Discontinuities for mental models - A source for difficulties with the multiplication of fractions

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Different theoretical approaches offer different ways of explaining students' well-documented difficulties with arithmetical operations like multiplication of fractions. The article recalls a conceptual framework that integrates approaches focusing on meanings of operations into conceptual change approaches It offers first results from an empirical study on discontinuities and continuities of models for the multiplication of fractions.

Keywords: Multiplication, fractions, conceptual change, mental models

1. Different theoretical approaches and an integrating multi-level model for discontinuities with fractions

Many empirical studies have documented enormous difficulties in students' competencies and conceptions in the domain of fractions (and decimals). Whereas *algorithmic competencies* are usually fairly developed, *understanding* is often weaker, as well as the *competencies to solve* word problems or realistic problems including fractions (e.g. Hasemann, 1981; Barash & Klein, 1996; Aksu, 1997).

Different theoretical approaches exist for explaining these difficulties. One common aspect of several approaches is the emphasis on *discontinuities between natural and fractional numbers*, for example the fact that multiplication always makes bigger for natural numbers (apart from 0 and 1), but no more for fractions (e.g. Streefland, 1984; Hartnett & Gelman, 1998). Among different theoretical approaches to explain students' difficulties with these discontinuities, the *conceptual change approach* (Posner et al., 1982) has gained a growing influence in mathematics education research (e.g. Lehtinen, Merenluoto, & Kasanen, 1997; Stafylidou & Vosniadou, 2004; Lehtinen, 2006). On the basis of a constructivist theory of learning and inspired by Piaget's notion of accommodation, the conceptual change approach has emphasized that learning is rarely cumulative in the sense that new knowledge is only added to the prior (as a process of enrichment). Instead, learning often necessitates the discontinuous reconstruction of prior knowledge when confronted with new experiences and challenges. Problems of conceptual change can appear, when learners' prior knowledge is incompatible with new necessary conceptualisations. The key point in the conceptual change approach adopted here is that discrepancies between intended mathematical conceptions and real individual conceptions are not seen as individual deficits but as necessary stages of transition in the process of reconstructing knowledge - in the sense of epistemological obstacles, in Brousseau's terms (1997).

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Up to 2006, this discussion on conceptual change was held nearly separately from a second influential theoretical approach that emphasized the importance of underlying mental models (Fischbein et al., 1985, Greer, 1994) or 'Grundvorstellungen' (GVs, see vom Hofe et al., 2006) for explaining students' difficulties. The notion *mental model* is used here as nearly synonymous to *Grundvorstellung*. It starts from Fischbein's use of model as a "meaningful interpretation of a phenomenon or concept" (Fischbein, 1989, p. 12) which is more specific than the often cited construct *mental model* as used by cognitive scientists like Johnson-Laird (1983). Within the theoretical approaches to which this article refers (Fischbein, 1989, vom Hofe et al., 2006), the formation of mental models is considered to be especially important for mathematical concept acquisition. Mental models constitute the meanings of mathematical concepts based on familiar contexts and experiences. They create mental representations of the concept and they are crucial for the ability to apply a concept to reality by recognizing the respective structure in real life contexts or by modelling a real life situation with the aid of mathematical structures (cf. vom Hofe et al., 2006, p. 2).

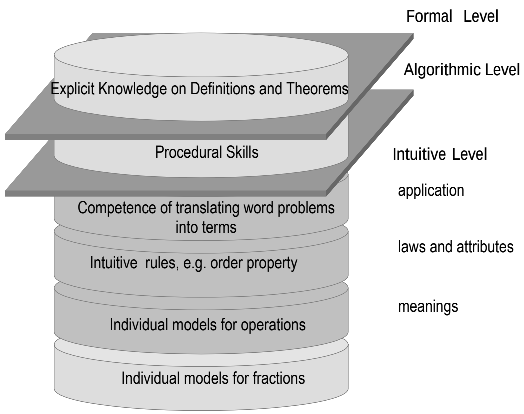


Figure 1: Obstacles can lie deeper – Different levels of students' difficulties

In Prediger (2008), these two so far competing theoretical approaches of conceptual change and mental models were integrated into a multi-level model for knowledge of operations (see Fig. 1). Its main purpose was to provide a conceptual tool for describing the precise locations of students' difficulties with discontinuities, i.e. the epistemological quality of the obstacles hindering students to master the necessary changes in the process of conceptual change.

