# "THREE EIGHTHS OF WHICH WHOLE?" - DEALING WITH CHANGING REFERENT WHOLES AS A KEY TO THE PART-OF-PART-MODEL FOR THE MULTIPLICATION OF FRACTIONS

Susanne Prediger & Andrea Schink – IEEM Dortmund, Germany

*One important meaning of the multiplication of fractions is the part-of-part-model, by which 4/5 x 2/3 is interpreted as 4/5 of 2/3. Students' understanding of this model is often constrained by the difficulty of changing referent wholes. The paper presents first investigations of a learning arrangement that was designed in order to deal with this obstacle and to increase students' awareness about changing referent wholes by associating different representations. The qualitative analysis of prospective teachers' products and processes gives insights into individual constructions of meanings and terms for part-of-part-situations.*

## THEORETICAL BACKGROUND AND EXISTING FINDINGS

### Theoretical background: Grundvorstellungen in different representations

The (individual and normative) *meaning of operations* can be conceptualized in different ways. This paper draws upon the notion of *Grundvorstellungen*, shortly *GV* (vom Hofe et al., 2006), being the cognitive building blocks for interpreting and mathematizing in processes of modelling (see Figure 1). We take this notion nearly synonymously to mental models in Fischbein's sense as a "meaningful interpretation of a phenomenon or concept" (Fischbein, 1989, p. 12). While mathematizing, GVs are activated to find models of a situation; while interpreting, GVs provide models for the formal mathematical expression.

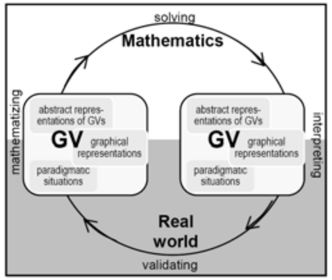
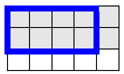



Fig. 1: GVs as translation - tools in modelling processes

GVs are not only represented by their abstract form, like by saying "multiplication of fractions can be interpreted by the part-of-part-interpretation", but also by paradigmatic situations or graphical representations. For example, the meaning of 4/5 x 2/3 as 4/5 of 2/3 is more accessible in a picture or a context: Jim has 2/3 of a pizza left from lunch. For dinner, he eats 4/5 of the rest of the pizza. So, he eats 8/15 *of the original pizza*.



Our research illustrates how individual processes of constructing GVs can be enhanced by intermodal transfer, i.e. by associating different representations for GVs. By this we follow Gerster & Schultz (2004) and others who conceptualize understanding of operations as a successful interplay between different modes of representations.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, C. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 409-416. Thessaloniki, Greece: PME.
(Slightly corrected version)



### Empirical starting point: Changing referent wholes as obstacle to understand the part-of-part-model for multiplication

Although the already mentioned part-of-part-interpretation is one of the most important mental models for the multiplication of fractions, empirical studies show limited success in students' acquisition of this GV (vom Hofe et al., 2006; Prediger, 2008).

Previous findings point at one important obstacle for students to construct the part-of-part-model: the change of referent wholes (e.g. Mack, 2000). In Schink (2008), this is exemplified by a student who approached parts of parts through paper-folding (original paper complete and zoomed in Figure 2). Having successfully folded 1/8 of 1/5, he correctly obtained 40 rectangles in his paper. The obstacle is manifest in his notation "1/8" for 8 rectangles (in the left 1/5-stripe of Figure 2). Realizing that the partner wrote 1/40, he corrected one of them into 1/40.

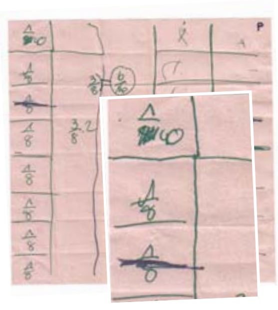


Fig. 2: $\frac{1}{8}$ of $\frac{1}{5}$ is $\frac{1}{8}$ or $\frac{1}{40}$?

