FROM TEACHING PROBLEMS TO RESEARCH PROBLEMS
Proposing a Way of Comparing Theoretical Approaches

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To appear in the Proceedings of CERME 5 (2007, Larnaca, Cyprus)

Abstract. We analyse how researchers with different theoretical backgrounds conceptualise a fuzzy teaching problem and reframe it as a research problem. Does this represent a useful way of comparing different theoretical approaches, and of evaluating their practical significance? This contribution is intended to initiate a discussion of theoretical approaches, first but not only in Working Group 11 of CERME 5.

How should the scientific community in mathematics education deal with what is perceived as diversity of theories within the field? Here we write 'perceived', because the community itself may not be well placed to step back and recognise important theoretical commonalities which reflect its shared and taken-for-granted assumptions, being more alert to those features which differentiate theories. Rather than the frequent demand for *unifying* theories, some researchers plead for the primacy of *understanding the differences and commonalities of different theories* (e.g. Dreyfus et al. 2006, Bikner/Prediger 2006). This process of understanding different theoretical approaches has always been an important part of the discourse of CERME-conferences and due to its complexity and due to the richness of different theories, it is far from being finished.

This paper explores one possible way *to compare different theoretical approaches* and their meaning for research practices and the relation between theory and classroom practice. It follows Charles Sanders Peirce's pragmatic maxim:

> "In order to ascertain the meaning of an intellectual conception one should consider what *practical consequences* might conceivably result by […it]; the sum of these consequences will constitute the entire meaning of the conception." (Peirce CP 5.9).

We interpreted Peirce in such a way that we tried to understand different theoretical approaches by considering their *expression in the practice of researchers*. Whereas the practice of research is often discussed only as an issue of research designs and methodologies, we are convinced that research practices are strongly influenced (but of course not completely determined) by the earlier stage of research, namely the way researchers conceptualise their field. That is why we focus our comparison not on different ways of analysing a given piece of data (like Gellert, Halverscheid or Maracci in these proceedings) but on an earlier step in research, conceptualisation of the problem.

Therefore, we asked researchers with different theoretical backgrounds to briefly describe, first how they would conceptualise a given teaching problem, and then how they would design an appropriate research study.

The initial reference point was a teaching problem, which we have often heard expressed along the following lines:

> How is it that some students can learn to tackle a particular type of mathematical problem successfully (as shown by their performance in the class), but be unable to do so two weeks or months later?
> What strategies can the teacher use to reduce the likelihood of this occurring?

In order to see how different approaches frame this teaching problem as a research problem and devise a research design, we asked the following questions:

a) How do you –a priori– answer this question and what are your basic assumptions?
b) How do you transform the raised problem into a research question starting from the question above?
c) What is your research design?
d) What type of results would you expect?

The complete responses of eight researchers or research teams can be found in the appendix to this paper.

Some authors informally synthesise different frameworks, although not always explicitly (Ruthven; Bikner-Ahsbahs; and Kaldrimidou, Tzekaki, & Sakonides). Others explicitly adopt some theoretical framework: Jungwirth uses the interactionist perspective, Artigue & Lenfant rely on the Theory of Didactical Situations and the Anthropological Theory of Didactics, Bosch & Gascòn on the Anthropological Theory of Didactics, and Dreyfus & Kidron on the Theory of abstraction in context with the RBC-Model. According to Eisenhart's classification (1991), these frameworks can be classified as theoretical frameworks, whereas Christer Bergsten proposed (in the discussion of the working group) to classify Arzarello & Robutti's use of different aspects of Semiotics, Anthropological Theory of Didactics and the perceptuo-motor approach as a conceptual framework in Eisenhart's (1991) sense.

CONCEPTUALISATIONS

How do the different authors conceptualise the given teaching problem? Most of the responses accept to a degree the terms in which the problem is posed, representing it as a 'banal phenomenon' (Artigue & Lenfant), a 'natural fact' (Arzarello & Robutti), 'a phenomenon already recognised by everyday commonsense and psychological science' (Ruthven); or couch it in new terms of 'meanwhile the students have worked on other problems [and] have just forgotten how to solve the problem' (Bikner-Ahsbahs); or offer confirming evidence for it, of 'the same students who very successfully factored expressions and solved equations in grade 9, [who] cannot do the same exercises any more a year later' (Dreyfus & Kidron). Nevertheless, all these responses also suggest that the original terms are inadequate to frame the problem: they identify various issues requiring clarification, or directly elaborate a range of alternative conceptualisations and explanations.

The response from Bosch & Gascón is rather different: it directly proposes a particular reframing, so that the 'teaching problem' becomes an 'institutional problem':

> "We postulate that these facts are different manifestations of a didactic phenomenon that we call '*the dis-articulation' of the school mathematics (the taught mathematical knowledge)*. …The kind of mathematical activity the students carry out (for instance, learning to solve a 'narrowly defined' type of problems for a short period of time and forgetting it afterwards) is mainly a consequence of the kind of mathematics that exist at school, which are affected by the phenomenon of 'dis-articulation'." (Bosch & Gascón)

This conceptualisation, taking an institutional perspective, is far reaching since it implies limitations for improvement strategies at other levels:

> "As consequence of our previous postulate, it does not seem that the didactic phenomenon associated with the fact mentioned can be easily modified only by changing teachers' strategies. The kind of solution we can think of is the implementation of new didactic organisations in a system that has strong traditions and imposes many constraints on the way changes can be carried out ... It is thus necessary to study the mechanism and the scope of the phenomenon." (Bosch & Gascón)

Bosch & Gascón's response provides a striking example of how a theoretical framework –in their case, the Anthropological Theory of Didactics– shapes conceptualisation of the given teaching problem and privileges certain types of research question.

Another definite, but distinct position, reframes the 'teaching problem' to emphasise that it is also a 'learning problem'. Following the concerns and perspectives of their RBC model of abstraction in context, Dreyfus & Kidron adopt an individual cognitive perspective with a focus on student learning factors:

> "Our research would rather start from the perspective of the student. What we want to know is how things are learned, not only how they are taught. We want to investigate … what are the learning processes by means of which […students] arrive at … connections between knowledge elements [… and] acquire … (or fail to acquire) explanatory power with respect to a cluster of mathematical concepts or processes." (Dreyfus & Kidron)

By adding "with respect to a cluster of …", they stress the possible domain specificity which is a basic assumption in the theoretical approach of abstraction in contexts used in their research group for analysing processes of knowledge construction. This background guides their formulation of the exact research question in a focused way:

> "What are the processes of constructing the knowledge under consideration, and what are students' emerging knowledge constructs? In what are these processes of knowledge construction for a given construct different for the learning processes of students who are successful with this specific construct after a year and those who are not? In what are

> these processes of knowledge construction of the same student different for constructs with which the student is successful after a year and those constructs with which she/he is not?"(Dreyfus & Kidron)

In another reponse, Jungwirth acknowledges that "there are so many explanations", but "prefer[s] a certain one… due to [her] interactionist stance towards the world". This distinctive theoretical stance traces the origins of the problem to the way in which "everyday, smooth-running interaction is established by the teacher's and students' adjusting to the acting of each other" with the result that "students can successfully participate without an understanding to be located in their 'heads'; for instance, by answering on questions by short, tentative utterances which seem to indicate understanding so that the teacher completes to the desired answer". It is typical of an *interactionist perspective* that explanation is sought in terms not of individual cognition but of social constitution of knowledgeableness through classroom interaction.

Although Dreyfus & Kidron, Bosch & Gascón, and Jungwirth all adopt a particular theoretical perspective which 'privileges' certain factors as its objects of study, none of the responses denies that other factors may play a part, and that other lines of explanation might be developed; they simply choose not to examine these. By contrast, two other responses identify a much wider collection of potentially important factors.