Following Fischbein et al. (1985), the model differentiates between algorithmic, intuitive and formal understanding. The *formal level* includes the definitions of concepts and of operations, structures, and theorems relevant to a specific content domain. This type of knowledge is formally represented by axioms, definitions, theorems and their proofs. The *algorithmic level* of knowledge is basically procedural in nature and involves students' capability to explain the successive steps included in various, standard procedural operations. Although solving word problems also has procedural aspects, it is assigned to the intuitive level since it necessitates interpretations of mathematical concepts.

Intuitive understanding is characterized as the type of mostly implicit knowledge that we tend to accept directly and confidently as being obvious. On the *intuitive level*, it is worth to distinguish between conceptions about concrete mathematical laws or properties, here called *intuitive rules* (like "multiplication makes bigger") from those about the *meanings* of concepts (like the interpretation "multiplication means repeated addition"). Nearly all studies dealing with conceptual change in the field of fractions have treated intuitive knowledge, but they have mainly focused on the level of intuitive rules. In contrast, they have neglected the level of meanings which consists of mental models. (Remark that the term is used her for domain-specific intuitions, not in a general sense like by Tirosh & Stavy, 1999.)

Natural numbers	Fractions
repeated addition (3x5 means 5+5+5, i.e. 3 wands of 5m length in a row)	???
area of a rectangle (3x5 is the area of a 3cmx5cm rectangle)	area of a rectangle (2/3 x5/4 is the area of a 2/3 cm x 5/4 cm rectangle)
????	part-of-interpretation (2/3 x 5/2 means 2/3 of 5/2)
multiplicative comparison (twice as much)	multiplicative comparison (half as much)
scaling up (3x5 means 5cm is stretched three times as much)	scaling up and down (2/3 x 5/2 means 5/2 cm compressed on 2/3 of it)
combinatorial interpretation (3x5 as number of combining 3 shirts + 5 trousers)	????

Figure 2: (Dis-)Continuities of mental models for multiplication in the transition from natural to fractional numbers

The level model allows to re-locate the exact place of the epistemological obstacles in the process of conceptual change from natural to fractional numbers. Most researchers in conceptual change research locate the problem on the level of laws and rules, conceptualizing the transfer of rules from natural numbers to fractions simply as a problem of hasty generalization. In contrast, some researchers (like Fischbein et al., 1985; Prediger, 2008; Greer, 1994) showed the importance of the underlying level of meaning as the more important level to locate discontinuities. Already in 1985, Fischbein et al. showed how many students adhered to the ‘repeated addition’ as the dominant model for the multiplication of natural numbers. In Prediger (2008), the author pleaded for widening the considerations to *all mental models* for multiplication.

A mathematical (not yet empirical) analysis of mental models as summarized in Figure 2 makes clear that not all mental models for multiplication have to be changed in the transition from natural to fractional numbers. The interpretation as an area of a rectangle or as scaling up can be continued for fractions as well as the multiplicative comparison. In contrast, the basic model ‘repeated addition’ is not sustainable for fractions, neither the combinatorial interpretation. Vice versa, the basic model of the multiplication of fractions, the part-of-interpretation, has no direct correspondence for the natural numbers (see Fig. 2 and 3 for examples explaining the models).

This mathematical analysis sensitizes for possible locations of obstacles in the process of conceptual change: Not the intuitive rules pose the most urgent problem, but the necessary changes of mental models. *Metaphorically speaking, the discontinuities that possibly generate epistemological obstacles in the transition process can be located in the flashes of Figure 2*.

2. Research questions, test items and research design

2.1. Research questions and test items

So far, the analysis of discontinuities of mental models was only conducted theoretically as a mathematical analysis of meaning. The study presented here tries to show its empirical relevance by treating the following research questions:

1. What mental models do students in Grade 7 and 9 activate?

2. What kind of situations can they describe by a multiplicative term?
3. Is there empirical evidence that those models that have to be changed are more difficult than those which might be continuously transferred from natural numbers to fractions?
4. For which of the models can we find a connection to the intuitive rule “multiplication makes bigger”?

