How to Develop Mathematics for Teaching and for Understanding
The Case of Meanings of the Equal Sign

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Abstract. What kind of mathematical knowledge do prospective teachers need for teaching and for understanding student thinking? And how can its construction be enhanced? This article contributes to the ongoing discussion on mathematics-for-teaching by investigating the case of understanding students' perspectives on equations and on meanings of equations and the equal sign. It is shown that diagnostic competence comprises didactically-sensitive mathematical knowledge, especially about different meanings of mathematical objects. The theoretical claims are substantiated by a report on a teacher-education course, which draws on the analysis of student thinking as an opportunity to construct didactically-sensitive mathematical knowledge-for-teaching for pre-service middle-school mathematics teachers.

This article adds to the ongoing process of defining and developing mathematics-for-teaching for the special situation of *listening, analysing and understanding student thinking*. The analysis is focused on the relevant, but only exemplary mathematical subject meanings of the equal sign. The article starts in Sections 1 and 2, presenting the background to the discussion on mathematics-for-teaching and diagnostic competence and continues in Section 3, by recalling the mathematical background of equal sign. Section 4, offers an empirical insight by investigating prospective teachers' analyses of episodes with students. Section 5, proposes an approach for teacher-education courses especially designed for enhancing prospective teachers' diagnostic competences.

"Listen to your students!" This claim is crucial for student-centred teaching agendas in mathematics classrooms. However, although there is general agreement on the need for more diagnostic competence for teachers, it is still a major concern to specify *what* kind of mathematical knowledge and pedagogical competences are needed for developing diagnostic competence and *how* they can be developed.

1. Defining and developing mathematics-for-teaching in general

1. 1 Background

It is widely accepted that a good mathematics teacher needs a deep and sound mathematical background (e.g. Shulman 1986, Cooney & Wiegel 2003). Nevertheless, there is still no consensus on the question of what kind of mathematical competences are needed and which are crucial for successful teaching in a student-centred paradigm (as underlying most projects that aim at change in mathematics education, like in the NCTM-Standards 2000).

> The development of a successful practice grounded in the principles that guide the current mathematics education reform effort requires a qualitatively different and significantly richer understanding of mathematics than most teachers currently possess. (Schifter 1998, p. 57)

The most frequently cited conceptualization of necessary teacher knowledge was given by Shulman (1986). He emphasized that it is not enough to acquire mathematical knowledge on the one hand and content-independent pedagogical knowledge on the other hand. He emphasized the need for linking these aspects by giving a framework for conceptualizing "content knowledge in teaching" in three central categories:

1. *Subject-matter knowledge*, which comprises facts and concepts of a domain, but also understanding of its structures grounds and warrants. Additionally, teachers should know why a given topic is central to a discipline (Shulman 1986, p. 9);
2. *Pedagogical-content knowledge*, which includes "the ways of representing and formulating the subject that make it comprehensible to others" and also includes typical difficulties and preconceptions of students of the most frequently-taught topics (Shulman 1986, p. 9);
3. *Curricular knowledge*, i.e. programmes and instructional materials and the "set of characteristics that serve as indications and contraindications for the use of particular curriculum or program materials in particular circumstances." (Shulman 1986, p.10)

Shulman (1986) underlined that there was too little attention paid to pedagogical content knowledge as the intermediate component.

Twelve years later, Shifter still complained of the lack of attention that was paid to the mathematical part of teacher education: "teachers' developing *mathematical* understandings and how those understandings affect instruction have not received sustained attention from researchers and teacher educators: What kinds of understandings are required of teachers working to enact the new pedagogy?" (Schifter 1998, p. 57) The title of this article was "Learning Mathematics for Teaching: From a Teachers' Seminar to the Classroom". This seems to be one of the first occurrences for the notion of *mathematics-for-teaching,* which was later used to signify two issues: first, it indicates an even stronger intersection between subject-matter knowledge and pedagogical content knowledge than in Shulman's conceptualization. Second, it emphasizes the special character of teacher subject-matter knowledge (see Davis & Simmt 2006).

Since 1998, there has been much effort to specify mathematics-for-teaching and to develop suitable teacher education programmes that use mathematics-for-teaching, with direct reference to the notion of mathematics-for-teaching (e.g. Cuoco 2001, Bass & Ball 2004, Davis & Simmt 2006, etc.) or without direct reference to it (Wittmann 2001, Hefendehl-Hebeker 2002b, Cooney & Wiegel 2003). The discussion is supported by various empirical studies, which show that teachers (especially primary teachers but also others) "often lack the kind of mathematical sophistication needed to support a reform-oriented agenda that goes beyond the acquisition of skills and procedural knowledge" (Cooney & Wiegel 2003, p. 802).

That is why the author of this article wants to contribute to the "ongoing project" (Davis & Simmt 2006) of identifying crucial elements of mathematics-for-teaching and of searching for pathways for its construction in in-service and pre-service teacher education.

1.2. A method for identifying mathematics-for-teaching

Bass and Ball (2004) proposed a fruitful practical-based method for identifying the necessary mathematical knowledge for teaching, the so-called "job analysis", in which they

> observe the mathematically implicated problems that a teacher regularly has to solve. Then we analyze the mathematical resources deployed (or potentially useful) for that problem solving, and the ways in which the teacher held and used these mathematical resources. … Following are some examples of core tasks:
>
> - setting and clarifying goals
> - evaluating a textbook's approach to a topic
> - selecting and designing a task
> - re-scaling tests
> - choosing and using representations
> - analyzing and evaluating student responses
> - analyzing and responding to student errors
> - managing productive discussions
> - figuring out what students are learning

> Our analysis of these tasks in mathematics teaching shows that they call upon substantial mathematics knowledge, skills, and sensibility. This sort of mathematical knowledge is often of a kind that is not typically taught in traditional mathematics courses. Thus our work has yielded new insights into the mathematics knowledge and resources for teaching. And it is knowledge of mathematics, not knowledge of pedagogy or of cognitive psychology. (Bass & Ball 2004, p. 296)

Bass and Ball follow their research programme of job analysis for the core task of *mathematical reasoning.* By analysing reasoning in classrooms, they specify the potentially useful mathematical competences for teachers. By doing so, they can show the richness of the demanded knowledge and competences. Their example makes clear that mathematics-for-teaching is not only the minor version of academic mathematics but a special variant with its own necessities.