Bikner-Ahsbas enumerates a wide spectrum of possible causes for the given teaching problem. Whereas the emphasis of most of the responses is on epistemological and cognitive factors, the factors proposed by Bikner-Ahsbahs explicitly include those of student affect and identity which traditional framing of the problem largely ignores. Perhaps, this creates a more holistic model closer to the lived experience of teachers and students, but one less amenable to controlled investigation. In designing a research project, Bikner-Ahsbahs also adopts an *interactionist perspective* on micro-situations in the classroom and poses the following research question: "What kind of conditions in everyday maths classes foster or hinder tackling a similar mathematical problem?" Consistent with her acknowledgement of different lines of explanation, Bikner-Ahsbahs' framing of a research question is more open and exploratory than that of the similarly interactionist, but more focused and transformative proposal from Jungwirth. Equally, in identifying factors that foster and hinder problem solving, Bikner-Ahsbahs envisages attention to individual students and task characteristics as well as to classroom interaction.

Sakonides, Kaldrimidou & Tzekaki emphasise *a priority of mathematical and epistemological issues* for the conceptualisation of the problem. As long as these issues remain unclear, they cannot develop a concrete research question or a research design. This priority reflects their *epistemological perspective*. They suggest that a constellation of issues must be taken into consideration including the 'particular type of the mathematical problem', 'epistemological features - involving concepts, definitions, properties, procedures, figures, symbols, several modes of representations', the 'classroom mathematical culture' (Sakonides, Kaldrimidou & Tzekaki):

> “It will be possible to claim that this particular ‘teaching problem’ is due to difficulties of cognitive nature (stereotypes that persist, or difficulties to treat information of a given way of representing data); of conceptual understanding type; of meta-mathematical nature (it might be that the students thought that something was not very important, so they didn’t learn to ‘tackle the problem’); of didactical nature; of social – cultural nature… [W]e need a hypothesis on why this happens, in order to … decide whether we need to focus on students (cognition and ways of learning) or on the didactical approach (content knowledge and teaching practices).” (Sakonides, Kaldrimidou & Tzekaki)

In contrast, the response from Ruthven does not accord priority to a specifically mathematical dimension; indeed, this feature distinguishes it from all the other responses. Rather, the main alternative conceptualizations that he proposes frame the problem in generic psychological terms. Most basic of these is that “retention of learned material tends ... to decline over time; in particular, in the absence of further use” (Ruthven); more elaborate, the idea that

> “learning is far from complete when students achieve assisted performance in a tightly framed setting; further learning – some of it quite different in character – is required for independent performance in a loosely framed setting.” (Ruthven).

Hence, it makes a difference whether a problem is originally offered in isolation or later in a new situation or combination that demands a related but modified use of learned material. In this case, retention could be improved by giving students experience of solving non-standard problems, tackling mixed revision etc. Other responses acknowledge these same issues, notably those from Dreyfus & Kidron and Artigue & Lenfant, but rather than pursuing these commonalities prefer to adopt a more specifically mathematical focus. Ruthven’s approach to the problem in terms of generic psychological terms which are not specific to mathematics reflects a *“practical theorising approach”* which seeks to find ways of framing the problem which are relatively accessible to practitioners and can be applied “to the design of practical means of addressing it” (Ruthven). Whereas the other responses all emphasise the domain-specificity of their research practices, Ruthven treats the “degree to which generic approaches can be effective, and to which more topic/setting-specific designs are required” as an open question.

Artigue & Lenfant bring out this issue, when they note that their opening suggestions “do[] not have a specific didactic flavour and could lead to look for explanations only at the level of the brain functioning or at the level of personal motivation for studying such or such topic, for learning to solve such or such type of task”, whereas a “didactic approach offers alternative or complementary perspectives, and will not necessarily lead to the same suggestions for improving the situation” (Artigue & Lenfant). Thus they take a quite different path to Ruthven when they suggest that while:

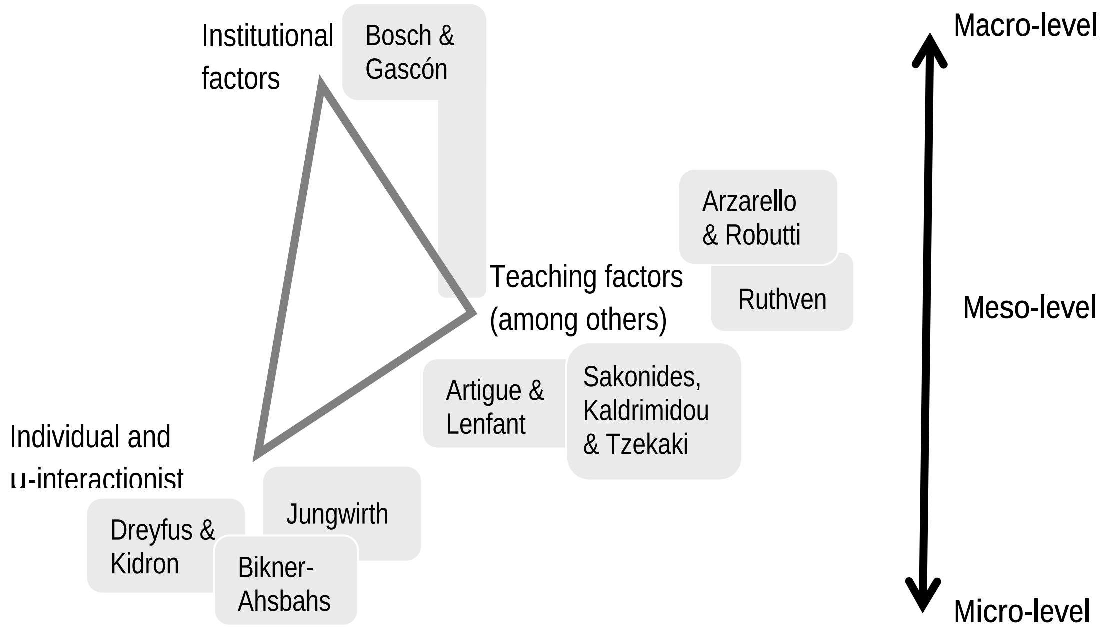
> “[t]here is certainly a lot of literature about such issues in cognitive research, [f]rom a didactic perspective, what seems more interesting to us is to transform the raised problem into a research question in such a way that the specificity of mathematics knowledge, of

> mathematical and didactical organizations could be taken into account, and that a systemic view could be developed, the 'forgetting student' being no longer the exclusive or central object of our attention." (Artigue & Lenfant)

Due to the situatedness of knowledge and learning, the context of tasks is crucial in their approach. 'Context' comprises the situations in which the knowledge was constructed as well as the learning history of the class. They start from the assumption that "the observed phenomenon [of forgetting], if not created, is highly reinforced by by didactical choices" (Artigue & Lenfant) concerning the environment of the task. The articulated focus on the mathematical problems and the learning contexts is influenced by the constructivist learning theory –the Theory of Didactical Situations– underlying their framework. Within their conceptualisation in terms of this holistic framework, they suggest that research questions and strategies may vary according to whether the aim is one of improved scientific understanding or of improved teaching practice (see below). They distinguish:

> "between research questions orientated towards the understanding of the system functioning and of the influence of its characteristics on the observed phenomenon on the one hand, and research questions associated to the elaboration and evaluation of didactical engineering trying to improve the current situation by playing on one or several levers, on the other hand." (Artigue & Lenfant)

A distinctive feature of the response from Arzarello & Robutti is the way in which they explicitly put the comparison of teaching strategies –albeit, conceptualised in terms rather different from those current among practitioners– at the centre of their research outline. Adopting a cognitive and semiotic perspective, they conceptualise the problem in terms of three aspects: the level of problems and knowledge (level 1 –knowing, understanding, applying– versus level 2 –analysing, synthesising, evaluating–), the way of thinking (analytical versus spatio-motoric thinking) and the way of teaching. They only focus on two distinct teaching strategies, the traditional one versus the perceptuo-motor approach. The aim of their research outline is to find correlations between the different aspects. One hypothesis is for example that the perceptuo-motor teaching approach produces better long-term effects. For this, teaching experiments and assessments are planned. This response is not only more concretely elaborated than others, but it is also different in the *aim* of giving empirical evidence for the superior long-term-effects of a certain teaching strategy (the perceptuo-motor approach). Long-term knowledge construction is conceptualised to depend on the level of knowledge as well as on the way of teaching and learning. In this respect, the Arzarello & Robutti response aligns with those of Artigue & Lenfant, Dreyfus & Kidron and Ruthven.



