"…BECAUSE 'OF' IS ALWAYS MINUS…" - STUDENTS EXPLAINING THEIR CHOICE OF OPERATIONS IN MULTIPLICATIVE WORD PROBLEMS WITH FRACTIONS

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The correct choice of operations is well known to be an obstacle for students when solving word problems. The presented study contributes to the discussion on possible explanations by investigating explicitly given reasons for choices in a written test with 269 German grammar school students. It shows that no uni-dimensional account can be given for the multi-faceted phenomenon of choice of operation.

Various empirical studies have documented difficulties in students' performance with word problems. Different obstacles were specified for a successful mathematization of word problems, some of them concerning the external appearance of word problems like length and readability of texts, others concerning their internal structure, like familiarity of contexts, necessary choice of operations, number type, didactic contracts for questions of validation etc (cf. Verschaffel et al., 2000, for an overview). Among all these important aspects, the choice of operations and their background gained a special attention for word problems with non-natural numbers. Many researchers studied the choice of operations for one-step multiplicative word problems with two decimal numbers (e.g. Fischbein et al., 1985; Bell et al, 1981; Bell et al., 1989; Harel et al., 1994). This study builds upon them, but extends them by

1. a focus on fractions rather than on decimals (demanded by Harel et al., 1994),
2. an enriched test design, including various models of multiplication and items for other layers of competence, and
3. a deeper analysis of reasons for choices given by the students themselves. By this, we attempt to enlarge usual quantitative research designs.

1. EXISTING RESULTS AND THEORETICAL EXPLANATIONS FOR DIFFICULTIES WITH THE CHOICE OF OPERATIONS

Solving word problems is only one aspect in a multi-faceted landscape of competences that are to be developed for fractions. Following Fischbein et al. (1985), this landscape can be structured in a *multi-level model for competence with fractions* (elaborated especially on the intuitive level in Prediger, 2008):

- *Formal Level,* including the definitions of concepts and of operations, structures, and theorems relevant to a specific content domain; formally represented by axioms, definitions, theorems and their proofs,
- *Algorithmic Level,* comprising procedural skills - here of multiplying - and the capability to explain the successive steps of the standard procedures,

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- *Intuitive Level*, characterized as the type of mostly implicit knowledge that is often accepted directly and confidently as being obvious, including different layers:
  - *mathematizing competence*, i.e. ability to translate word problems into terms,
  - *intuitive rules*, i.e. individual conceptions about existing laws and coherences,
  - *individual interpretations of operations*, and
  - *individual interpretations of numbers* (decimals, fractions etc.).

Individual interpretations of operations and numbers have been conceptualized as (mental) models (Fischbein et al., 1985; Greer, 1994; Usiskin, 2008) or 'Grundvorstellungen' (GVs, see vom Hofe et al., 2006; Prediger, 2008). They constitute the meanings of mathematical concepts based on familiar contexts.

In order to give more precise accounts for students' deficits in their mathematizing competences for multiplicative word problems, various researchers investigated into students' choice of operations. A robust finding is that number types involved in the problem statement strongly affect the difficulty of the mathematization process, the so-called multiplier effect: For multiplicative problems with an integer multiplier, the correct choice of operation is easier than for decimal multipliers > 1, and those are easier than for problems with multiplier < 1, from which one knows that the result must be smaller than the factors (e.g. Bell et al., 1981; Bell et al., 1989).

Basically, two different theoretical accounts have been given for the multiplier effect: First, Bell et al. (1981) emphasized the importance of the intuitive rule "multiplication makes bigger" (here shortly called the 'order property') and its generalization from natural to fractional numbers as the main obstacle for choosing multiplication for word problems with multiplier < 1 (cf. Bell et al., 1981; vom Hofe et al., 2006). Fischbein et al. (1985) gave empirical evidence for an explanation situating the difficulty one layer underneath: They emphasized that the pertinacity of the *intuitive rule* "multiplication makes bigger" is often connected with the continuing maintenance of the *interpretation* of multiplication in the repeated addition model (which does not work for decimal or fractional multipliers). Both accounts can be integrated, as shown in Prediger (2008), since the intuitive rule is often based upon uncompleted conceptual changes on the layer of interpretations of multiplication (similarly Greer, 1994). That means, that those students who have widened their repertoire of interpretations for multiplication (and hence mastered the discontinuity of the repeated addition model) can also change their intuitive rules concerning the order property of multiplication.