In this article, the focus is on the core task of *analysing and understanding student responses and errors* in the classroom discourse. Whereas Bass and Ball (2004) concentrate on the first part of their programme, namely identifying important competences, this article deals with both parts: the analytical work of identifying, and the developmental work of constructing a sequence for teacher education, exemplified by a sequence in the course entitled *school algebra and its teaching and learning* for second-year, prospective middle-school teachers.

2. Analysing student thinking as an aim and opportunity to learn

2.1 The necessity of diagnostic competence or: why understanding students is so crucial

The importance for teachers to manage the core task of analysing and understanding student responses and errors, shall be shown by an episode (1) that happened to the author while teaching a grade 5 class (10-11 year old children) some time ago (here translated from German by the author):

Episode 1: Emily and the equal sign:

> In order to work on strategies for flexible mental arithmetic, students in grade 5 were asked to solve the following task:
>
> *Lisa calculates 24 x 7 by decomposing: 24 x 7 = 20 x 7 + 4 x 7 = 140 + 28 = 168.*
>
> *i) Did she calculate correctly? How would you have done it?*
> *ii) Calculate 54 x 6, like Lisa did.*
>
> Emily (ten year old) is sceptical: “Lisa calculates wrong, 24 times 7, does not equal 20! And what is that after the 20?” Due to her difficulties with the unfamiliar symbolic representation, Emily does not continue with the task although she usually uses the same strategy of decomposing 24 x 7 into 20 x 7 and 4 x 7.

It is not at all trivial to react in a didactically-sensitive and adequate way to students’ utterances like Emily’s. What is wrong with the equation in Emily’s view? Why doesn’t her view correspond to the official view? What could a teacher do in order to reconcile the individual and the mathematical view? Without answers to these questions, teachers are hardly able to help Emily to change her conception in a sustainable way.

In Germany, this issue is referred to as *diagnostic competence*. This notion (that, in English, might have some medical connotations) is used for conceptualizing a teacher’s competence to analyse and understand student thinking and learning processes without immediately assessing them.

Many authors have emphasized the importance of teachers’ ability to analyse and understand student thinking in these kinds of situations. This concept is linked to a student-centred teaching style, which is being discussed throughout the world and for which the NCTM Standards and Principles give only one example, among others: “Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (NCTM 2000, cited from the online-resource).

Hence, diagnostic competence is especially needed in two situations: first, the necessity to recognize student thinking as the individual starting point of learning processes, which is derived from the constructivist insight that learning always builds upon existing mental constructs and experiences. In this way, the claim for adaptive teaching is directly linked to diagnostic competence. Second, sustainable learning processes must be accompanied by sensitive teacher support. For example, teachers can only moderate students' discussions on different ideas if they first understand the students' ideas.

Despite these necessities, empirical studies show that there is insufficient development of teachers' diagnostic competences; German teachers, especially, appear to have enormous difficulties in properly analysing their students' thinking (Baumert et al. 2001). That is why the development of diagnostic competence has now become an important issue in German teacher education. More and more in-service and pre-service teacher-education programmes follow this important aim, to enhance teachers' competence to understand and analyse students' conceptions (Helmke et al. 2003, Wollring et al. 2003, Peter-Koop 2001).

Of course, this activity of understanding and analysing students' conceptions must not be understood in an epistemologically-naïve or positivist sense. Students' conceptions cannot be directly understood but must be reconstructed by interpreting their utterances. This basic idea underlies the whole article.

2.2 Constituents of diagnostic competence

In this section, four constituents of diagnostic competence are explained: 1. interest in student thinking, 2. interpretative attitude of understanding from an inner perspective, 3. general knowledge on learning processes, providing sensitizing constructs for a deeper analysis, and 4. domain-specific mathematical knowledge, especially about meanings.

1. Interest in student thinking

Diagnostic competence is based on an *interest in student thinking* as a necessary (yet not sufficient) precondition. Without being curious about student thinking, teachers cannot seriously take it into account.

If this interest in student thinking is only encouraged by simple studies of errors and misconceptions in teacher education classrooms, it risks stabilizing a deficit-oriented attitude towards students without being able to "look through children's eyes" (Selter 2001). Section 4 will illuminate this theory using some examples that will show that it is not sufficient.

2. Interpretative attitude of understanding from an inner perspective

Scholars, who use interpretative methods in their own research, have emphasized that the experience of interpretative analysis is valuable in developing diagnostic competence that emphasizes the need to understand thinking from an inner perspective (e.g. Jungwirth, Steinbring, Voigt, & Wollring 2001, Scherer & Steinbring 2006, Peter-Koop 2001). Systematic, joint reflection on small excerpts of classroom situations, clinical interviews or teaching experiments (mostly given in transcripts or videos) can substantially sensitize prospective teachers.

> Interpretative studies rely on detailed documentation of authentic teaching and learning situations for analysis and reflection. They aim to reveal what is hidden in practice … [e.g.] intuitive conceptions. … The participating teachers and student teachers can experience how scientific analyses may help them to enlighten those aspects of their classroom practice that they perceive as important. (Jungwirth et al. 2001, p. 49)

Temporarily adopting the role of a researcher is fruitful in order to get to know careful, in-depth analysis instead of forming hasty judgements. Most important is the *change of perspective* from deficit-oriented judgements, according to normative aspects ('What is wrong in the student's answer?'), to *understanding the inner rationality* of student thinking, even if it is divergent to conventional mathematics ('In which ways might the student's ideas make sense?', cf. Smith, diSessa, & Roschelle 1993).

3. General knowledge on learning processes

Beyond this interpretative attitude, theoretical *knowledge* is needed, as Helmke et al. emphasize:

> A description of persons (or their utterances) only becomes a diagnostic analysis when it is based on an *explicit theoretical fundament*, i.e. when based on a categorical system or sensitizing concepts. (Helmke et al. 2003, p. 19, translated by the author, italics added, see also Jungwirth et al. 2001)

This shows why research-based, professional teacher training must offer opportunities to learn the theoretical backgrounds that are needed for analysing and understanding student thinking.

Whereas the general necessity of theoretical backgrounds is, again, a shared conviction, authors have different viewpoints concerning the choice of theories that are useful for mathematics teachers' diagnostic competence (see Even & Tirosh 2002, for an insightful overview). Tsamir (2005), for example, proposes Fischbein's theory of intuitive knowledge (1987), Tall and Vinner's model of concept definition and concept image (1981), and most importantly, Tirosh and Stavy's intuitive rules theory (1999). These theoretical approaches are interesting and useful examples of pedagogical content knowledge (Shulman 1986), which is specific to mathematics, but not yet to specific domains in mathematics, since they offer domain-independent categories for conceptualizing mathematical thinking processes.