One way of thinking about the conceptualisations underlying these responses is in terms of the framing system to which they appeal: a first level system is that of individual learner and associated task environment; a second level system is that of classroom activity and its associated social interaction; at the third level is educational system as a social institution with associated curricular and pedagogical discourses.

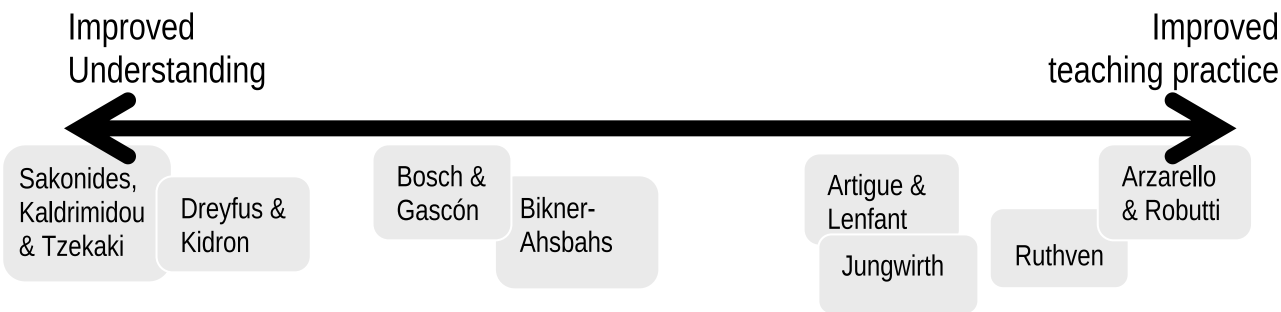
The responses can be located in relation to three idealised poles. One pole, marked by concern with the *micro-level* of individual is most closely represented by Dreyfus & Kidron focusing on domain specific processes of knowledge construction; the micro-interactionist tradition followed by Bikner-Ahsbahs and Jungwirth, which focuses on the fine grain of processes of knowledge construction and communication is probably also most appropriately placed here. Another pole, marked by concern with a *macro-level* of institutional factors is represented by Bosch & Gascón. Finally Artigue & Lenfant, Arzarello & Robutti, Ruthven and Sakonides, Kaldrimidou & Tzekaki define a position which can be thought of as at a *meso-level* in relation to these poles, but also as differing from Bosch & Gascón and Dreyfus & Kidron in a willingness to accept the *particular concern posed with teaching strategies*. In this respect it is notable that such distinctions arise despite the fact that Arzarello & Robutti appeal to the same theoretical perspective as Bosch & Gascón. Equally, Artigue & Lenfant allude to some of the same epistemological and institutional factors as Bosch & Gascón, but tackle questions on the meso-level as well as the macro-level.

RESEARCH AIMS AND DESIGNS

Understandably, the research outlines could not be very detailed with the given "fuzzy" teaching problem and the restricted space of two-three pages of response. Nevertheless, it is interesting to consider the synopsis (on the next page) of research aims and designs as well as the results expected by the researchers.

Without discussing the expected results of research in detail, we can see differences in the major intention of research, reflecting the dual character of the original request to offer explanation and to advise on teaching strategies. Whereas some responses emphasise the theory-building purpose of mathematics education research, i.e. the increase of understanding for the phenomenon, others stress the theory-applying purpose of developing instructional designs and teaching strategies (see Bergsten 2007 on the double nature of mathematics education research). Depending on the degree to which researchers consider that adequate explanatory frameworks are already available, the balance between seeking improved explanations and converting available explanations into transformative actions can be expected to vary. Ultimately, of course, these two purposes are not opposed but complementary; in particular, research in 'Pasteur's Quadrant' (Stokes 1997) seeks to combine them; indeed all these contributions can be seen as situated there, which is one strong commonality between them.

Without forgetting this complementarity, the responses can, nevertheless, be located at different places on the continuum between pure emphasis on improved scientific understanding or pure emphasis on improved teaching practice in the light of already available explanations.



	Research aims and designs	Explicitly expected results
Artigue & Lenfant	different according to concrete research questions, e.g. • "Are different types of tasks equally sensitive to the 'forgetting phenomenon'?" Explain differences? (by enquiries among teachers and students, questionnaires, analyse syllabus…) • "evaluate the influence on this phenomenon of a specific didactic strategy" (by standard methodology of didactical engineering)	"A better understanding of the didactic characteristics of this phenomenon, and of the possibility of action."
Arzarello & Robutti	"RQ1. Does students' specific knowledge … change according to the level of the task and how does (can) it change? RQ2: Does the knowledge depend on the way the students learn it and how?" Hypothesis for a concrete design: "correlation between … ways of learning and … ways of thinking: …the perceptuo-motor learning produces long-term effects." Design: Comparing concurrent teaching styles	"We expect to find some positive correlation between some … couples of variables: • … Long-time and level 2 knowledge; • Long-time knowledge and B methodology of teaching… • Long-time knowledge and spatio-motoric thinking
Bikner-Ahsbahs	What kind of conditions in every day maths classes fosters or hinders tackling a similar mathematical problem? qualitative research on classroom videographs (not necessarily in specific learning environments) and interviews afterwards	Improved understanding: Specify "ideal types of situations, task aspects, or personal aspects which disturb or foster solving a problem. … Improved teaching: • enhance diagnostic competences of teachers • give concrete micro-strategies for interaction
Bosch & Gascón	"(a) Didactic transposition problem: What are the mechanisms of didactic transposition that can explain the phenomenon of disarticulation? … (b) Ecology of didactic praxeologies: What … didactic praxeologies can be introduced … to allow the development of more 'articulated' mathematical activities…?" *Methods*: Curriculum analysis and design of epistemological reference model, set up and experimentation of a designed research study course, data analysis	1. *"Ecology of mathematical praxeologies:* new ways of curriculum organisation around powerful generative questions … 2. *Ecology of didactic praxeologies:* characterisation of possible didactic devices and strategies to manage the different moments and dynamics of the RSC; description of the didactic constraints …
Dreyfus & Kidron	Qualitative in-depth-analysis of processes of knowledge construction with respect to the connections to retention.	1. characterize differences between the learning processes of students who 'forget' and who don't 2. This "might lead to a welcome modification/expansion of the theory". 3. No focus on teaching designs. Hope to derive design principles for constructing and consolidating "in the long run".
Jungwirth	I make videos and transcriptions of some lessons of the teacher and analyse them with respect to patterns in the interaction that have been reconstructed by interactionist research before.	I design a teacher education or individual coaching of the teacher … to make her/him realize the pattern and its routines in order to change her/his part.
Ruthven	"Develop measures for improving retention of material – both in isolation and combination" as a starting point for theorized design of teaching approaches (general approaches and topic/setting specific design) and their analysis (in different research designs: classical experiments, design experiments, action research cycles).	1. "Refinement of original theorised measures, and generation of new ones" 2. theorised design of teaching approaches
Sakonides Kaldrimidou, & Tzekaki	Empirical clarification of the extent to which the teaching problem occurs (design depends on concretisation). Afterwards, development of classroom interventions	"provide ... insights into the … ways in which the classroom learning … acts, shaping the mathematical knowledge negotiated and … individual … learning trajectories."

CONCLUSION

Bergsten (2007) raises the question "How does a theoretical basis chosen for a study influence the nature of the purpose, questions, methods, evidence, conclusions, and implications of the study?" In our analysis of eight responses to the starting task, we have initiated some lines of thinking in answer to this question which are worth discussing further. The analysis we have presented is a tentative one, and involving limited testing and iteration. In the working group, these initial thoughts stimulated other members to generate alternative interpretations of the responses, to draw attention to other significant features, and to propose new analyses of the material. We hope that this process continues.

As an answer to Bergsten's (2007) final question, "But how does this contribute to compare and integrate the contributions of these studies, and others, to a deepened progression of our didactical knowledge?", we plead for patience: this exploratory analysis shows that integrating theories is much more than a simple triangulation of research methods. The crucial point is the conceptualisation of the problems in view. Comparing, networking or even integrating theories starts from understanding each other's problem definitions, something which requires extended communication. At the same time, the complementarity of perspectives gives hints that the process is worth it.