The problem with changing referent wholes appears for interpretations in all representations: Whereas the 2/3 in the pizza-situation refers to one original whole pizza, the second factor, 4/5, refers only to the rest of the pizza, thus it has another referent whole. But the result 8/15 again refers to the original pizza, i.e. the whole one. Hence, constructing meaning for the part-of-part-model necessitates a clear orientation on the question "What is the whole?" (cf. Mack, 2000).

As mentioned by many researchers, problems with changing referent wholes or units also appear in non-multiplicative contexts, and not only for fractions (e.g. in Harel/Confrey, 1994). Well known is that when students formulate word problems for given additions, one of the most typical mistakes is to join parts of two different referent wholes (Prediger, 2008).

### Design of a learning arrangement built upon associating representations

Starting from these empirical findings on typical difficulties, a learning arrangement was designed that allows students to develop or enlarge their individual GVs of addition and multiplication of fractions and to gain awareness for different referent wholes and the question "What is the whole for this fraction?".

The so-called Excursion-Problem shown in Figure 3 is one part of this learning arrangement that demands the construction of terms for single-step and two-step additive and multiplicative situations. It builds upon the interplay of different representations by giving three texts of (paradigmatic) situations and three pictures which have to be associated. The case study presented here investigates how this arrangement supports the construction of adequate GVs and terms.

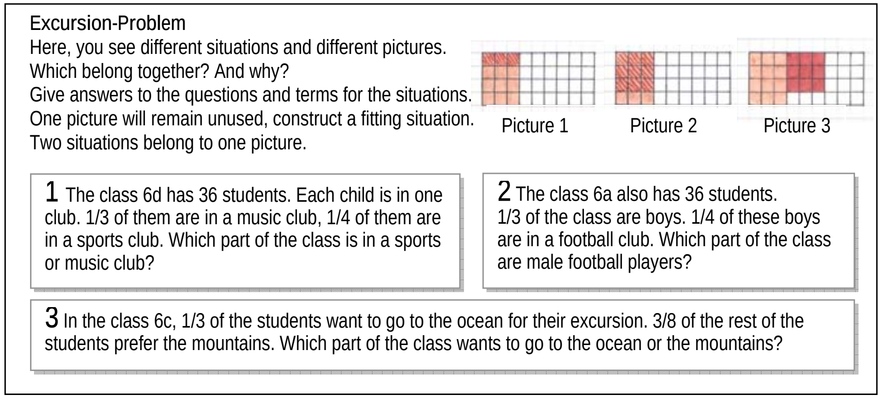



Fig. 3: “Excursion-Problem”, designed for enhancing students’ GVs

## RESEARCH QUESTIONS AND DESIGN

As a first step in an ongoing design research project, this study investigates the didactical potential of the above problem in a preservice teacher education course for prospective (mostly middle-school) teachers in their 2nd or 3rd year. Their working processes in groups of 2-3 were observed, partly video-taped and transcribed. 66 written answers were collected that document the products. The data analysis of products and processes followed four research questions:

(1) How do the participants relate situations, pictures and terms? Which terms do they construct especially for the more complex Situation 3?

(2) How do the participants deal with changing referent wholes, and in how far do they gain an increasing awareness during the process?

(3) How does the requested activity of associating representations (situation, pictures, terms) influence the process of constructing or choosing models and terms?

(4) Which obstacles hinder participants in their construction processes?

Research question (1) and (2) were addressed for all participants by analyzing and categorizing the 66 written documents (see quantitative overview in the next section). Research question (3) and (4) were in the core of a more detailed case study on Laura and Paul, two prospective middle-school teachers. The transcript and the video of their process were analysed qualitatively turn by turn, then coded and categorized in a procedure of open coding. Selected dimensions and results of the analysis are presented here.