When considering an example of analysing prospective teachers (in Section 4), it will become evident that these domain-independent theories on (mathematical) learning processes must be complemented by domain-specific mathematical categories. Also, Tsamir (2005, p. 469) points out that, besides general theories for mathematics learning, specific mathematical knowledge, on the specific domain in view, is needed.

4. Domain-specific mathematical knowledge for teaching and analysing: focusing on meanings

Domain-specific mathematical knowledge is a crucial constituent that has to complement general theories on mathematical thinking and learning. For example, Tall and Vinner's (1981) distinction between concept image and concept definition helps to make clear that there can be a big difference between the conventional definition of a mathematical concept and the individual, partially idiosyncratic, concept images that students possess. But in order to analyse the difference for a given mathematical concept and a special student, it is helpful to know, not only the mathematical definition, but also the different meanings within the mathematical view and typical concept images. This knowledge is specific to each mathematical concept and is not yet acquired when one knows the general idea of concept definition and concept image. Similarly, Fischbein's (1987) mental models and implicit models are important sensitizing constructs to focus on local meanings of mathematical objects. For each concrete topic, the knowledge that different models exist, must be complemented by the knowledge of the *specific*, mental and intuitive models (see the example of Maja in Section 4, who knows the general idea of generating a cognitive conflict but has no means to find where the exact point is located).

These examples already give a hint of what mathematical knowledge is important for analysing student thinking. Since the differences between students' individual thinking and their intended mathematical thinking is often characterized by different meanings given to mathematical objects and operations, mathematics-for-teaching must comprise knowledge about different meanings.

Meaning is a widely used notion, understood in different ways (see Kilpatrick, Hoyles, & Skovsmose 2005 for a profound discussion on meanings). For the specific mathematical content chosen for this article, the equal sign or equality, there has been a long tradition of reflecting and researching on meaning, which was described as a process of broadening the sources of meaning by Kieran (2006). In this article, the notion of meaning is restricted to local interpretations of mathematical objects (see next section).

Here, meaning is emphasized as an important part of mathematics-for-teaching, which resulted from the job-analysis while examining the requirements for diagnostic competence. Davis and Simmt (2006) also emphasized the importance of meaning and presented a way of making it explicit in in-service teacher training. In line with Bass and Ball (2004), they started from the insight

> that, whereas the work of research mathematicians might be described in terms of 'compressing' information into increasingly concise and powerful formulations, the work of teachers is more often just the opposite: teachers must be adept at prying apart concepts, making sense of the analogies, metaphors, images, and logical constructs that give shape to a mathematical construct. (Davis & Simmt 2006, p. 300f)

When this kind of knowledge is conceptualised as being crucial for mathematics-for-teaching and understanding, the border between subject-matter knowledge and pedagogical-content knowledge, as specified by Shulman (1986), is transcended in favour of an *integrated understanding of mathematical knowledge for teaching*. Hefendehl-Hebeker (2002b) underlined the importance of such an integrated understanding and called it a *didactically-sensitive understanding of mathematics*.

The investigation of prospective teachers' analysis in Section 4, will illustrate the importance of these four constituents for diagnostic competence. Especially, the empirical job-analysis will affirm the insight into the importance of *meanings* when the job entails understanding and analysing student thinking.

3. Mathematical background: different meanings of variable and equality

This section presents the necessary domain-specific knowledge for teachers in their core task of understanding and analysing student thinking in algebra classrooms where the equal sign is in view. It has often been emphasized that learning algebra is influenced by the *ambiguity of meanings* of signs (Freudenthal 1983, p. 474). The following clarifications on the meanings of variables and the equal sign (or more holistically, equality) will be set out in order to make explicit the mathematical basis of the case study and, at the same time, the curricular basis of the teacher education course, presented in Section 5.

3.1. Meanings of variables

Malle (1993) distinguished five interpretations (see also Küchemann 1981 and Drijvers 2003 who suggested a similar distinction): the variable

- (I) as a *place holder* in the so-called substitution aspect,
- (II a & b) as an *unknown* or a *generalized magnitude* in the so-called situational aspect
- (III) as a *meaningless symbol*, which can be manipulated according to given rules in the so-called calculation aspect
- (IV) as a *changing quantity* in the algebra of functions.

The difference between these interpretations of variables can be best explained by considering its impact on the different interpretations of the equivalence of two terms like $(x-2)(x+3) = x^2+x-6$ and $(a-b)(a+b)=a^2-b^2$, which applies generally to all x, a and b.

If, in the calculation aspect, the variables are considered to be *meaningless symbols*, terms are also meaningless expressions and equivalent terms are those which can be transformed into each other according to the transformation rules. For example, $(a-b)(a+b)$ is transformed into a^2-b^2 by applying the distributive law, twice. If, in the situational aspect, the variables are considered to *signify unknowns or generalizable magnitudes*, terms serve as descriptions for specific situations, like geometric constellations. Then, equivalent terms are those which describe the same situation. For example, terms can describe the same figure in different ways. When, in the substitution aspect, the variables are considered to be *place holders* into which numerical values can be inserted, equivalent terms are those which generate the same value for all inserted numbers (this perspective is adopted, for example, while using spread sheets, cf. Drijvers 2003). When the variables are interpreted as *changing quantities* in the algebra of functions, the equivalence of the terms inform us that the functions $f_1(x) = (x-2)(x+3)$ and $f_2(x) = x^2+x-6$ describe the same functional dependency.

3.2. Meanings of equality and the equal sign

Many researchers have emphasized the important distinction between two meanings for equality, namely the *operational* and the *relational meaning* (Kieran 1981, Winter 1982, Wolters 1991, Filloy, Rojano & Soares 2003, Theis 2005, Knuth et al. 2006):

> That the equal sign is a ‘do something signal’ is a thread which seems to run through the interpretation of equality sentences throughout elementary school, high school, and even college. Early elementary school children … view the equal sign as a symbol which separates a problem and its answer. (Kieran 1981, p. 324)

Primary school arithmetic often privileges the operational meaning of the equal sign by exclusively focusing on asymmetrically-considered equations like 24:6-3 = 1. As a consequence, the later relational and symmetric use of the equal sign in middle-school algebra poses problems to students, e.g. in expressions like 24 x 7 = 20 x 7 + 4 x 7 or algebraic equations like $x^2 = -x+ 6$.