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APPENDIX

Kenneth Ruthven & Susanne Prediger

Material for the activity

COMPARING THEORETICAL APPROACHES WITH RESPECT TO THEIR WAY OF FRAMING TEACHING PROBLEMS AS RESEARCH PROBLEMS LINKED TO RESEARCH DESIGNS

One aim of the working group 11 is to deepen our insights on theories, their underlying assumptions, relationships and differences. For this purpose, the organizing committee agreed to prepare a set of questions based on an exemplary teaching problem.

We invite all participants of working group 11 to submit a report (individually or together with other colleagues) outlining their responses to questions (a) to (d) below. This will provide us with interesting material for discussion during the CERME 5-conference.

The initial reference point is a teaching problem, which we have often heard expressed along the following lines:

> How is it that some students can learn to tackle a particular type of mathematical problem successfully (as shown by their performance in the class), but be unable to do so two weeks or months later? What strategies can the teacher use to reduce the likelihood of this occurring?

What we are interested in discussing is your approach to framing this teaching problem as a research problem and devising a research design:

a) How do you – a priori – answer this question and what are your basic assumptions?

b) How do you transform the raised problem into a research question starting from the question above?

c) What is your research design?

d) What type of results would you expect?

If you decide to participate in undertaking this preparatory task (which is not mandatory), please send us your answers along these questions on max. 2 pages before the 15th December. Return them to Susanne Prediger (prediger@math.uni-dortmund.de).

On behalf of the organizing team of Working group 11,

MICHÈLE ARTIGUE & AGNÈS LENFANT: THEORY OF DIDACTICAL SITUATIONS

A) HOW DO YOU – A PRIORI – ANSWER THIS QUESTION AND WHAT ARE YOUR BASIC ASSUMPTIONS?

This question is a priori a question arising from a rather banal phenomenon: what we learn is most often not definitively learnt, and if we do not use what we have learnt, generally, more or less quickly we forget it. Most of us are certainly no longer able to tackle a lot of mathematical tasks, they were used to tackle years ago, and some of us perhaps share the experience of grasping the content of some math courses they were not especially interested in within a few weeks, getting excellent marks, very little remaining from this learning some months later, and especially not the technical ability quickly developed on some precise tasks. And, we also all know the recurrent complain of teachers saying that during summer holidays or even shorter holidays students forget everything.

What is written just above does not have a specific didactic flavour and could lead to look for explanations only at the level of the brain functioning or at the level of personal motivation for studying such or such topic, for learning to solve such or such type of task.

A didactic approach offers alternative or complementary perspectives, and will not necessarily lead to the same suggestions for improving the situation.

First, it leads us to question the question itself. What is meant by having learnt to tackle a particular type of problem successfully? Up to what point can we say that the task proposed two weeks or months later is the same as the initial task? This cannot be inferred just by looking at the mathematical text of the task, without taking into account the context for this task and the ways the teacher manages it.

Second, a didactic approach leads us to question the didactic strategies used for organizing the students' learning of this particular piece of knowledge, and for organizing its relationships with other related pieces of knowledge, hypothesizing that the observed phenomenon, if not created, is highly reinforced by didactical choices: how this type of task was introduced to the students with what mathematical motivations, how techniques for solving it were developed, how did the respective responsibilities given to the students and the teacher in the solving of this type of task progressively evolved, up to what point some particular techniques were trained and routinized, how the variation around this type of tasks was organized taking into account its didactic variables, up to what point the mathematical knowledge at stake was explicitely pointed out, justified, institutionalized and how the necessary decontextualization of knowledge was worked out, how this type of task was related with other ones in wider mathematical organizations, what opportunities were given to make the students' relationship with this task evolve beyond the necessarily short period of its official teaching… All these characteristics of the teaching process can seriously affect the personal relationships the students will develop with this type of task, the resistance to time of their ability of solving it, by using a memorized technique or by reconstructing it. Let us add that from this didactical point of view, a distance of two weeks and several months from initial learning cannot treated exactly the same; they correspond to different scales in the didactic organization.

Third, if the period of time considered includes some institutional change (change in teacher, change in institution), we can look for other types of answers, relying for instance on the notion of didactical memory, or on the notion of institutional relationship to knowledge, hypothesizing that the introduced change have partially blocked the ordinary functioning of didactical memory, or changed the institutional relationship to this type of task. But we will not develop more this kind of answer as the way the question is phrased does not seem to suppose the possible existence of an institutional change.

B) HOW DO YOU TRANSFORM THE RAISED PROBLEM INTO A RESEARCH QUESTION STARTING FROM THE QUESTION ABOVE?

There is certainly a lot of literature about such issues in cognitive research. From a didactic perspective, what seems more interesting to us is to transform the raised problem into a research question in such a way that the specificity of mathematics knowledge, of mathematical and didactical organizations could be taken into account, and that a systemic view could be developed, the "forgetting student" being no longer the exclusive or central object of our attention.

Several research questions can emerge from the tentative answers proposed above. Moreover it is certainly interesting to distinguish between research questions orientated towards the understanding of the system functioning and of the influence of its characteristics on the observed phenomenon on the one hand, and research questions associated to the elaboration and evaluation of didactical engineering trying to improve the current situation by playing on one or several levers, on the other hand.

We will limit to a few ones.

Q1: Are different types of mathematical tasks equally sensitive to the "forgetting phenomenon" and what can explain observed differences if any?

Q2: What are the strategies that mathematics teachers tend to use for limiting or controlling the "forgetting phenomenon"? What is the rationale underlying these and what are their effects?

Q3: Are there characteristics of the usual mathematical organizations which tend to reinforce the "forgetting phenomenon" and, if so, what are the mechanisms underlying this reinforcement?

Q4: Does an engineering design where specific attention is paid to the balance between the different moments of the study (according to the TAD) and to the completeness of mathematical praxeologies can make a difference?

These remain very general questions that should have to be localized and thus can lead to a great variety of specific research projects.

C) WHAT IS YOUR RESEARCH DESIGN?

The research design of course depends on the question and on the way this question will be more specifically phrased. For instance, looking for Q1, one could try to create, through enquiries among teachers and students whose extent would be to be defined, a set of potentially contrasted mathematical tasks in that respect, then use another methodology for instance several questionnaires in order to check what tasks are really contrasted, and if so investigate possible explanations for similarities and differences in the nature of the tasks and in their institutional life through the analysis of syllabus and official texts, textbooks, copybooks, teachers' material… Of course, all of this supposes the existence of some regularities… But one could also on the contrary, use the analysis of the characteristics of the institutional life of different types of tasks for conjecturing that they can be more or less affected by this phenomenon and then test these conjectures through adequate questionnaires.

The research design will be different if the question is to evaluate the influence on this phenomenon of a specific didactic strategy, and in this case it could obey for us the standard methodology of didactical engineering.

D) WHAT TYPE OF RESULTS WOULD YOU EXPECT?

A better understanding of the didactic characteristics of this phenomenon, and of the possibility of action.

FERDINANDO ARZARELLO & ORNELLA ROBUTTI

A) HOW DO YOU – A PRIORI – ANSWER THIS QUESTION AND WHAT ARE YOUR BASIC ASSUMPTIONS?

First a general comment.

In each cognitive performance, particularly in mathematical ones, there are two aspects:

- one is linked to the techniques, which require a continuous training to be performed properly (e.g. how to solve a Riccati differential equation),
- the other is linked to the ideas behind the techniques she is asked to perform, namely the technologies and the theories, in the terminology of the ATD frame (e.g. the basic concepts concerning the differential equations).

Maybe that a person many years after she ended the school remembers something about the theories but has forgotten everything concerning the techniques (it is our case for differential and Riccati equations) and so is not able to solve the problem (if it requires to solve a Riccati equation). Maybe a “feeble” student remembers the technique but not the technology and the theory: so she is not able to solve the problem for different and opposite reasons.

It is a question of level at which the knowledge related to the problem must be known to solve it. It is clear that without a continuous training many abilities linked with techniques and technologies become lower. This may cause lower performances and is a natural fact. Of course this depends on the type of performances asked and on the level of assimilation of the techniques, technologies and theories required by the performance itself.

Hence to tackle the question the teacher must distinguish carefully at which level the performances of a task are situated.