In order to answer these questions, twelve test items were constructed. This paper is focused on those eight items which referred to the intuitive level (see Figure 3, all items were of course given without headline). Item 2 operated on the *level of intuitive rules*, asking in a multiple choice format for the order property of multiplication. Items 5 and 6 operated exploratively on the level of meaning. It was given in an open item format in order not to impose a presupposed mental model but to exploratively gain a great variety of really existing *individual mental models*. Item 7 to 11 referred to the competence of finding multiplicative terms to given word problems. They differed in the necessary mental model that had to be activated.

Selected test items with reference to the intuitive level

Item 2: Order property of multiplication
Which statement is correct (mark with one or more crosses): When I multiply two fractions
- ○ the solution is always bigger than the two fractions
- ○ the solution is always smaller than the two fractions.
- ○ the solution is sometimes bigger, sometimes smaller than the two fractions.

Item 5: Find word problem for an additive equation
When solving word problems, you are supposed to find calculations for given everyday situations.
Here, you are asked to do it vice versa.
Find a word problem that can be solved by means of the equation $\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$.

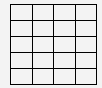
Item 6: Find word problem for a multiplicative equation
Find also a word problem that can be solved by means of the following equation: $\frac{2}{3}\cdot\frac{1}{4}=\frac{2}{12}$.

Item 7: Mathematize a situation with multiplicative comparison
a.) One kilogram tangerine costs € 1.50. Kate wants to buy $\frac{3}{4}$ kg. How can she calculate her price to pay? (Mark with one or more crosses)
☐ $1{,}5 - \frac{3}{4}$ ☐ $1{,}5 : \frac{3}{4}$ ☐ $\frac{3}{4}\cdot 1{,}5$
☐ none of these, but this:
b.) Justify your answer given in a)

Item 8: Mathematize a situation with repeated addition (natural times fraction)
a.) Every child eats on average $\frac{2}{10}$ kg of mashed potatoes. How can we calculate what 15 children would eat? (Mark with one or more crosses)
☐ $15+\frac{2}{10}$ ☐ $15:\frac{2}{10}$ ☐ $\frac{2}{10}\cdot 15$ ☐ $15\cdot\frac{2}{10}$
☐ none of these, but this:
b.) Justify your answer given in a)

Item 9: Mathematize a situation with part-of-whole number (given verbally)
a.) How can we calculate $\frac{2}{3}$ of 36? (Mark with one or more crosses): ☐ $36-\frac{2}{3}$ ☐ $36:\frac{2}{3}$ ☐ $\frac{2}{3}\cdot 36$
☐ none of these, but this:
b.) Justify your answer given in a.)

Item 10: Specify part of a fraction and mathematize



a.) Colourize $\frac{3}{4}$ of the rectangle.
b.) Colourize now $\frac{2}{5}$ of these $\frac{3}{4}$ with another colour.
c.) Give the fraction that describes the part of the rectangle that is double coloured now.
d.) With what calculation could you come to this fraction?

Item 11: Mathematize a situation of scaling down

a.) An African elephant has a body height of 3.60 m.
Anna has a model of the elephant which is scaled down to $\frac{3}{40}$ of the original body height.
Give the height of the model elephant.
Explain your way how you found it.
b.) Can you solve the task in a.) also with one single operation? (if not already done) Which one?

Figure 3: Selected test items

2.2. Design – sample and data analysis

The study was designed as a 60 minutes paper and pencil test, written by 269 students in five Grade 7 classes (age about 12 years) and five Grade 9 classes (age about 14 years) in German grammar schools which comprise the higher achieving 40% of students in Germany.

The students' answers were evaluated quantitatively in a points rationing scheme. Furthermore, the answers to Item 5 and 6 and reasons given in Item 7 to 9 were analysed qualitatively by categorizing the manifested individual conceptions about operations on fractions. In a pre-test, the developed coding scheme achieved an interrater agreement of Cohen's kappa of 0.81 to 0.94.

3. Most important results

3.1. More difficulties for interpreting multiplication than for addition

The comparison of Item 5 and 6 offers well expected results: much more students could find an adequate word problem for a given additive equation than for a multiplicative equation (see Figure 4).