This distinction gives a clue to the understanding of Emily’s difficulties in Episode 1 of Section 2.1. Emily is not aware of the possibility to interpret the equal sign in a relational way. She only seems to have constructed the operational meaning of the equal sign in her primary school experience. This can be concluded from her idiosyncratic reading rule “Read equations only until the first sign after the equal sign”, which might have been strengthened by the missing brackets in the term. However it is perfectly consistent with the operational interpretation of the equal sign. This seems to fit with the equal sign’s use in her primary school classes. In contrast, German secondary school classes allow the use of more than one equal sign in one line. Only the change of meanings, from the operational to the relational, justifies the conventional reading rule, “Read the complete terms on both sides of an equation” (Kieran 1981, p. 323f), which Emily doesn’t seem to know.

In reaction to the earlier, empirical results on difficulties with the development of a relational meaning in the transition from arithmetic to algebra, many researchers in mathematics education claimed to include a symmetric and relational use of the equal sign in primary school curricula (Kieran 1981, Winter 1982, Wolters 1991, Theis 2005). Nevertheless, usual classroom practices still seem to emphasize only the operational meaning, even in arithmetic contexts of middle-school textbooks, as has recently been shown by McNeill et al. (2006). Since algebra learning processes are influenced by the fact that the relational meaning, in itself, has different interpretations (e.g. Malle 1993, Cortes et al. 1990), the distinction is further elaborated here, into six categories (see Figure 1).

The first category, *operational meaning*, stands for the mentioned asymmetric use: “operation equals answer” (McNeill et al. 2006), which is often described for elementary arithmetic. It also applies, in higher mathematics like in calculus when the derivative of a polynomial function is calculated and then notated by $f'(x) = (3x^2)' = 6x$.

The second category, *relational meaning*, focuses on a symmetric use of the equal sign. It has four subcategories, which substantially differ from each other. From Kieran (1981), the author's categorization borrowed the term *arithmetic identities* (subcategory 2a). Kieran invented it for the communication with students in order to explicitly distinguish symmetric arithmetic equalities from those involving variables. The symmetric use of the equal sign in arithmetic contexts can help to express general relations (like commutativity in 3+5 = 5+3), but also numerical identities which are trivial to calculate in one direction, but difficult to figure out in the other direction, for example: whereas $10^2-9^2=19$ is easy to calculate, decent number theoretical knowledge is needed to find a representation of 19 as a difference of two square numbers.

The *formal equivalence* (subcategory 2b in Figure 1) refers to the equivalence of algebraic terms involving variables like in $(x-2)(x+3) = x^2+x-6$ and $(a-b)(a+b)=a^2-b^2$, being satisfied for all x, a and b. As explained in section 3.1, this formal equivalence can be interpreted in different ways, depending on the interpretation of the variables involved.

It is important for algebra classrooms to distinguish formal equivalence with its different interpretations from *conditional equations characterizing unknowns* (subcategory 2c in Figure 1). Although the equation $x^2 = - x + 6$ is symbolically near to the former $(x-3)(x+2) = x^2+x-6$, it has a completely different character, since it does not apply to all x. Instead, it characterizes specific unknowns, and one can (by solving the equation) even figure out which unknowns they are.

The subcategory 2d, *contextual identities in formulae,* is challenging since, on the one hand, formulae like the volume formula of the cone or the equation of the Pythagorean theorems $a^2+b^2=c^2$, are general statements, but on the other hand, they do not apply to all a, b like in formal equivalences as: $(a-b)(a+b) = a^2-b^2$. They are only general in specific contexts, e.g. binding a, b, c to lengths in a right triangle and r, h, V to measures of a cone.

Cortes et al. (1990, p. 28) suggested introducing a third category, *specification*, in which identities are not described but stated, like in definitions. The difference to, for example, a contextual identity, is an epistemological one: it helps to distinguish between definitions and propositions.

In sum, there is a very rich diversity of interpretations of the equal sign. Siebel emphasizes the importance of this diversity by including it in her global characterization of elementary algebra:

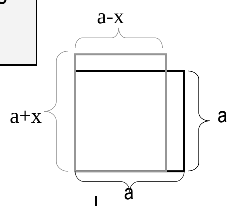
> Elementary algebra is the theory of calculating with general numbers which permits good descriptions of quantifiable coherences. … numbers and variables are combined to units of thinking and connected by different conceptions of equality. (Siebel 2005, p. 73, translated by the author)

Different meanings for equality

1. Operational meaning: operation equals answer, e.g.
   "24:6-3 = 1" or "$f'(x) = (3x^2)' = 6x$"
2. Relational meaning:
   - 2a. symmetric arithmetic identity, e.g. "5+7 = 7+5" or "$19 = 10^2 - 9^2$"
   - 2b. formal equivalence describing equivalent terms, e.g. "$x^2+x-6 = (x-2)(x+3)$", "$(a-b)(a+b)=a^2-b^2$"
   - 2c. conditional equation characterizing unknowns, e.g. "Solve $x^2=-x+6$"
   - 2d. contextual identities in formulae, e.g. volume formula for a cone: "$V = \frac{1}{3}\pi \cdot r^2 \cdot h$" or "Right triangles with hypotenuse c and legs a, b satisfy $a^2+b^2=c^2$"
3. Specification, e.g. "m: = ½ (a+b)" or "y: = 2x + 52"

Figure 1: Categorization of meanings for equality

Isoperimetric Problem:
Show that among all rectangles with equal perimeter, the square has the largest area.



Solution with derivatives:

			Interpretation of the variables	Interpretation of the equal sign
(1) A square with length a of its sides has the perimeter P=4a. Each rectangle with P=4a has the length a+x and a-x for its sides.			generalizable magnitudes	contextual identities
(2) Then the area satisfies	A	$= (a-x)\,(a+x)$	generalizable magnitudes	contextual identity
(3)		$= a^2 - x^2$	meaningless symbol	formal equivalence
(4) Consider the function	$A(x)$	$= a^2 - x^2$.	changing quantity	specification
(5) Calculate the derivatives	$A'(x)$	$= -2x$	meaningless symbol	operational sign
(6)	$A''(x)$	$= -2$.	meaningless symbol	operational sign
(7) Every local maximum satisfies	$A'(x_E)$	$= 0$ and $A''(x_E) < 0$.	generalizable magnitudes	contextual identity
(8) Let be	$A'(x_E)$	$= -2\,x_E = 0$	Unknown	conditional equation for unknowns
(9) hence, we have in a potential extremum.	x_E	$= 0$.	unknown found	operational sign
(10)	$A''(0)$	$= -2$	substitution	operational sign
(11) Thus, A has a local maximum at $x_E=0$. Hence, the square is the rectangle with maximal area.			Interpretation for the original problem.	