From the point of view of the student, her performances depend on the training she has got in the techniques required to solve the problem and on the level of conceptualisation she got in the ideas and theories related to such techniques. E.g. in acknowledging that a certain technique is suitable to solve that problem, in transferring a technique from a context to another, and so on.

Hence students’ performances depend on the task, on the students and on the didactical story of those students in that classroom.

Moreover students performances are not an abstract concept: the way a teacher (and the student herself) interprets them is intrinsically linked to the methodology of assessment used in the classroom.

There are different levels of performances and implicitly or explicitly a teacher has a taxonomy in her mind. For example, this could be Bloom’s taxonomy (or something different). Let us take this for the sake of an example (but what we mean does not depend on which taxonomy we are using, what we are proposing could be based on PISA taxonomies as well).

According to Bloom's taxonomy there are 6 levels of performances. Suppose that we divide the performances in two groups according to their levels:

(1) knowing, understanding, applying;

(2) analysing, synthesising, evaluating.

Also this subdivision is arbitrary and is taken here for the sake of simplicity but is meaningful; for example in PISA they use three levels of performances: reproduction, connection, reflection.

A first work should consist in classifying the problems according to the two groups (of course our example is very crude here because we use only two groups), namely according to the level of performances required to solve it. This classification should clarify that teaching and assessing is not only a problem of content but also a problem of levels of performances related to some content.

Hence the teacher could fix her didactical objectives in term of contents to teach and in term of level of performances at which she wants her students perform, for example in problem solving.

A third variable in our discussion could be the methodology of teaching that the teacher is designing for a certain content. For example we could distinguish between:

A) a traditional approach, based on the sequence: explanation-exercise-repetition-assessment;

B) a more innovative approach, where the knowledge is constructed by students in suitable learning situations, based on the use of laboratory and ICT.

These two approaches can be analysed according to the different ways of teaching-learning they produce from a cognitive point of view.

For this, two related types of analysis can be developed, based on some recent researches, which point out different modalities of learning and of thinking:

- some researchers distinguish between a perceptuo-motor and a symbolic- reconstructive way (Antinucci, 2001);.
- others distinguish between spatio-motoric and analytical thinking (Kita, 2000).

The methodology A is typically based on a symbolic-reconstructive approach, which may produce analytical thinking while the methodology B can be based on a perceptuo-motor approach, which may trigger spatio-motoric thinking. For a general discussion on this point, focussed on mathematics learning, see: Nemirovsky et al. (2004) and Arzarello et al. (2005).

Careful observation of teacher's and students' performances can point out the different modalities, according to which teaching and learning happen.

In short, our basic assumptions are based on the analysis of the links among teaching, learning, methodologies and assessment. In fact we have pointed out the following variables:

- different specific mathematical contents k_i ($i = 1, 2, ...$);
- the level (1 or 2) at which the performances for a specific knowledge k_i in the task are required;
- the methodology of teaching (A, B) for each k_i.

B) HOW DO YOU TRANSFORM THE RAISED PROBLEM INTO A RESEARCH QUESTION STARTING FROM THE QUESTION ABOVE?

The research question is:

RQ1. Does students' specific knowledge that we measure as a performance in some task change according to the level of the task and how does (can) it change?

As related questions:

RQ2: Does the knowledge depend on the way the students learn it and how?

RQ3: How can we verify if there is a relationship between the way of learning and the way of thinking?

An hypothesis to validate could be the following:

It exists a correlation between the ways of learning and the ways of thinking, namely:

i) the perceptuo-motor approach produces more spatio-motoric thinking;

ii) the perceptuo-motor learning produces long-term effects

iii) the symbolic-reconstructive one produces short-term effects.

We could investigate the previous question using the data we already have (they concern mainly teaching experiments with methodology B) and designing a teaching experiment as sketched in point c.

C) WHAT IS YOUR RESEARCH DESIGN?

We prepare two equivalent assessment tests based on structured items, the one to be given immediately after the teaching sessions and the other some months later.

We can suppose to organise the testing and the groups of students to whom the test is given so that we can distinguish among:

- different specific mathematical contents k_i (i = 1, 2, ...);
- the level (1 or 2) at which the performances for a specific knowledge k_i in the task are required;
- the methodology of teaching (A, B) for each k_i;

The data could give us some information on the short- and long-term knowledge and on the ways they correlate with the levels of performances and with the methodology of teaching.

More data should be collected to answer our research hypotheses, namely:

a) Observation through videos of the two types of teaching in order to point out the perceptuo-motor and the symbolic-reconstructive performances that are required by the students.

b) Observation of processes in a sample of students while solving problems in order to classify them according to the dichotomy analyitical Vs/ spatio-motoric thinking. To get this we should organise some specific problem solving session, where they solve some problem working in group and interacting.

With all these data we could interpret them in order to test our research hypotheses.

D) WHAT TYPE OF RESULTS WOULD YOU EXPECT?

We expect to find some positive correlation between some of these couples of variables:

- Short-time and level 1 knowledge
- Long-time and level 2 knowledge;
- Short-time knowledge and A methodology of teaching
- Long-time knowledge and B methodology of teaching
- Short-time knowledge and analytical thinking
- Long-time knowledge and spatio-motoric thinking

Of course the research project, as it is stated here is too crude. We should need to elaborate it further before starting the research. In particular we should choose carefully the arguments to teach and to test, in order to avoid the interference of other variables (e.g. epistemological obstacles). An idea could be to start comparing the data got teaching some fresh subject, not usual in the curriculum, e.g. discrete linear dynamic systems, with some standard argument, e.g. second order equations. In any case, the task should graduate carefully the technical abilities, which it requires to be solved.

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ANGELIKA BIKNER-AHSBAHS

A) HOW DO YOU – A PRIORI – ANSWER THIS QUESTION AND WHAT ARE YOUR BASIC ASSUMPTIONS?

A phenomenon like this is often observed by teachers. They think – as a teacher I have thought this as well sometimes – that a topic or an aspect is clear and has been taught so that every student should be able to repeat, know, use, … it. There are many reasons why this could not be the case:

(1) Teacher and students have a small basis of understanding: This might depend on the different thinking, working, argumentation styles of the teacher and the student, teacher and students might be of different preference types, might have different cultural backgrounds or languages, different aims, …

(2) The students need to know the context the task could be embedded:

- The student needs to know, why, what for and how the task has to be done.
- The task is not recognised as known because the story around it is different.
- The usual tools are not available.
- The social interaction process appears differently,

(3) The student needs special cognitive, social, psychic assistance or more challenge:

- The students might are weak concerning some partial performances like weak figure-ground perception or are dyslexics.
- They might need an atmosphere without pressure, anxiety, or fear but experienced pressure, anxiety or fear.
- They might give up their attempts too early because they have low self confidence. That is why they have to be encouraged.

(4) The problem is posed a little bit differently. The students might need to be supported to become more flexible.

- The teacher should provide a variety of material which might help or prepare a systematic list of heuristics or specific questions.
- If the problem is posed more complex then the teacher could reduce complexity. Instead of using variables he/she could use simple numbers.

(5) Meanwhile the students have worked on other problems. They have just forgotten how to solve the problem.

- The teacher has to prepare assistance to call a proto example to mind. That assumes that the teacher has used paradigmatic examples as prototypes.

- The teacher could make the necessary knowledge more available by repeating pieces of knowledge before posing the problem.

(6) The students might be disturbed somehow, are frightened, ill, struggle with personal problems, problems in their family, the social group, etc.

- The teacher could ask, what is wrong or if the student feels well.

I think that there are necessary and unnecessary obstacles of this kind. Unnecessary obstacles of this kind are concerned with the atmosphere in the lesson, and with the memory. They can be changed easily. Obstacles which cannot be avoided depend on the personalities of the teacher and the students, they have to be handled in a suitable way such that the teacher and the students are able to work together. They can depend on the institutional context (for instance just before their holidays), as well. However, there are necessary obstacles which the students have to overcome in order to become more flexible and better problem solvers. Since a teacher works together with different students he/she should arrange the lesson in order to be able to support the students the way they need and provide different similar tasks, provide a list of heuristics, questions, or help according to different categories. Students could be encouraged to choose the kind of help they need. This way they learn how to overcome this kind of necessary obstacles and experience more competence and autonomy.