Although later studies started to widen the structure of situations in view (e.g. Bell et al., 1989, considered not only repeated addition models, but also prices, speed and currency-conversion), the great variety of other individual models for the multiplication of fractions and naturals are still to be explored more systematically.

2. CENTRAL METHODOLOGICAL IDEA OF THE PRESENT STUDY

In order to specify aspects in students' thinking and in word problems that influence students' choice of operations, three main research strategies have been adopted so far:

1. Studying effects of factors by comparing difficulties when systematically varying operation-choice test items (e.g. Bell et al., 1989; de Corte / Verschaffel, 1996).
2. Searching for statistical coherences in a written test, covering different layers of competence (e.g. vom Hofe et al., 2006; Bell et al., 1981; Prediger, 2008).
3. Qualitative in-depth analysis by clinical interviews (for example Bell et al., 1981 in their first phase, Wartha, 2007).

Research strategies 1 and 2 can only give statistical coherences (by comparing, in contingency tables or with correlations), but no account for causal connections. That is why some quantitative studies have been complemented by qualitative in-depth studies in clinical interviews, but they only allow small numbers of participants.

4. Qualitative deeper analysis of written answers
   This study tries to combine advantages of qualitative and quantitative strategies by applying an intermediate strategy: We conducted a written test with open items, coded answers in an explorative procedure and quantified frequencies of constructed codes afterwards. This generated insights beyond statistical coherences.

3. RESEARCH DESIGN: CORE ITEMS, PARTICIPANTS, DATA ANALYSIS

The empirical material presented here was part of a study conducted by a written test with twelve test items (see Prediger / Matull, 2008). This paper focuses on two core items (the other ten items that are shortly characterized in Table 1):

Item 7 a.) One kilogram tangerine costs 1.50 €. Kate wants to buy 3/4 kg. How can she calculate the price?
☐ 1,5 – 3/4 ☐ 1,5 : 3/4 ☐ 3/41,5 ☐ none of these, but this:
b.) Give reasons for your answer given in a)

Item 9 a.) How can you calculate 2/3 of 36? ☐ 36-2/3 ☐ 36 : 2/3 ☐ 2/3 · 36 ☐ none of these, but this:
b.) Give reasons for your answer given in a.)

Item 7 and 9 follow the choice of operation methodology (cf. Fischbein et al., 1985, Bell et al., 1989), in which students are asked to give or choose a term without calculating the answer. Crucial for research strategy 4 is the added part b.), asking for *reasons of choices* in an open item format.

Item 7 asks for a mathematization in a situation acting across quantities, which is (according to Usiskin, 2008), especially difficult for students. Item 9 has the same structure, but refers to a situation in which the multiplication is used for taking a part of a whole number, one of the most important models for the multiplication of fractions.

The paper and pencil test was written by 269 students in five Grade 7 classes (age about 12 years) and five Grade 9 classes (age about 14 years) in German grammar

schools which comprise the (assumingly) higher achieving 40 % of students.

The students' answers were evaluated quantitatively in a points rationing scheme. The values were used for statistical investigations on correlative coherences and contingencies of performances for different items.

For a deeper exploration, the self-constructed word problems to Item 5 and 6 and the reasons given for operation choice in Item 7 and 9 were analysed qualitatively by coding the manifested individual conceptions and strategies. Whereas a coding scheme for Item 5 and 6 pre-existed (from Prediger, 2008 with an interrater agreement of Cohen's kappa 0.93), the coding scheme for the reasons in Items 7 and 9 first had to be constructed from the data. In an explorative coding procedure, categories were built by comparing answers due to their similarity. Some could be anticipated by the existing literature (like the pertinacity of the order property "multiplication makes bigger and division makes smaller", see Bell et al., 1981), but other interesting, unforeseen codes (see Table 2, e.g. restructure strategy) had to be constructed in the process. Once finally established, the coding scheme of Item 7 and 9 achieved an interrater agreement of Cohen's kappa 0.83.

4. RESULTS

4.1. Statistical results

Table 1 gives an overview on the scores and frequencies of complete solutions for all test items. Item 7 and 9 are among the most difficult, with only 0.7 % complete solutions for Item 7 and 4 % for Item 9 and average scores of 14 % and 12 %, resp. The distributions of performances in Item 7a and 9a are compared to other operation choice items in Figure 1.