## ANALYSIS AND DISCUSSION OF PROCESSES AND PRODUCTS

### Variety of constructed terms - Quantitative overview on products

Nearly all participants could assign Situation 1 and 2 to suitable pictures. The describing additive (1/3 + 1/4) and multiplicative terms (1/3 x 1/4 or 1/4 x 1/3) were successfully constructed in 45 of 66 (and 42, resp.) documents. Hence, the big majority of participants could activate a part-of-part-model in this setting (although only 22 % of them chose a part-of-part-situation in an earlier problem when asked to invent a situation for a given multiplication of fractions).

As anticipated, Situation 3 was more challenging for the participants, since it demands a complex combination of GVs to construct the term 1/3 + 3/8 x 2/3. In sum, 23 different terms (not including multi-step calculations) were constructed, with varying appropriateness for describing the situation (see Figure 4). Only 15+4 documents gave a complete term description for the group of ocean- and mountain-fans, in fractions or absolute numbers.

The categorization also took revised terms into account. As far as they appeared in the written documents (e.g. scratched out or later corrected), they allow interesting insights into the processes of conjecture and refutation. The fact that 12 participants notated wrong terms but revised them afterwards, gives a first evidence that the designed Excursion-Problem offers the desired potential to affiliate the intended processes of developing GVs. Only 5 documents ended with terms in which wrong operations were chosen or fractions of different referent wholes were combined. However, these 5 and 11 more documents without terms emphasize the importance of the issue.

Most documents (7+11+9+3+1=31) contain partly adequate solutions which refer only to a subgroup (e.g. the subgroup of mountain-fans) or describe the situation in a

| Categories: appropriateness of term | Subcategories: What and how does term describe? (abs. num. = absolute number) | Examples for written terms | Frequency of occurrences of this type of term as final results (in brackets: frequency of revised terms) |
|---|---|---|---|
| Adequate terms | group as part | “1/3+2/3x3/8” or “1/3+3/8x(1-1/3)” | 15 (1) |
| Nearly adequate terms | group in abs. num. | “1/3x36+3/8x24” or “1/3x36 + (36-1/3x36)x3/8“ | 4 (0) |
| Partly adequate terms or calculations | subgroup (as parts) in more steps or one single term | e.g. “2/3x3/8” or “(36-36/3)x3/8 [(36-36/3)x3/8]/36=1/4” | 7 (1) |
| | subgroup (in abs. num.) in more steps or one single term | e.g. “36x2/3x3/8” or “36:3 36-24 24x3/8” | 11 (0) |
| | group (as part) in more steps | e.g. “1-1/3 2/3x3/8 1/3+9/36” | 9 (0) |
| | group (in abs. num.) in more steps | e.g. “1/3x36=12 36-12=24 3/8x24=9 21” | 3 (0) |
| | Others | “1/3+1/4” [fraction taken from picture] | 1 (2) |
| Wrong or No terms | no term or fractions | no written term or all scratched out | 6 |
| | only fractions or verbal descriptions | “21/36”; “1/3 of the whole (=12 children)+3/8 of the rest (=9 children)” | 5 (1) |
| | wrong term | e.g. “1/3x3/8”, “1/3+3/8” or “1/4x1/3” | 5 (12) |

Fig. 4: Overview on terms, constructed for Situation 3 in 66 documents

multi-step calculation instead of a single term. Hence, most participants were apparently able to give meaning to the situation itself and to realize the importance of changing referent wholes, but were nevertheless not able to give a complete term description. The underlying reasons cannot be reconstructed by an analysis of products alone; a case study gives more insights into the process and its obstacles.

**A long search for a term – The case of Laura's and Paul's process**

Laura and Paul, two prospective middle-school teachers, intensively worked on the Excursion-Problem for 25 minutes. A detailed analysis of their interesting case provided insights into patterns and obstacles of an unfinished process (cf. research question (3) & (4)). Due to place restrictions, only the key results can be sketched which were drawn from the qualitative coding procedure (shortly documented in Figure 5).