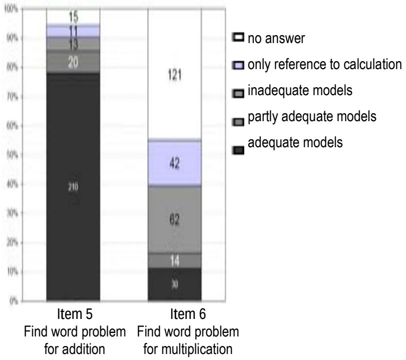


Figure 4: Comparison of occurences of interpretations for additive and multiplicative equations in Item 5 and 6

Whereas 210 of 269 students (78%) were able to find an adequate model for the addition, only 30 out of 269 (11%) found one for the multiplication, like a word problem concerning "2/3 of 1/4 l milk". In contrast, 15 students gave no answer and 11 referred only to the calculation itself for addition (in sum 10%), but 61% did one of both for multiplication.

In Item 6, a middle group of 14 students gave interpretations which showed partly adequate multiplicative models, but in an incomplete way. A middle group of 20 students gave traces of adequate models for the addition, all of them trying to join parts of different wholes, like in this example:

Lisa completed 2/3 of her English homework and 1/6 of her math homework. Which part has she completed in sum?

Among the 62 wrong answers (23%) with inadequate models for the multiplication, there were 38 answers with a word problem that referred to an additive situation, e.g.

Mr. Miller sells 2/3 of his bread s on one day and ¼ and the next day. He wants to know how much he sold together.

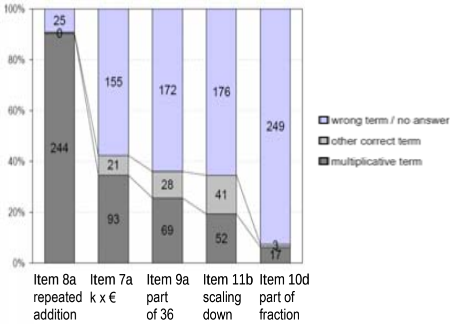


Figure 5: Comparison of chosen terms for different items

3.2. Not all models for multiplication equally difficult

The difficulties of connecting multiplications with situations became equally visible by the five items with the inverse question, in which students should choose terms for mathematizing different given situations.

Figure 5 gives an overview on the decreasing frequency of reached scores for those five items. The far best results were reached for Item 8a, in which 91% of the students could activate the well-known model of repeated addition and chose the multiplicative term $\frac{2}{10}\cdot 15$ or $15\cdot\frac{2}{10}$. (More precisely, 38% chose one, 53% chose both.)

In contrast, in all other items, no more than 93 of 269 students (i.e. 35%) chose the multiplicative term. Only 17 of 269 students (6%) could mathematize the verbally and graphically given $\frac{2}{5}$ of $\frac{3}{4}$ in Item 10d by the multiplication $\frac{2}{5}\cdot\frac{3}{4}$.

3.3. Only a singular connection between mental models and intuitive rule on order property

Due to the low rates of correct answers, it is difficult to determine correlations between different items. Only for Item 9, we find a clear connection to Item 2:

109 students expressed the intuitive rule "When I multiply two fractions, the solution is always bigger than the two fractions." in Item 2, 88% of these 109 (i.e. 96) chose a wrong term for calculating $\frac{2}{3}$ of 36, most preferably (64 answers) the division 36 : $\frac{2}{3}$.

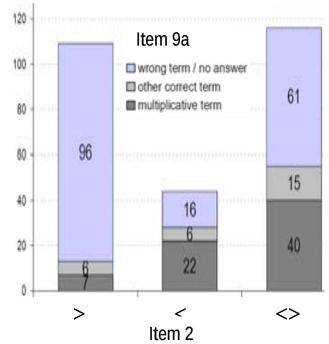


Figure 6: Connection between order property (Item 2) and choice of f operation for part of 36 (Item 9a)

In contrast, among those 116 students who expressed the correct rule "sometimes bigger, sometimes smaller", only 61 chose a wrong term, i.e. only 53%. Vice versa for those who chose the multiplication $\frac{2}{3}\cdot 36$ correctly: It was chosen by 40 out of 116 students (i.e. 35% of those) with correct order property, and only by 7 of 109, i.e. by 6% of those with false order property (even better was the ratio for the "overgeneralizers" who believe that for fractions, every multiplication makes smaller: 22 of 44, i.e. 50%).