Item	Content	Frequency of complete solutions	Average of reached scores	
1	Multiply fractions (technically)	62 %	3,15 of 4	79 %
12	Confirm commutativity of multiplication	54 %	1,55 of 2	77 %
5	Find a word problem for a given equation with addition	54 %	1,39 of 2	70 %
4	Explain the meaning of a given fraction	49 %	1,37 of 2	68 %
3	Identify a multiplication in rectangle picture	67 %	1,33 of 2	67 %
8	Mathematize situation allowing repeated addition (natural multiplier)	20 %	1,21 of 2	60 %
2	Order property (multiplication makes bigger?)	34 %	0,77 of 2	38 %
10	Specify part of a fraction and mathematize	1 %	1,24 of 5	25 %
11	Mathematize a situation of scaling down	1 %	0,30 of 2	15 %
9	Mathematize situation with part of whole number (2/3 of 36)	4 %	0,27 of 2	14 %
7	Mathematize situation acting across quantities (kg x €)	0,7 %	0,24 of 2	12 %
6	Find a word problem for a given equation with multiplication	3 %	0,11 of 2	6 %

Table 1: Scores of items, ordered due to average scores

The far best results were reached for Item 8a, in which 91% of the students activated the well-known model of repeated addition and chose the multiplicative terms 2/10 · 15 or 15 · 2/10 (More precisely, 38 % chose one, 53 % both.). In contrast, in all other items, no more than 93 of 269 students chose the multiplicative term (< 35 %, the guess probability was 33% for Item 8 and 9).

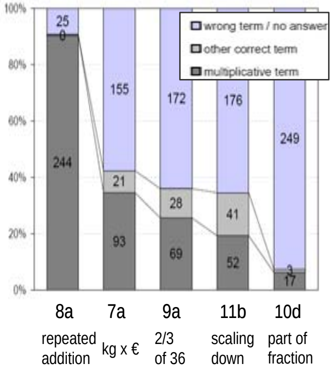

Figure 1: Comparison of chosen terms for Items 7-11

Reasons for these operation choices can first be studied by statistical methods. Chi-squared tests for independence in the contingency tables of Item 7 and 2 (or Item 9 and 2, resp.) gave evidence for an association between unsuccessful operation choice and wrong intuitive rules about multiplication making bigger: The null hypothesis of independence of outcomes in Item 2 and 7 could be rejected with a chi-square of 9.65, being highly significant ($p < 0.008$). For Item 2 and 9, a chi-square of 42.34 allowed to reject the null hypotheses of independences with $p <0.001$. In contrast, the contingency tables between Item 7 and 3, and Item 9 and 3 showed no significant dependence of item outcomes. By these results, we could confirm classical findings on the importance of the intuitive rule "multiplication makes bigger", shortly said the order property of multiplication.

Nevertheless, contingency tables and chi-squared tests cannot account for *causal connections* between layers of competence. That is why the analysis was deepened by coding the reasons given by the students.

4.2. Deeper analysis of reasons

The answers in Item 7b and 9b were coded according to the reasons given for the choices of operations. As we were especially interested in those answers that gave access to the choosing strategy behind the given reason, we filtered all answers that did not allow the interpreters any access to the strategy, as for example "Because you have to take this." (Kim).

Table 2 gives a quantitative overview on the answers that were filtered or coded according to the reconstructable choosing strategies behind the given reasons. The explorative coding process ended with a categorization of codes into three main categories: order strategies, restructuring strategies and keyword strategies, by which most of the reconstructable answers could be captured (79 %). The categories shall be explained by examples in the following.

In Item 9, 69 of 269 students chose correctly a *multiplicative term* for mathematizing 2/3 of 36. Only in 5 of the reasons given, the researchers could identify points that allowed any access to their thinking. One of the students activated an order strategy, i.e. she successfully made use of her intuitive rule: "For fractions and multiplication,

	Item 7				Item 9				
	Mult	Div	Sub	Oth	Mult	Div	Sub	Oth	
Chosen Term	93	81	36	59	69	120	19	61	Overall part of reconstructable strategies
Answers allowing no access to strategy for interpreter	90	72	29	55	63	73	11	51	
Reconstructable strategies behind the formulated reasons:									
Order Strategy: multiplication makes …	1	7	1	0	1	2	0	0	12 / 95 = 13 %
Keyword Strategy: of-tasks are… tasks	0	0	0	0	4	25	3	0	34 / 95 = 35 %
Restructure Strategy Use other parts	0	3	6	0	0	17	5	0	28 / 95 = 29 %
Others	2	0	0	3	0	5	0	10	20 / 95 = 21%
Guessing Strategy	0	0	0	0	0	0	0	0	0 / 0 = 0 %

Table 2: Operation choice and frequencies of reconstructable strategies of choice

you always get less." (Liza) (Although Liza's rule is not of sufficient generality, it worked here).