Figure 2: Changes in the interpretation of the variables and the equal sign – the example of the isoperimetric problem solved by derivatives

Many researchers emphasize the context specificity of meanings and the long-term shift in meaning, i.e. the conceptual change in the transition from arithmetic to algebra (e.g. Kieran 1981, Winter 1982, Cortes et al. 1990 and others): whereas elementary arithmetic focuses on the operational meaning; middle-school algebra needs different relational meanings and specifications. Successful college students are able to switch between different meanings according to the necessities of the context

This perspective is very important and could explain many difficulties, which students encounter. Nevertheless, it neglects the important fact that changes of meanings not only appear in the long-term perspective (or by switching from arithmetic to algebraic contexts) but also, even *within* one problem.

Similar to the ambiguous change of meanings of variables (see Freudenthal 1983, p. 474 or Malle 1993), the change of meaning for the equal sign not only represents an obstacle in the learning process but also represents an important characteristic and strength of mathematical problem-solving. The importance of these changes, within one problem, can be exemplified by the justification of the isoperimetric problem for rectangles using derivatives, as shown in Figure 2.

The statement P=4a in line 1 is a typical contextual identity, which is generally valid, but only when P is interpreted as a perimeter of a square with side a, hence a and P are generalizable magnitudes. The same holds for A = (a-x)(a+x) in line 2. In line 3, the variables change their meaning from a generalizable magnitude to a meaningless symbol, which is part of a term that can be transformed by rules and guarantees the formal equivalence of terms. Line 4 is a typical obstacle for students: by considering A(x) as a function of the variable x (and parameter a), x and a change their character: whereas a is still considered to be a generalizable magnitude, x is now considered to be a changing quantity and examined, in its functional dependency, to A(x). The equal sign in line 4 specifies this new function, which shall be treated hereafter by methods of calculus. For generating lines 5 and 6, the x and a in line 4 are again considered to be meaningless symbols that can be transformed by derivation rules. Hence, the equal signs in lines 5 and 6 are operational signs, indicating the operation "derive A and A' ". In line 7, a general proposition is stated, which holds whenever x_E is the extremum of a derivable function (hence it serves as a generalizable magnitude), the equal sign here serves to formulate a contextual identity. Since this x_E is to be found, it is considered as an unknown in line 8, and the proposition from line 7 is used as a conditional equation in order to find out the unknown. In line 9, the equation $x_E = 0$ signifies: x_E was to be found, now the problem is solved, it is zero. Hence, the unknown (the potential extremum) is now found, and the equal sign serves here as an operational sign. Testing the potential extremum in line 10 by inserting into A''(x) is the last step in which x_E is interpreted to be a place holder (substitution). The analysis is condensed in the two right columns in Figure 2. The columns show that in nearly every line, the interpretation of the variable, or the interpretation of the equal sign, changes.

This flexibility of meaning is a typical strength of mathematics, since it is exactly this ambiguity of meaning that allows us to describe a *geometric* problem *algebraically* and solve it by methods of *calculus*.

On the other hand, Hefendehl-Hebeker (2002b) has shown how such implicit ambiguities of meaning can provide obstacles for understanding when they are not made explicit. Hence, it is crucial for teachers to be able to communicate explicitly on differences between meanings whenever the situation demands (as in Episode 1 and 2, see Section 5.1).

4. Empirical snapshots of differences in prospective teachers' diagnostic competences

In this section, the importance of the mathematical component of diagnostic competence is underlined (see Section 2.2) by giving some empirically-based insights into remarkable differences in prospective teachers' diagnostic competence that can be discerned before their systematic training.

The data presented here is taken from the beginning of a course entitled *school algebra and its teaching and learning* for second-year, prospective, middle-school teachers. At the time of data collection, the 45 participants had already worked on different meanings of variables and had had some interpretative experiences, but not yet dealt with meanings of the equal sign.

The participants were shortly confronted with Episode 1 concerning Emily and the equal sign (see Section 2.1) and each one was asked to answer the following questions in a short written form:

a. What does Emily mean?
b. In which view is she right?
c. How would you answer her if you were her teacher?

Prospective teachers' answers (translated by the author)	1. Interest for student thinking	2. Interpretative attitude of understanding from an inner perspective	3. General knowledge on learning processes	4. Mathematical knowledge concerning meaning
Irene "a. Emily … only considers Lisa's assumed result directly behind the equal sign. ... The equivalence arrow would have been more appropriate here, then Emily could have understood the calculation more easily. b. … With respect to the way of writing, Emily is right. But Lisa's calculation is nevertheless correct, when you remember the distributive laws. *[no answer to c.]*"	shown by following Emily's thinking	might be existent, but cannot be activated due to mathematical mistakes	not activated here	equivalence arrow mathematically wrong. Lisa's writing is not false!
Morton "a. Perhaps, Emily has not seen that Lisa has decomposed 24 x 7. Probably, she does not understand the decomposition of the calculation. b. 24 x 7 ≠ 20. How, in which view, is she right? c. Re-ask her to explain again what she means; only after that would I answer. Without that, I do not know. But perhaps she already solves her problem on her own during re-explaining."	shown by following Emily's thinking	not able to interpret her utterance but activates useful, everyday, diagnostic strategy: re-ask her and hope that verbalization might help her to solve problem on her own	not activated here	not activated (considers only calculation, not meaning)
Maja "a. Emily only considers the first number after the equal sign, hence the equation 24 x 7 = 20. For her, x 7 + 4 x 7 is a term that does not belong to the equation. b. Emily is right when considering the equation 24 x 7 = 20 (as she sees it). c. If Emily knows what the terms are, I would ask her to read the terms and to explain her ideas. I would generate a cognitive conflict, in the hope that she can realize her mistake on her own without my help."	shown by following Emily's thinking	existent	activates 'cognitive conflict' as a useful general psychological construct	teaching strategy of cognitive conflict is not yet filled with mathematical substance – which conflict to generate?
Joanne "c. Dear Emily, your answer is logical, in itself, and legitimate, since the number 20 comes directly after the equal sign and appears to be the result of the calculation. But you should consider both sides completely and refer to them completely. Then, not only 20 is the result, but all signs which came before the next equal sign."	shown by following Emily's thinking	general interpretative attitude and careful analysis	uses knowledge on student-oriented teacher-feedback (start positively)	only focused on the technical use of the equal sign, not on its meaning

Figure 3: Investigating prospective teachers' analyses of Episode 1

All the written answers were collected, qualitatively analysed and classified according to the manifested diagnostic competence. For this analysis, the framework of four components, presented in Section 2.2, gave the guiding normative orientation. To illustrate the results of this analysis, Figure 3 shows four prototypical answers and outlines a short summary of the resulting evaluation.