Within 6 minutes, Laura and Paul successfully assign Picture 1 and 3 to Situation 2 and 1, resp., and find the terms 1/3 x 1/4 and 1/3 + 1/4. Unlike many colleagues, Laura immediately activates the part-of-part-model for multiplication.

Their work on Situation 3 starts by drawing individual pictures (based upon the assumption that 3/8 refers to the whole class, not to the non-ocean-group). When they hear that they should only use existing pictures, they restart by re-reading the text:

104 L …oh, 3/8 of the **rest of the students** prefer…

Once having realized that the remaining group of non-ocean-fans was meant (abbreviated by "nog" in Figure 5), Laura immediately associates Picture 3. So, 8 minutes after having started with Situation 3, the right picture is found, and they recognize the difference between the class and the non-ocean-group as referent wholes for the first time. The remaining 11 minutes are dedicated to the search of a term.

110 L *[writes down* "1/3 x 3/8, because you take 1/3 of the class and then 3/8 of the rest". *She types on the calculator, apparently receives 3/24, later she scratches out the equation]*
112 L *[looks on her picture]* No, these are 9/36. Thus 1/4.
114 L So 1/4 times 1/3 *[writes* "1/4 x 1/3 = 1/12", *types on the calculator, scratches it out]*

Although she is aware that 1/3 and 3/8 have different referent wholes (evidenced by her verbal formulation in line 110), she combines the fractions unconventionally, apparently because "3/8 of the rest" signals multiplication, even if this rest is not described by 1/3. The calculator serves her (here and later) as important tool for falsifying constructed terms.

As the fractions extracted from the text do not work, Laura controls their meaning in the picture. Keeping the multiplication, she exchanges 3/8 by 1/4 in her next term, because she extracts 9/36 by counting from the picture and by syntactically reducing 9/36 = 1/4. Although all fractions in this second term refer to the class, the calculator does again not produce the desired results.

Paul studies the text intensively and recognizes the abstract structure "join of parts" that is to be mathematized by an addition (119). Adopting this insight (125), Laura





| Transcript Line and Actor | Intermediate terms (w=written, s= only spoken, p=only pointed at already written terms, c= apparently entered into calculator) | Referent whole for each number (nog=remaining non-ocean-group, abs.n.= absolute number) | Activated GV for chosen operation | Associated Representation (most often: number taken from… t = text, p = picture, r= already obtained result, tpr = t or p or r) | Prompt for dropping the term (f calc = falsified by calculator, p=control of result in the picture, T/P= communication with teacher /Paul) |
|---|---|---|---|---|---|
| 110 L | w "1/3 x 3/8" | class x nog | x part of ? | t x t | f calc, p |
| 114 L | wsc "1/4 x 1/3" | class x class | x part of part | p x tpr | f calc |
| 119 P | s take together | ? | + join parts | t | (not dropped) |
| 125-128 L | ws "1/3 + 3/8" | class x nog | + join parts | tpr + t | f calc, p |
| 152 L | s "3/8 of 24" | rest of abs.n. | (no term) | t pr | does not help |
| 152 L | ws "4/12 + 3/12" | class + class | + join parts | pr + pr | no fit to text |
| 154 L | w "36 – 1/3" | abs. n. – class | - not reconstructable | r – tpr | f calc |
| 159 L | p "4/12 + 3/12" | class + class | + join parts | pr + pr | no fit to text, T |
| 169-181 L | ws "1/3 + 3/8" | class + nog | + join parts | tpr + t | L: f calc<br>P: diff ref wholes,T |
| 184 L | c "1/3 x 3/8" (?) | class x nog | x part of part | tpr x t | f calc |
| 194 L | s "3/8 ref. to 36" | | (no term) | tp r | does not help, T |
| 198-200 L | wsc "24 x 3/8" | abs. n. x nog | x part of whole | pr x t | (not dropped), T |
| 202 L | s "36 ÷ 9" | abs. n. ÷ abs. n. | 1/9 as part | r | (not dropped) ,T |
| 205 L | ws "1/3 + 1/4" | class + class | + join parts | r + tpr | no ideal fit to text, used as one step |
| 208 L | s "1/3 ref. to 36" | | (no term) | r | P |
| 211 L | s "24 x 3/8" | abs. n. x nog | x part of whole | r | P |
| 211 L | s "1/3 + 1/4" | class + class | + join parts | r | (not dropped) P |
| 212 P | w "3/8 x 24" | nog x abs.n. | x part of whole | r | (not dropped) |
| 213 L | w "1/3 + 1/4 = 7/12= 21/36<br>↗ 3/8 of 24,<br>referred to the whole" | class + class<br>abs.n. | + join parts<br>x part of whole | r | final expression |