Five of the students referred to the keyword "of". For them, this seems to be a rule guiding their operation choice, for example, "because of is the same as times" (Eddie) or "of-tasks are times-tasks" (Sam). We subsumed these answered under the so-called keyword-strategy.

Whereas Liza, Sam and Eddie drew upon their strategies successfully, order strategy and keyword strategy were more often used for choosing the wrong operation *division*: Among all 120 wrongly chosen divisions, 49 were justified in a way that allows access to the underlying strategies and conceptions. 25 of these 49 belonged to misleading order strategies: "When you multiply, it becomes more, when you subtract, it also becomes less, but just wrong." That is why Karen chose division.

17 other participants restructured the situation in an idiosyncratic way, by mixing wholes and parts like Paul, writing "because you need a part". Anna's answer was also categorized as restructure strategy, as she tried to solve the following task: "You must calculate, how often 2/3 fit into the 36 for getting the new fraction."

The rate of restructure strategies was even higher for *subtraction*: 19 students wrongly chose subtraction for Item 9, and among the 8 reasons that gave access to their thinking, there were 5 with restructure strategies. Most of them explained their constructed term 36 – 1/3 while referring to false referent wholes, like Ali "When you want to have e.g. 2/3, then 1/3 is left! They are subtracted (in this case) from 36."

The strategy that was most often reconstructable for the operation choice in Item 9 was the keyword strategy: 4 choices of multiplication were (correctly) explained by this (see above), but even more (25) choices of division, like in Terry's answer: "Well I want to know how much is 2/3 <u>of</u> 36, not 36-2/3 or 2/3 ·36, though, division." With the same argument, 3 students explained their choice of subtraction by the keyword strategy, like Eve: "2/3 <u>of</u> 36, thus minus", another one already cited in the title.

Two of the three main strategies reconstructed for Item 9 also reappeared in Item 7: Whereas the keyword strategy was not that important here, restructure and order strategies also appeared in the reasons given for Item 7 (see Table 2).

In conclusion, we resume that order strategies were not as important as suggested by Bell et al. (1981) for our sample. Only 12 of 96 interpretable answers showed order strategies (13 %). In contrast, keyword strategies (35 %) and restructure strategies (29 %) were found in significantly more cases. This relativizes other findings.

Guessing Strategies have not been formulated by any student of this sample (of higher streamed students). This is very different in another sample of 561 lower streamed students, in which 14% of the answers to the same two items were "I have guessed" (see Prediger / Matull, 2008). Apparently, higher streamed students know that this answer is not accepted in mathematics classrooms for higher achieving students, so even if they guessed, they did not write it.

5. DISCUSSION

The findings of this study affirm the thesis (formulated in Prediger, 2008) that difficulties on one layer of competence (here the mathematizing competence, operationalized as choice of operations) cannot be explained uni-dimensionally because they might be located on different layers of competence. The reconstructed categories of choosing strategies have their roots on different layers:

- The *order strategy* comprises all reasons given with reference to a sustainable or non sustainable intuitive rule on the order property of multiplication (making bigger or smaller). But although being privileged in existing studies, like Bell et al., 1981, it could only be reconstructed in 13 % of the accessible cases in this study.
- In contrast, the restructure strategy (that was reconstructable in 29 % of all accessible cases) does refer to misconceptions on the layer of interpretations of fractions, a layer that has not yet had sufficient attention in empirical studies.
- The guessing strategy (not appearing in the grammar school sample) and the keyword strategy (reconstructable in 35 % of all accessible cases) are interpretable as (sometimes misleading) schemes on the layer of mathematizing competence itself. The use of a keyword strategy alone cannot be taken as empirical evidence for deficits on layers underneath, but it shows that the students did not activate a deeper layer of interpretations in this situation. But additional cross-references in the raw data table showed that none of the 25 students who activated a keyword strategy for choosing division in Item 9 had been able to find a correct word problem for a given multiplication in Item 6. This indicates that wrong keyword strategies might often be connected to missing interpretations for the multiplication of fractions.

To sum up, the study could show some connections between layers of competence that can account for wrong operation-choices better than pure correlative results. This

gives hints for new research strategies in this field.

Obviously, the short paper must leave many questions unanswered, especially more precise connections between the reconstructed choosing strategies and the answers to other items. In further research, we plan to investigate a larger sample of students of all achievement levels.

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