Before analysing the details of the evaluation, a possible objection shall be anticipated: Of course, the teacher educator's evaluation of prospective teachers' interpretations on student thinking must be guided by the awareness that in many classroom situations, like Episode 1, more than one interpretation for a student's thinking is possible and appropriate. But this insight does not justify *unfruitful relativism*, which says that any interpretation is acceptable. In contrast, the identification of misleading or superficial interpretations (often given by beginners) is necessary in order to help them develop their diagnostic competence.

The prototypical answers in Figure 3 show the importance of all components of diagnostic competence for tackling the core tasks of understanding and analysing student thinking, with the aim of enhancing students' understanding.

All prospective teachers' answers reflect an interest in Emily's ideas, and the author's general impression of the course suggests that this interest is not only driven by the demands of the task. In all other components, the answers vary.

Irene (who has not specialized in mathematics) reproduces Emily's misconception by saying that Lisa's way of writing 24 x 7 = 20 x 7 + 4 x 7 = 140 + 28 = 168 is wrong (see Figure 3). Although Irene's writing might allow for other interpretations; from discussions with her during her coursework it is clear that her own suggestion of using equivalence arrows (which usually apply to equations, not to terms), actually reproduced a mathematical misconception and she was not able to clarify the mathematical situation. This mathematically-problematical answer is prototypical of one group of prospective teachers in the course who could not develop a didactically-based diagnostic competence due to their own deficits in mathematical knowledge. At worst, Irene might even irritate Emily more than help her.

Morton is the prototype for a group of students who do not try to understand the child's perspective. His first tentative analysis of Emily's thinking focuses on decomposition and explicitly contradicts information given later in the Episode, namely that Emily was able to make the same decomposition.

So far, Morton has not learned to activate any theoretical knowledge for understanding, neither with respect to learning processes nor with mathematical backgrounds. But he reacts to this deficit with a well-developed inquiring attitude, which is connected to a sometimes useful practical strategy: ask her again and hope that she might even solve her problems alone while verbalizing them. Although Morton does not yet show a highly developed diagnostic competence, he might eventually succeed in a classroom situation, at least to establish the correct reading rules for equal signs.

In contrast, *Maja* analyses Emily's idiosyncratic reading rule in a mathematically-correct way and with a certain ability to enter into Emily's thinking. Beyond that, she proposes a domain-independent strategy for affiliating conceptual change, namely the generation of cognitive conflicts. Without judging whether this strategy is suitable here, the teacher educator can infer that Maja already has a certain capability for reacting to situations. However, she reaches her limits when a more concrete localization of the problem is necessary: which cognitive conflict to generate and how? To specify this, it would be necessary to leave the syntactical level and to activate domain-specific knowledge on typical difficulties and divergent interpretations of the equal sign. Maja represents another prototype, which again underlines the importance of the fourth component.

Also *Joanne* is able to enter partially into Emily's thinking. She demonstrates some knowledge of the importance of positive teacher feedback, but since she only focuses on the technical use of the equal sign, instead of on its meaning, she cannot yet find an appropriate resolution.

Both prototypical examples, Joanne and Maja, show that mathematical knowledge of the meaning of the equal sign cannot be developed by simply analysing examples from other topics (which is what these prospective teachers have already done). A general, interpretative attitude must be complemented by domain-specific mathematical knowledge on the concrete topic, in this case, using knowledge of different meanings of the equal sign.

What is missing in these four answers given in Figure 3 becomes apparent when they are compared to Sam's answer, which was formulated forty-five minutes later in the course, after the different categories of meanings had been introduced:

> Emily interprets the equal sign as 'makes' and not as a sign that can have several meanings like connecting terms.

Sam's very short answer shows that these categories can be a clue to the situation: in contrast to the other four answers, Sam was able to find the exact point at which the further discussion with Emily should start.

Although Sam's answer might suggest it, the presentation of mathematical categories of meaning alone is not enough to enhance the diagnostic competence of most prospective teachers. According to the author's experience, only a minority of prospective teachers are able to directly and successfully activate the acquired theoretical constructs for the analysis of student productions. For most prospective teachers, who are not used to applying scientific concepts to the description of real phenomena, learning will need more support (see also Leufer & Prediger 2007). The following section presents how this support might be given, during a sequence of a teaching course.

5. Acquisition of mathematics-for-understanding in teacher education courses

For the ongoing project of defining and developing mathematics-for-teaching and understanding, this article goes beyond the analytical work of *identifying* necessary constituents of diagnostic competence. It starts the developmental work of *constructing* sequences for teacher education. The following sections give examples of how to affiliate this competence (Section 5.1) and draw on some general strategies (Section 5.2).

5.1. An example for a sequence on meanings of the equal sign in a school algebra course

How do we develop mathematics for teaching and for understanding? Before trying to formulate some general strategies, this section gives an example of the concrete sequence of learning, which was used in the mentioned course entitled *school algebra and its teaching and learning* for second-year, prospective, middle-school teachers.

The first activity in the sequence was the writing of a spontaneous, individual analysis of Episode 1 (cited and investigated in Section 2.1 and 4, respectively). This activity was followed by a discussion of different suggestions that should provide the opportunity to "experience the limits of an intuitive analyse of problematic situations" (Hefendehl-Hebeker 2002a, p. 58, translated by the author) since this can, according to Hefendehl-Hebeker, "motivate the development of more precise scientific constructs" (ibid). In order to generate this experience of limits, the prospective teachers were asked to evaluate the answers of their colleagues by collectively speculating on how Emily would have reacted to their interventions. This was not just speculative because the author of this article, being not only the professor of the teacher-education course but also Emily's teacher in grade 5, could report on her own different attempts to answer to Emily in the real situation. In this way, suggestions that were based on a too superficial diagnosis were rejected by the participants and the teacher educator. During the course, the individual need for deeper analysis evolved.

In the second activity, the learning situation was enriched by another episode, video graphed in a grade eleven calculus course (sixteen year old students). Please note that according to the used transcription rules, the sign … is used for a short break of less than 3 seconds

Episode 2: Hanna and the transitivity of the equal sign

Hanna, sixteen years old, stands at the blackboard and explains how they found the functional equation f_y (x) = 10 x + 110 for a given realistic linear situation.