Fig. 5: Chronology of Laura's & Paul's search for a term for Situation 3

varies the term into 1/3 + 3/8. In this variation, she keeps the numbers from the text and changes the operation (128). Again, she falsifies the result 17/24 given by the calculator by counting in the picture.

128 - 130 L *[writes* "1/3 + 3/8", *enters it into the calculator, writes* "=8/24 + 9/24 = 17/24", *counts in the picture] [laughs]* I still cannot calculate

Paul extracts 3/12 from the picture as a fraction that describes the part of the mountain-group when referred to the class. His attempt to associate this fraction also to the text of Situation 3 raises a new question: How to receive the 3/12 by a term involving 3/8? Paul knows in principle what to do, but cannot mathematize it:

151 P Well, we should first calculate what these 3/8 *[3 sec. break]* are of the whole
152 L ...3/8 of 24 are 3/12 because there are 36 – err, 24 – no 9/24 *[scratches out the "1/3 = 4/12" and "3/8=9/24"]* I cannot!
Anyway, the result is *[writes at the same time]* 1 err 4/12 + 3/12 = 7/12, isn't it?

Although already having distinguished the class and the non-ocean-group as different referent wholes for 1/3 and 3/8, the relation between both is still not completely clear. Line 152 shows the conceptual difficulty in a nutshell: within one utterance, Laura refers the mountain-group (described by 3/8) to the non-ocean-group (described by the absolute number 24), then to the whole class (described by the absolute number 36), and then (with the 9/24) to the non-ocean-group (this time conceptualized as 1). In contrast, they never describe the non-ocean-group as 2/3 or as 24/36, which would allow to conceptualize the non-ocean-group as part of the class (as 1), so that the referent wholes could be nested in each other. This constraint hinders them to relate 3/8 to 1/4 by mathematizing 3/8 x 24/36 = 1/4.

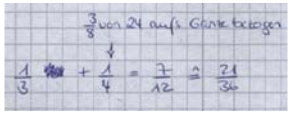
Although the term 1/3 + 1/4 at least gives the right result 21/36, they consider it not to be sufficient, because they cannot relate it to the text:

| | | |
|---|---|---|
| 162 | P | Well, from the picture, we can justify it *[he means the term* "3/12 + 4/12"*]*, but we do not find a term that associates it more directly |
| 163 | L | Because this is not direct, it is only read off *[meant is from the picture]* |

As they do not succeed, Laura comes back to 1/3 + 3/8, although the calculator falsified it, and Paul explicitly states that it is not appropriate:

| | | |
|---|---|---|
| 181 | P | But you cannot add 1/3 and 3/8, because the 3/8 do not refer to the whole picture here |

After trying other terms and discussing with the teacher (see Figure 5 for a complete chronology), the teacher signals that the time is over. Laura is finally satisfied with a verbal description of the connection as printed here:




Translation: "3/8 of 24, referred to the whole"

**Discussion: First answers to the research questions in the case of Laura & Paul**

(1) Like most of the participants, Laura and Paul quickly relate situations and pictures. Unlike many of their colleagues, both immediately mathematize Situation 2 by a multiplicative term. We conclude that they are familiar with the part-of-part-model for multiplication and also the join-model for addition. But these building blocks alone do not enable them to combine them in one complete term for the more complex Situation 3; so they end with an intermediate result that associates 3/8 of 24 to 1/4 only verbally.