B) HOW DO YOU TRANSFORM THE RAISED PROBLEM INTO A RESEARCH QUESTION STARTING FROM THE QUESTION ABOVE?

Since this problem is complex, a research study could look at the student and his/her inner world, at the social situations within the lesson and its social interactions and connect both. Since the question posed is oriented towards the development of support I would try to do research in a comparative way. I would try to compare hindering and supporting aspects:

What kind of conditions in every day maths classes fosters or hinders tackling a similar mathematical problem?

C) WHAT IS YOUR RESEARCH DESIGN?

Since the problem is very complex I would prefer carrying out deep analyses with a small group of students in a cyclic way. Every cycle should consist of a data collecting and data analysing step. During the process I would try to narrow the question according for instance to cognitive, mathematical, epistemic, social or other aspects in order to reduce complexity.

1. Collecting video data of the lessons

I would take every day lessons at a starting point because I want to gain knowledge for teachers who have to handle this problem in every day lessons. I would choose a specific topic to teach which is taught according to the curriculum. I would collect video data of all lessons with this topic until the test. I would take the test as another piece of data and find out students who did well and some who did wrong.

2. Interviews with students who did surprisingly well or bad and students who were expected to do well or bad, two in each group

I would ask, what conditions fostered or hindered them to do well solving this problem. The interviews are supposed to be narrative. I would try to build a trusting situation without any harm and inform the students, that I want to find out how students could get suitable help in solving a problem. The analyses of these data would be done in a comparative way and would lead to first hypotheses.

3. Analyses of the video data

With the help of these information, I would analyse the video data, try to prove the hypotheses, and gain new hypotheses.

4. Posing a similar problem in the lesson

According to suitable hypotheses I would develop a similar problem which the teacher poses two weeks later in a lesson. The technical part of the following research design depends on the teacher's kind of teaching (class discourses or group work). I would take video data from the lesson and the students which were interviewed and try to arrange the cameras so that two students of the group could be observed with one camera. This way, I would need at least four cameras which is a lot.

5. Stimulated recall.

I would watch the video together with every student and tell him/her to stop the video when he/she observes something that fostered or hindered him/her solving the problem. During the pauses the interviews would take place. Again, these interviews were video recorded.

6. Analyses of the interviews

Based on the hypotheses these interviews were analysed again.

7. Repeating 4, 5, and 6 two and four month later

D) WHAT TYPE OF RESULTS WOULD YOU EXPECT?

Depending on the students I would try to develop ideal types of situations, task aspects, or personal aspects which disturb or foster solving a similar problem. I would take these types as background concepts and reinterpret the stories of the children. The result could be a diagnostic view on every child according to the question what conditions fostered or hindered him/her solving the problem. This could be the basis of investigating the question: How could a teacher be able to handle the problem.

Remarks: I would not build a cyclic design process of finding out the student's problems and implementing and proving a special kind of teacher behaviour because deep analyses takes a long time which practice does not have. Teachers have to react immediately. Therefore we could help the teacher to become more sensitive, offer him/her some suitable possibilities to act in ideal type situations, but whether or not the this kind of action is suitable next time is an open question. The first step is to develop ideal types (see research design above) but one small study is not enough to get an overview. Based on this "theoretical" knowledge we could develop suitable types of teacher behaviour. Again, this work is not done in one study, it is a research programme.

MARIANNA BOSCH & JOSEP GASCÓN: ANTHROPOLOGICAL APPROACH

A) HOW DO YOU – A PRIORI – ANSWER THIS QUESTION AND WHAT ARE YOUR BASIC ASSUMPTIONS?

Two questions are raised: (1) “how it is that…” and (2) “what strategies can the teacher use”.

(1) The fact considered here is an aspect of a broader fact that can be described as follows: at school, students are rarely conducted to perform a mathematical activity that goes beyond the resolution of very tightly delimited types of problems, studied in a quite isolated form. They use to work in a narrow ‘mathematical space and time’, where topics come one after the other only weakly connected. Once the study of a topic is finished, all can be forgotten because a completely new activity is starting. We can mention other aspects of the same fact:

- Knowledge built up in the study of previous topics is rarely reinvested in the construction of the new one;
- Students are rarely asked to explore the borders of a type of problems or the limitations of the techniques used to solve them as a way to motivate the passage from one topic to another;
- The identification, description, delimitation, evaluation, connection, etc. of techniques and types of problems is commonly the teacher’s responsibility and rarely “transferred” to the students.
- Problem solving is being assigned by most curriculum reforms as a way to connect different topics and content areas. See for instance the following quotation from the Principles and Standards for School Mathematics of the National Council of Teachers of Mathematics (http://standards.nctm.org): “Problems and problem solving play an essential role in students’ learning of mathematical content and in helping students make connections across mathematical content areas”.

We postulate that these facts are different manifestations of a didactic phenomenon that we call *“the dis-articulation” of the school mathematics (the taught mathematical knowledge)*. In other terms, we assume that the kind of mathematical activity the students carry out (for instance, learning to solve a “narrowly defined” type of problems for a short period of time and forgetting it afterwards) is mainly a consequence of the kind of mathematics that exist at school, which are affected by the phenomenon of “dis-articulation”.

(2) As a consequence of our previous postulate, it does not seem that the didactic phenomenon associated with the fact mentioned can be easily modified only by

changing teachers' strategies. The kind of solution we can think of is the implementation of new didactic organisations in a system that has strong traditions and imposes many constraints on the way changes can be carried out – at least if we expect long-term changes, and not only local and temporary modifications. It is thus necessary to study the mechanism and the scope of the phenomenon.

B) HOW DO YOU TRANSFORM THE RAISED PROBLEM INTO A RESEARCH QUESTION STARTING FROM THE QUESTION ABOVE?

According to the phenomenon that we postulate as an "explanation" of the considered fact, we can formulate the two following research problems:

(a) *Didactic transposition problem*: What are the mechanisms of didactic transposition that can explain the phenomenon of the disarticulation of school mathematics as described above? Why is the current situation as it is? What constraints make things be like this?

We can mention here some generic didactic constraints coming from the necessity for school to show the work done in it. The didactic contract cannot concern the whole mathematical curriculum: the "mathematics to be taught" has to be split up into pieces in order to form a "study programme".

(b) *Ecology of didactic praxeologies:* What kind of didactic praxeologies can be introduced at school, and under what conditions, in order to allow the development of more "articulated" mathematical activities, that is, to allow the construction of more "complete" and "connected" mathematical praxeologies?

Some current researches of our team are focusing on these kinds of questions. They are using the notion of "Research and Study Course" (RSC) as a reference didactic praxeology and studying the function of mathematical modelling as a tool to build up more articulated (or connected) mathematical praxeologies [Bosch, García, Gascón, Ruiz Higueras (2006), *Proceedings of PME 30*, Vol. 2, pp. 209-216].

C) WHAT IS YOUR RESEARCH DESIGN?

We are focusing on problem (b) and on a specific topic or theme.

Stage 1. Curriculum analysis and design of a "reference epistemological model"

- Choose a theme or topic in the curriculum; describe the mathematical organisations (MO) that can be put into correspondence with the syllabi instructions and look for "generative questions" that can have some of these mathematical organisations as a possible answer. Describe the way(s) these mathematical organisations can be structured and obtained as the answer of questions that cannot be solved in a previous MO. This leads to the *a priori mathematical design* of a Research and Study Course that may articulate different curricular mathematical organisations, linking them through a dynamic of questions/answers.

- Sometimes this a priori analysis shows that the initial chosen topic was not well delimited or that some curricular constraints were assumed without any questioning. It is thus necessary to come back to the curriculum design and study the transposition phenomena that can explain the particular “map of praxeologies” that is traditionally taught at school.

Stage 2. Set up and experimentation of the designed “Research and Study Course”

- Propose a concrete generative question and the necessary didactic resources to make the RSC “viable” at a chosen level and under particular school conditions.
- Experiment the RSC in real classrooms.
- Observe the study process (data collection), with special attention to the way the different *moments of the study process* are managed, the share of responsibilities between teacher and students, etc.