Hanna: Yes, we have done it in such a way. First, the points, I mean the distance, I mean, 1 times 10; the second, for example, is 2 times 10. Makes 20 and then plus 110, because
[writes on the black board 1 x 10 = 10 + 110 = 120
2 x 10 = 20 + 110 = 130]

Teacher: And how have you found the equation?

Hanna: Well, the equation is: f_y of x equals 10 x + 110. *[writes $f_y (x) = 10·x + 110$]* Yes, because you have to … 10 times and then plus 110...

Teacher: Perhaps just a little moment for this here. *[points at Hanna's first equal sign]* Mathematicians are a bit nitpicking with the equal sign. Here you have 1 times 10 and then it equals 120… that is…

Hanna: Why?

Teacher: Or did you mean a colon instead of the equal sign?

Hanna: No, that is an equal sign.

Teacher: That equals 20, but then you **must** not write 'equals 110' behind it. I will put this part into brackets, because when you use the equal sign, you really have to have the same on the left and on the right side. But your idea is clear. It is just, that mathematicians always get nervous when there is not the same on the left and on the right side…

Hanna: But I am not a mathematician...

Teacher: But this also has advantages because… *[4 sec silence]*
Well, Vera and Sarah have another way to come to the equation – can you please explain it, Vera?

In this episode, the conflict between the operational and the relational interpretation of the equal sign becomes more explicit than in Episode 1, although the teacher in Episode 2 stops her explanation about why the so-called chain notation is not allowed.

The prospective teachers in the school algebra course were asked to discuss the episode in smaller groups based on the following questions: Can we help the teacher in her argumentation? Why is it important that there is the same value on both sides of the equal sign? What is Hanna's conception of the equal sign? For which context is it important that the value on both sides of the equal sign is the same?

In their discussion of Episode 2, most of the prospective teachers realized that Hanna referred to an operational interpretation of the sign, to which many students refer while producing chain notations and using the well-known strategy of "keeping a running total" (Kieran 1981 and cf. Section 3.2). In this way, the prospective teachers discovered the existence of different interpretations of the equal sign (operational versus relational), which so far, they had only known implicitly. They realized that it was the transitivity of relationally-interpreted equations that made it necessary to forbid the chain notation.

But Hanna is a challenging case insofar as she insists on her perspective by saying "But I am not a mathematician." Can we allow Hanna to keep her perspective when she does not want to adopt a relational interpretation? The prospective teachers started an intense discussion in which the urge for better arguments in favour of the relational interpretation emerged.

In the beginning, most participants insisted that the conventional notation rule must also be compulsory for Hanna. However, some objected: "But when I make auxiliary calculations and feel unobserved, I also prefer the chain notation. It is much easier to handle! It is a pity that it is forbidden, just because of the transitivity." The following suggestion was then made: "Why can't we allow different notations in different contexts? Why don't we allow chain notation in arithmetic contexts, and forbid them in algebraic contexts?"

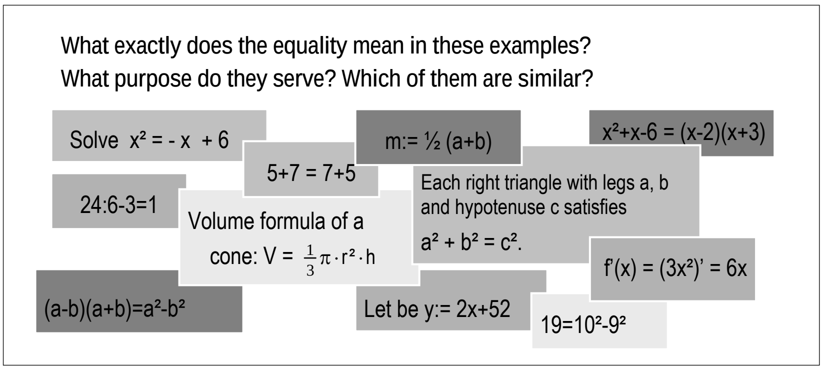


Figure 4: Independent work for systematizing meanings

It was at this moment that it became important to embed the discussion in a wider context. In the third activity, the participants were asked to work on the task in Figure 4, providing a carefully-chosen pool of examples for different uses of the equal sign. In this way, they were supposed to genetically become aware of different interpretations of the equal sign by considering similarities and differences in the examples. For this process, the verbalization of equations very much helped (see Figure 5).

Afterwards, in order to consolidate the systematization, the ^small group’s attempts to systematize the different meanings, were collected. In this whole-group interaction, the partially vague ideas on differences between the examples could be sharpened. This was crucial for enhancing the participants’ depth of understanding.

Since many of the prospective teachers were not aware that the interpretation of the equal sign can change due to the purpose of the equation within a single problem, the fourth activity in the course was a collective analysis of the changes of meaning for the equal sign, using the example of the isoperimetric problem solved by derivatives (see Figure 2). This analysis was challenging for the participants since it demanded a deep mathematical understanding.

During this activity, the participants again became aware of the interdependence of their mathematical understanding with their diagnostic competence. One said in the end: “So far, I have been able to use the calculus of differentiation for solving extremum problems, since this worked in a mechanical way; but when I was asked about the meaning of the procedural steps, it really became difficult. In all honesty, I did not know it.”

By the end of this sequence (that took about 140 minutes), the systematization of different meanings of the equal sign was summarized in an overview (Figure 5) and was noted, together with the examples and the verbalizations of different meanings.

Although there was no isolated evaluation for this teaching sequence, the assessment at the end of the course gives at least some hints about its success. Forty-one of the forty-five participants took part in the assessment of the course. The participants could choose four out of five tasks, one of which was to analyse a multifaceted situation, similar to the one given in Figure 2. This difficult task was chosen by twenty-seven of the forty-one prospective teachers, out of which only three were not able to analyse the situation adequately. All the others showed that they had acquired a domain-specific diagnostic competence, based upon mathematical knowledge of different meanings of the equal sign, which is crucial for the core task of analysing and understanding student thinking in this topic.