(2) Whereas Paul is quickly aware of different referent wholes involved in Situation 3, Laura first confuses them. Associating representations helps her to gain awareness for the difference between the whole class and the non-ocean-group, but she continues to mix parts and absolute numbers. Finally, she gains awareness for the referents.

(3) Although Laura seems to know abstract representations of elementary GVs ("of is multiplication"), she cannot activate them for mathematizing the more complex situation. In this challenging constellation, the interplay of a *greater variety* of representations for GVs in pictures and paradigmatic situations is a big help for Laura and Paul. By linking representations, they construct the meaning of the context and gain awareness of different referents. Each interpretation or construction of a symbolic element can always be validated in another representation. However, representations offer benefits *and* difficulties: The picture also suggests a misleading absolute view





instead of focussing on parts; the calculator is an important tool to falsify terms, but also distracts Laura from controlling the meaning of her terms.

(4) Two further obstacles hinder Laura and Paul to find the term: Their unquestioned implicit premise that a term always consists of only two numbers, and the missing attempt to make use of *nested referent wholes* instead of varying referent wholes: They refer to 24 or to 36, but never to 24/36 or 2/3 as the part of the non-ocean-group *of* the whole class. We hypothesize that Laura and Paul could have found the term if the teacher would have given them an impulse to overcome these constraints.

## CHANGING REFERENT WHOLES – A MANAGEABLE CHALLENGE

Although the Excursion-Problem was originally designed for students of age 11, it proved to be productive for enhancing learning processes even for prospective teachers. It might be received as disappointing that the prospective teachers could only find a complete term in 19 of 66 documents. On the other hand, the first insights into their processes give hope that well designed learning arrangements can help them to manage the challenge (this is in line with Greer 1992 who gives kindred recommendations for designs with emphasis on construction of models and meanings).

Our empirical findings on the complex interplay of representations might be of major importance far beyond the concrete subject "part-of-part-model for fractions". Learners can fruitfully rely on different representations, but we are far from understanding in detail what happens in these processes. Future research should focus on this point more intensively.

## References

Gerster, H.-D. and Schultz, R. (2004) *Schwierigkeiten beim Erwerb mathematischer Konzepte im Anfangsunterricht*, Research Report, PH Freiburg, http://opus.bsz-bw.de/phfr/volltexte/2007/16/pdf/gerster.pdf [12.1.2009]

Greer, B. (1992) 'Multiplication and Division as models of situations', in Grouws, D. A. (ed.) *Handbook of Research on Math. Teach. & Learn.*, Macmillan, New York, 276-295.

Fischbein, E. (1989) 'Tacit Models and Mathematical Reasoning', *FLM* 9(2), 9-14.

Harel, G. and Confrey, J. (1994) (eds.) *The development of multiplicative reasoning*, Albany: SUNY.

Mack, N. K. (2000) 'Long-term effects of building on informal knowledge in a complex content domain: the case of multiplication of fractions', *J. Math. Behav.* 19(3), 307-332.

Prediger, S. (2008) 'The relevance of didactic categories for analysing obstacles in conceptual change: Revisiting the case of multiplication of fractions', *Learning and Instruction.* 18 (1), 3-17.

Schink, A. (2008) 'Vom Falten zum Anteil vom Anteil - Untersuchungen zu einem Zugang zur Multiplikation von Brüchen', *Beiträge zum MU 2008*, WTM, Münster, 697-700.

vom Hofe, R. et al. (2006) 'On the Role of 'Grundvorstellungen' for the Development of Mathematical Literacy', in Bosch, M. (ed.) *Proc. of 4th Congress of ERME*, Spain 2005.