Stage 3. Analysis of collected data

- It depends on the kind of data obtained, the initial didactic problem and the available didactic knowledge concerning the problem or the topic considered.

D) WHAT TYPE OF RESULTS WOULD YOU EXPECT?

- *Ecology of mathematical praxeologies:* new ways of curriculum organisation around powerful generative questions that can give a raison d’être to the mathematical praxeologies to be taught.
- *Ecology of didactic praxeologies:* characterisation of possible didactic devices and strategies to manage the different moments and dynamics of the RSC; description of the didactic constraints (coming from different levels of determination) that hinder the experimented study process.

TOMMY DREYFUS AND IVY KIDRON

A) HOW DO YOU – A PRIORI – ANSWER THIS QUESTION AND WHAT ARE YOUR BASIC ASSUMPTIONS?

Ken Ruthven has answered this part so well and comprehensively that we only add a comment:

At least with respect to high school algebra, and in the Israeli curriculum, with which we are familiar, 'consolidation' exercises, 'productive practice', and 'non-standard problems' are rarely used, and 'revision' exercises, though used, appear to have little effect. We surmise that the situation is similar with respect to other content domains, but we specifically relate to algebra because we have research evidence for the fact that the same students who very successfully factored expressions and solved equations in grade 9, cannot do the same exercises any more a year later, even if the first three differences in Ruthven’s "Previously" versus "Currently" table are avoided [Hoch and Dreyfus (2006): PME 30, Vol. 3, pp. 305-312].

B) HOW DO YOU TRANSFORM THE RAISED PROBLEM INTO A RESEARCH QUESTION STARTING FROM THE QUESTION ABOVE?

Here as well, Ruthven’s remarks are to the point, as far as teaching approaches are concerned: curriculum design, textbooks and teacher action such as coherently organizing the new material, emphasizing key elements, activating students, etc.

Our research would rather start from the perspective of the student. What we want to know is how things are learned, not only how they are taught. What we want to know is whether students' knowledge, their recognition of previously encountered ideas, concepts, processes and strategies, their connections between knowledge elements, explanatory power, and flexibility are excellent, adequate, wrong or lacking. We want to investigate how students reach a state in which, say, their flexibility with respect to a particular cluster of mathematical concepts or processes are excellent or lacking; or what are the learning processes by means of which a student (or a group of students) arrive at excellent (or at only partially correct) connections between knowledge elements; what are the learning processes by means of which a student (or a group of students) acquire (or fail to acquire) explanatory power with respect to a cluster of mathematical concepts or processes.

We included the term 'with respect to a cluster of mathematical concepts or processes' because we surmise that the answers to the above questions are likely to be different for different content domains.

More specifically with respect to the problem raised, we would want to investigate not only the learning processes concerning the relevant cluster of mathematical contents and processes, but also the students abilities to deal with problems requiring this cluster at various points in time after the learning experience.

When investigating such learning processes, we consider the context in which the learning process takes place to be of great importance, and therefore it must be observed and form part of the data. Context is considered in a comprehensive sense, including students' and classes' learning history, social context of learning (classroom, groups, individuals), the physical context of learning (including the availability of manipulatives and/or computer software and the manner in which these are used, etc.

As a side remark, yes, obviously, such a program of research requires instruction, and instruction needs to be designed. However, in the short term, our choice is not to focus on instructional design as a topic to be researched but to use or adapt an existing design, the choice being based on intuition and past experience of team members (the team around Rina Hershkowitz at the Weizmann Institute has well over 30 years of experience). In the long run, we would hope to also derive design principles for constructing and consolidating, derived from experience with RBC analyses.

The above are general aims of our research program; more specific research questions will be formulated in the next part, within the framework of the research design.

C) WHAT IS YOUR RESEARCH DESIGN?

Stage 1: Content analysis and instructional decision

In order to be able to ask questions that make sense, we start from an analysis of the contents (cluster of mathematical concepts and processes) under consideration. We need to analyze the contents in terms of the goals to be achieved: What concepts and strategies do we want the student to have acquired and be able to use, and in which circumstances (contexts)?

As a first step in our research design, we would therefore produce an analysis of the contents to be learned into principles that can form a basis for analysis of the data we will have. In other words, these principles should be operational. Our focus would be on students' constructing, and later building-with (using, or failing to use) these principles in a given context.

We would then choose (and possibly modify) a teaching design that has the potential for constructing these principles. This would be based on long-time experience of designers and teachers but not usually a new design which we want to subject to experimentation (i.e. not a design experiment). In other words, the design is being chosen ad hoc and then possibly refined by successive approximation.

Stage 2:

Next we identify the elements of the context in which we are interested in observing the learning process, including the students' prior learning experience, the social situation in which we want to make the observations (classroom or laboratory; often we would first collect data in a laboratory situation and then "scale up" to regular classrooms), the kind of teacher (very experienced or "regular", used to let students work in groups or not, etc.).

Given the above cluster of mathematical concepts and processes, design of instruction, and context, some of our research questions are:

- What are the processes of constructing the knowledge under consideration, and what are students' emerging knowledge constructs? In what are these processes of knowledge construction for a given construct different for the learning processes of students who are successful with this specific construct after a year and those who are not? In what are these processes of knowledge construction of the same student different for constructs with which the student is successful after a year and those constructs with which she/he is not?
- How do contextual factors including prior knowledge and experience, available technology, social interaction, teacher guidance, etc influence the constructing process? I what way, if at all, are these contextual influences different for the learning processes of students who are successful after a year and those who are not.
- Is the knowledge under consideration being consolidated during problem solving and reflecting activities, and possibly during further processes of constructing, and how is it consolidated in cases where the student is / is not successful with the specific construct after a year.
- Are there some constructs which have and others which have not been constructed / consolidated in cases in which a student is not successful after a year?

Some of the hypotheses underlying these research questions are that in cases of lack of success after a year

- constructs may have been constructed but not consolidated,
- constructing may have led to partial knowledge constructs,
- specifically, some “deeper” connecting principles may not have not been constructed or consolidated
- contextual factors such as a student’s personal history, peers with whom he or she collaborated, or computerized tools may have had beneficial or detrimental influence on the constructing or consolidating processes.

The kind of data to be collected and the period of data collection have to be determined. In the case at hand, namely students "forgetting" what they have learned in earlier months or even years, this will evidently require long term observation. So far, our experience in the framework of the RBC paradigm is with intensive observation over periods of up to two months. In the present research design, a longer time period is required, and therefore, for practical reasons, we would opt for selective observation: Intensive observation during the period of initial learning; selected observation during class periods when the planned activities require or give the option to use the relevant concepts and processes.

Additional data would be constituted by all examination questions during the period, in which the relevant contents are similarly required or optional.

Population: If possible, we would choose two populations, one from two or three schools, in which the teachers report that typically the contents under consideration are "forgotten", and a parallel one from a school in which teachers report no such problem of forgetting.

Stage 3: Data collection.

During the learning phase, we would try to have very detailed data. Typically, if we work in a classroom, we might have two video cameras, one focused on a group of students (always the same group) and the other on the teacher (or a student who speaks to the entire class). In addition, a researcher would take classroom notes. Throughout the period following the initial learning phase, and for the time span of interest, possibly about one year, the lessons specified above would be similarly observed. Examinations would be collected.

D) WHAT TYPE OF RESULTS WOULD YOU EXPECT?

We hope to be able to characterize differences between the learning processes of students who are successful after a year and others who are not. For example, we might observe that only some students have reached constructing processes with respect to some of the deeper connecting principles, whereas others have been able to carry out all required task while only building-with the component constructs. And we might see that the same students who have constructed these "deeper principle" are therefore able to make use of opportunities for consolidation which the curriculum offers, whereas other students are not, and that this consolidation leads to the effect that they "remember" a year later.

We may also need to look at re-constructing a principle, i.e. a student going through a second process of constructing the same principle she or he had constructed at an earlier period, possibly a year ago, without having had the opportunity to consolidate this principle, or without having used the opportunities for consolidation that were offered. Thus like most experiments, this one might lead to a welcome modification/expansion of the theory.

HELGA JUNGWIRTH: AN INTERACTIONIST PERSPECTIVE

A) HOW DO YOU – A PRIORI – ANSWER THIS QUESTION AND WHAT ARE YOUR BASIC ASSUMPTIONS?