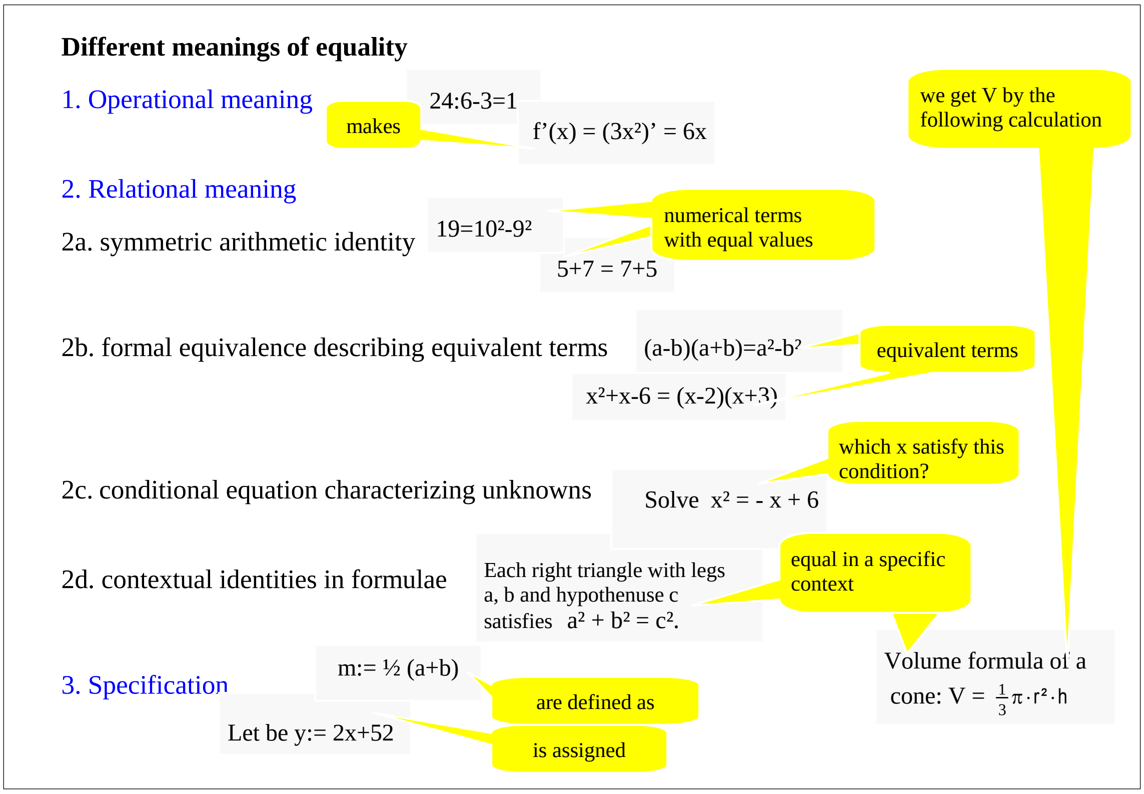


Figure 5: Systematization on the white board at the end of the sequence

5.2. Strategies for teaching meanings in general

Davis and Simmt (2006) reported on a sequence in an in-service, teacher-education course in which the participants were asked to work out different meanings of multiplication. In their course, the long experience of some teachers and the heterogeneity of the group were important resources; as a result, there was nearly no need for input from the teacher, they only made explicit their common, implicit knowledge about meanings of multiplication and hence collectively constructed a productive learning situation.

In the teaching sequence described here, the author also built upon the resource of implicit knowledge, which did not have to be constructed anew, only made explicit. This was done during the third activity. However, since prospective teachers in the second-year of education do not have such multifaceted, implicit knowledge (as experienced teachers in the in-service training of Davis and Simmt), the situation was enriched by giving examples of equations that had to be interpreted. Comparing and contrasting these well-chosen examples and asking the prospective teachers to verbalize them, proved a good strategy to make their knowledge explicit.

Furthermore, Weinert (2000) has emphasized that diagnostic competence is not only a bundle of knowledge items but a capability, which should be *action-guiding* with respect to goal-oriented, student-teacher-interaction in classrooms. Due to diagnostic competence being embedded in a teacher's classroom activities, the teaching sequence provided opportunities for action-guided diagnostic necessities instead of the isolated learning of mathematical categories of meaning. It is for this reason that the first and second activities developed for this approach were based on authentic, critical classroom situations and that the prospective teachers were asked to develop many possible responses.

That is why the first and second activity for the prospective teachers started from authentic critical classroom situations and asked them to develop possible ways of acting.

To summarize, the described sequence for teaching meanings to prospective teachers was guided by the following strategies:

- acquire necessary mathematical knowledge in diagnostic situations
- let prospective teachers experience the limits of missing diagnostic tools
- build upon prospective teachers' implicit knowledge by providing opportunities to make it explicit
- genetically develop mathematical categories while analysing a set of examples.

6. Conclusion: analysing student thinking needs mathematics and develops mathematics

"Listen to your students!" The author hopes to have given some ideas on how to approach this generally-formulated claim by contributing to the ongoing process of defining and developing mathematics-for-teaching, in this case, in relation to the job of *listening, analysing* and *understanding* student thinking.

The job-analysis showed that diagnostic competence has at least four constituents, one of which is domain-specific, mathematical knowledge on typical difficulties with and different meanings of mathematical concepts. One important conclusion is that teacher courses, which aim at diagnostic competence for mathematics classrooms, cannot be held by general educators alone, since they need a mathematical focus. The chosen topic, the equal sign, is only one example among many others, which are crucial for classrooms that aim to facilitate understanding instead of pure routines (e.g. Prediger 2008, for the case of multiplication of fractions).

The analysis of student thinking not only needs mathematics but it is also an interesting task that helps to develop mathematical knowledge. In this point, this article follows Shifter (1998) who promoted the examination of student thinking as an avenue to develop teachers' mathematical knowledge. In sum, the task of "analysing student thinking" can, at the same time, be an interesting opportunity to learn for pre-service mathematics education courses.

Teacher educators who subscribe to this aim should be aware that this is not in any way a trivial task, for two reasons. First, dealing with a variety of meanings is unfamiliar for many prospective teachers, and it needs a mathematical sovereignty, which is not naturally given to all candidates. These affective and cognitive challenges can only be overcome by intensive work in well-organised programmes.

Second, this intensive work is restricted by the usual time limits in mathematics-teacher-education courses. It is obvious that we cannot work on all relevant topics in the intensive way as presented in this article; although this would be desirable. As time is limited, the curricula must be restricted to some well selected core objects, like variables and the equal sign. Beyond that, we can only hope that the exemplary approach enables prospective teachers to continue their life-long learning process for all other important topics. Choosing these core examples and preparing them in the most transferable way is a major issue for the ongoing 'project' in mathematics for teaching,

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