Formulas for geometric areas generally apply for all concrete drawings, he identified the triangle and rectangle in the shape and calculated their area. However, Niclas’ conception about the variable was still limited, as became evident when they negotiated what to insert for the variables:

63 Niclas: …can I just do it with units, that I count this [he first touches the lower side and afterwards the height of the triangle] so or just six units?

…

69 Teacher: How long could they be, the sides?

70 Jan: Different, as you can actually choose, x-variable.

71 Teacher: Mhm.

72 Niclas: Or maybe one unit as one meter, that are 16 meters [points to side a] that are 9 meters [points to side b, gives a shrug], aren’t they?

At that point, both students are guided by the theorem-in-action <For calculating the area of the given shape, I can insert values for the variables>, but, whereas Jan holds the concept-in-action Variable as generalized number, Niclas still activates the concrete Variable as place holder for specific numbers.
(IV) Different possibilities for the shape of the terrace

The 3 x 6 m terrace shall be enlarged around the corner; the family discusses different possibilities.
a) Draw two more terraces (terrassen) on your own.
b) Calculate the area of the different shapes.
c) Can you calculate all possibilities for the terrace by the general expression 3 \cdot 6 + x \cdot y?

Snapshot from students’ solution:
Maike calculates different terraces for evaluating c)

Figure 4: Task IV for coordinating generality in geometric figures and insertions

While the newly inserted Task III helped Niclas to widen his understanding of a figure in the geometric context, his understanding of the variable itself stayed still concrete. These divergent conceptions later hindered him from changing between registers so that he could not relate the symbolic and graphic register.

As a consequence of this observation for the redesign, we developed a new activity in which the generalization of variables and figures was parallelized in both registers more consequently (see Task IV in Figure 4). This activity builds upon students’ resource to insert for specific numbers (the outcome of the deep analysis to the transcripts from Cycle 2a) and widens their conceptions to more general ones.

Design Experiment Cycle 3: Testing the complete prototype
In the Design Experiment Cycle 3, we tested the complete prototype learning arrangement in classroom settings with regular teachers and much more learning time with respect to practicality under normal classroom conditions (especially research question Q1 and Q3). The videos and written products from classrooms were triangulated by short design experiments in laboratory settings with some students of these classes in which the focus was set to the new activities and critical moments in the process with respect to question Q2 and Q3. The aims were, on the one hand, to see how the prototype of the teaching and learning arrangement worked and to enhance the redesigned arrangement and, on the other hand, to develop the local theory regarding the learning process and its elements.

A typical outcome in terms of practicality was the observation that teachers and students had difficulties understanding the three drawings in one picture of Task IV. As a consequence for the redesign, we split a similar picture into three for the Design Experiment Cycle 4.
However, the main idea of the task served for the intended purpose very well, and, in this way, we could verify and deepen the local theory of teaching and learning different meanings of algebraic equivalence: most of the students could develop their conceptions of generality in parallel for variables and geometric figures, as the analysis of students’ products in the classroom showed.

On the base of this learning success, the learning pathway towards description and insertion equivalence was opened, as the following short scene from Maike and Jenny (grade 8) shows: having worked on tasks similar to Tasks III and IV in the classroom, Maike and Jenny work on a tasks similar to Task I in a design experiment in a laboratory setting. Maike reflects on how to solve the task.

8 Maike: But when we don’t have such values to calculate - then we don’t know if it’s equal - huh?
9 Jenny: She wanted to calculate that, if it’s equal [points to the expressions].
10 Maike: Yeah, but you can’t do without [without given numerical values].

Maike wanted to use the insertion equivalence, but switches to description equivalence (see start of this section for the meaning of these concepts) when realizing that she did not have numbers to insert. Her practices seem guided by her theorem-in-action <To evaluate expressions as correct, I can calculate them or describe their connection to the figure>. She activated and combined the concepts figure as general and variable as general.

These observations were in line with the hypothesized learning pathway of understanding generality in graphical and symbolic registers as a precondition towards understanding algebraic equivalence. This was an important part of the local theory of teaching and learning algebraic equivalence.

Apart from isolated optimizations (like drawing three pictures instead of all in one, see above), the prototype teaching and learning arrangement could now be finalized in the last cycles that are not treated here in detail (see table 1).

3. Concluding remarks

Although the case study can only sketch selected aspects of a comprehensive, several-year-long project, it might have shown that Didactical Design Research is an ambitious endeavor in which for each of the four working areas (see figure 1), substantial theoretical and methodological background is needed. The iterative interplay between the different working areas (see figure 1) enables the researcher-designers to intensively coordinate the different working areas and to concretize the general design principle for giving students access to conceptual understanding of mathematical topics; teaching-learning arrangements should generally start from meaningful (often out-of-school) contexts, provide adequate representations, and allow students to build adequate models of the mathematical topics.

Specific to didactical research is the perspective that the learning content is not considered as pre-given. Instead, its specification and re-structuring are also an important result. For this aim, the method of mediating between students’ perspectives and academic perspectives has again proven to be fruitful (Prediger, 2008; Duit, Gropengießer, & Kattmann, 2005). Here concretely, we became aware of the importance of also generalizing the geometric shapes, not only the numbers, for developing a profound conceptual understanding of algebraic equivalence.
On the other hand, it is the orientation at feasibility in normal classrooms that helps the designers not to overemphasize singular aspects resulting from isolated empirical projects. Although the generalization of geometric shapes is important, the learning arrangement needs to focus on many other aspects, too. For assuring ecological validity of the learning arrangements, it is crucial to triangulate the design experiments in laboratory settings by design experiments in realistic classroom settings. In this second step, experienced teachers are very important partners.

The next steps in our project will deepen the results in both directions: The design results will be implemented in a widely organized project of implementation and professional development (KOSIMA-project, see Hußmann, Leuders, Barzel, & Prediger, 2011); the research results will be deepened by another study that focuses on the transition from generational to transformational activities, the next step in the algebraic learning arrangement.

**Key sources**


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Susanne Prediger (1971) is a full professor in mathematics education at IEEM (Institute for Development and Research in Mathematics Education) of TU Dortmund University. She has led various projects in development, research and continuous professional development for mathematics education in secondary schools, resulting in many publications on the development of students’ mathematical conceptions. Currently, she is speaker of the leading team of the interdisciplinary graduate school FUNKEN on didactical design research for all subject didactics.

Email: prediger@math.uni-dortmund.de

Larissa Zwetzschler (1986) is a PhD-student in mathematics education at IEEM (Institute for Development and Research in Mathematics Education) of TU Dortmund University. She gets a scholarship of interdisciplinary graduate school FUNKEN on didactical design research for all subject didactics and is a member of the organization team. In her Phd thesis she is interested in the development of students’ conceptual understanding in a topic of algebra in a design research framework.

Email: Larissa.Zwetzschler@math.uni-dortmund.de