For the determination of a relative part (two third of 36), it seems hence to be helpful to know that multiplication can (sometimes) make smaller.

As the following answer to Item 7b shows, some (singular) answers in other items also refered to the order property:

She would have to calculate 1,5 : ¾ because she wants to buy less than 1kg tangerines.

But this connection could not be found for a statistically significant number of students in Item 7, 8, 10 and 11.

4. Discussion

Why do students have difficulties with the application of multiplication for word problems? Although a quantitative study based on a paper and pencil test can only specify coincidences but no reasons, the results of the study give distinct tendencies which are interesting to discuss.

The first finding ”more difficulties for interpreting multiplication than addition” fits perfectly to the explanations given in the here presented *integrated conceptual change approach*: The empirical phenomenon that much more students can formulate a word problem for an additive equation than for a multiplicative equation can be explained by the mathematical fact that there is no epistemological obstacle for addition, that means no conceptual change is necessary in the transition from natural to fractional numbers. In contrast, the large number of students who were not able to find any adequate interpretation for the multiplication in Item 6 can be explained by the epistemological obstacle given by the discontinuity of interpretations for multiplication in the transition from natural to fractional numbers. Fischbein et al. (1985) emphasized already in 1985 that one difficulty lies in the mathematical fact that the most dominant mental model, the repeated addition, cannot be continued for $\frac{2}{3}\cdot\frac{1}{4}=\frac{2}{12}$. This interpretation of the empirical result can be supported by the comparison of difficulties in finding correct terms for differently structured multiplicative situations (second finding “Not all models for multiplication equally difficult” in Figure 5). The word problem asking for repeated addition (natural times fraction) was mathematized significantly better than all other multiplicative situations in which the repeated addition could not be activated. Due to too similar results in Items 7,9,10, and 11, we do not go further in the ranking of different discontinuous models for the multiplication of fractions (see Figure 2). Additionally, we cannot exclude the interpretation that the lowest rates in Item 11 and 10 might also be explained by the fact that there were no pre-given answers as in the multiple choice Items 7 to 9. Even the slight change from decimals to fractions might have contributed to the results.

However, the discontinuity of mental models seem to be more crucial in these cases than the pertinence of the intuitive rule “multiplication makes bigger”, since we could only find a statistical connection between Item 2 and Item 9a, but not for Item 7a, 8a, 10d and 11b. Additionally, the low rates of correct answers in the whole test sample might have contributed to the absences of correlations.

As it is not in line with well-known research results, e.g. by Bell et al. (1981), this phenomenon will need further research with a more differentiated look on different mental models, their discontinuities and their connection to intuitive rules on the order property.

5. Outlook: Beyond the example of multiplication of fractions

Although this paper reports on a concrete empirical study on one special operation for a special number set (namely the multiplication of fractions) the author cannot conclude without emphasizing that it should be taken as an example for a wider learning problem and also a wider research program. Although we know relatively much on students’ conceptions concerning the numbers, *the interpretations of operations in different situations* is still not enough in view, neither in the view of researchers nor in the view of teachers and text book writers.

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Appendix: Scores of items

Item	Content (in order of difficulty)	Frequency of complete solutions	Average of reached scores absolutely	in %
5	Find word problem for an equation with addition	57%	1,40 of 2	70%
8	Mathematize situation with repeated addition (natural x fraction)	31%	1,35 of 2	67%
2	Order property (does multiplication make bigger?)	43%	1,04 of 2	52%
11	Mathematize situation of scaling down	14%	0,62 of 2	31%
10	Specify part of a fraction and mathematize	4%	1,49 of 5	30%
7	Mathematize situation with multiplicative comparison	9%	0,54 of 2	27%
9	Mathematize situation with part of whole number	7%	0,46 of 2	23%
6	Find word problem for an equation with multiplication	11%	0,30 of 2	15%