Why should I have an answer at hand? There are so many explanations, and probably more than one will hold in the respective case. They may focus on students, the teacher, their interaction, on contextual events within the classroom, within the school …

B) HOW DO YOU TRANSFORM THE RAISED PROBLEM INTO A RESEARCH QUESTION STARTING FROM THE QUESTION ABOVE?

But I prefer a certain one, anyway. It is due to my interactionist stance towards the world. (Much of my research I have done on this basis). Don’t ask me why I favour it. I had an affinity to this stance, from the beginning. It is a viable belief of mine, that is, I have a good rationale for it. In particular, it has proved relevant in initiating steps towards a “better” teaching practice.

My preference in the given case, however, is underpinned by a hint in its description: students did well “in the class”; which I interpret that I cannot assume that they did well in a test, an exam as well. They performed well in the ongoing process. So their performance can be localized there. If it is sensible to understand the process as an interaction being established by the teacher and the students, which is the case presumably not from my point of view only, interactionism will be on the agenda.

C) WHAT IS YOUR RESEARCH DESIGN?

My research design is quite simple; my design activities, however, are not confined to research only. I make videos and transcriptions of some lessons of the teacher and analyse them with respect to patterns in the interaction that have been reconstructed by interactionist research before.

D) WHAT TYPE OF RESULTS WOULD YOU EXPECT?

In short, the argument is: Interaction –everyday, smooth-running interaction– is established by the teacher’s and students’ adjusting to the acting of each other. So students can successfully participate without an understanding to be located in their “heads”; for instance, by answering on questions by short, tentative utterances which seem to indicate understanding so that the teacher completes to the desired answer (just “recalling” what the students already “know”). As their competence is a phenomenon of the interaction (as I have called that once), an event existing between people, not in people, it is not surprising that some students cannot repeat neither former solutions nor the solution game later without any break-downs.

I design a teacher education or individual coaching of the teacher where the videos and transcripts are in the centre (this is not hypothetical!) The aim is to make her/him realize the pattern and its routines in order to change her/his part (because if one side does no longer act in the common way the other cannot keep to his; emergence of events in interaction put aside). We develop alternatives for utterances which do not allow students to perform well at the surface. The strategies for the teacher evolve from the concrete, detailed video or transcript reflection together with the teacher.

KENNETH RUTHVEN: PRACTICAL THEORISING APPROACH

A) HOW DO YOU – A PRIORI – ANSWER THIS QUESTION AND WHAT ARE YOUR BASIC ASSUMPTIONS?

A preliminary caution is that we need to make sure that the phenomenon has been well described. For example, there may have been important differences between the two occasions referred to:

Previously	**Currently**
The setting is one in which students can take for granted that they are required to solve this particular type of problem.	The setting is one in which students must now identify for themselves that they are required to solve this particular type of problem
The problem type is presented to students in particular forms with which they have become familiar.	The problem type is now presented to students in a variant form with which they are unfamiliar.
Forms of assistance are to hand for students, such as worked examples on the board or in their text or exercise book.	Earlier forms of assistance are no longer to hand for students.
A carefully structured teaching and learning process has led up to students' successful performance.	Students are now expected to achieve successful performance without preliminary 'reteaching' or 'relearning'.

Such differences indicate that learning is far from complete when students achieve assisted performance in a tightly framed setting; further learning –some of it quite different in character– is required for independent performance in a loosely framed setting. Indeed, some design features of existing curriculum materials represent attempts to support such further learning:

- having students tackle 'consolidation' exercises as part of the unit of work on a topic, intended not only to provide practice in solving the problem type in its standard forms, but to give experience of a wide range of variant forms.
- giving students 'productive practice' through the recurrent subsidiary tasks in 'substantial learning environments'.
- having students tackle non-standard problems, which relate in unpredictable ways to more standard problem types, requiring students to learn to match and adapt familiar techniques to unfamiliar situations.
- having students tackle mixed 'revision' exercises covering work on a range of topics, providing experience in recognising and solving particular problem types without the various forms of cueing and

assistance available during a unit of work itself; and providing opportunities for 'rehearsal' and 're-learning' of the topic.

However, if no differences of the types noted above are operative, then a simple answer is that this is a phenomenon already recognised by everyday commonsense and psychological science. Both give credence to the idea that retention of learned material tends, other things being equal, to decline over time; in particular, in the absence of further use (or rehearsal, or indeed relearning). At the same time, such sources also suggest that subsequent retention of newly learned material is likely to be more successful if attention is given to:

- avoiding learning being inhibited by students' lack of fluency with necessary prior material;
- organising new material into what students can appreciate as a coherent and connected system;
- identifying and addressing areas of student uncertainty and confusion about the material;
- engaging students intensively with the material, and in actively thinking about it;
- and most specifically:
- identifying and emphasising key elements of the material, and giving explicit attention to means of remembering and/or reconstructing them.

Finally, it should be added that the phenomenon under discussion also exercises researchers seeking to evaluate the effects of teaching interventions on student learning. Indeed, in well-designed studies of this type, it is now the norm to administer both immediate and delayed post-tests. These are intended to establish to what degree material is retained beyond the end of the instructional intervention (with some decline normally expected from immediate to delayed performance); however, such measures sometimes provide evidence of further learning (with delayed performance actually superior to immediate performance). The attribution of such improvements to 'incubation' seeks to explain them in terms of an extension or stabilisation of cognitive (re)organisation precipitated by the teaching intervention beyond the period of the intervention itself. (It should be noted, however, that the immediate post-test can itself be seen as constituting a further instructional intervention.)

B) HOW DO YOU TRANSFORM THE RAISED PROBLEM INTO A RESEARCH QUESTION STARTING FROM THE QUESTION ABOVE?

A' practical theorising' approach seeks to link theorisation of the problem to the design of practical means of addressing it. For example, the measures suggested above as means of improving retention of material –both in isolation and combination– provide a starting point for theorised design of teaching approaches and their subsequent analysis and evaluation.

An important issue is the degree to which generic approaches can be effective, and to which more topic/setting-specific designs are required.

C) WHAT IS YOUR RESEARCH DESIGN?

Analysis and evaluation of theorised designs in action involves forms of experimentation –including classical experiments, design experiments, and action research cycles. The latter two emphasise this as a cyclical process in which a theorised design is repeatedly refined.

D) WHAT TYPE OF RESULTS WOULD YOU EXPECT?

Refinement of original theorised measures, and generation of new ones.

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In discussing and/or answering the four questions (a, b, c, d) in relation to this particular ‘teaching problem’, several issues need to be taken under consideration:

1. Which is the particular type of the mathematical problem? Do we face the same ‘teaching problem’ in all problems of the same type (same content? same strategies? same category? same epistemological features - involving concepts, definitions, properties, procedures, figures, symbols, several modes of representations, several “registers”? same difficulty level -easy, moderate, difficult, complex, usual or unusual?)?
2. Do we have the same teaching problem across the various types of mathematical problems?
3. Who are these “some students”(are they always the same students or do they vary according to the problem at hand)?
4. Which is the ‘usual’ classroom mathematical culture with reference to which we make sense of this particular ‘teaching problem’? (practices of teaching, ways of working with the students, of learning, of argumentation…)
5. Which is the means we exploit in order to assess students’ performance? How these means affect and are affected by our teachers beliefs and conceptions about mathematics, its learning and teaching?

With the above concerns, here is our attempt to express some thoughts along the questions a, b, c, and d.

A) HOW DO YOU – A PRIORI – ANSWER THIS QUESTION AND WHAT ARE YOUR BASIC ASSUMPTIONS?

It is impossible to answer a priori this question without any reference to the above mentioned issues. Or, to say it better, the a priori answer will vary, depending on the perspective adopted for points 1, 2, 3, 4 and 5.

As for the basic assumptions, they will also vary. For example, it will be possible to claim that this particular ‘teaching problem’ is due to difficulties of cognitive nature (stereotypes that persist, or difficulties to treat information of a given way of representing data); of conceptual understanding type; of meta-mathematical nature (it might be that the students thought that something was not very important, so they didn’t learn to ‘tackle the problem’); of didactical nature; of social – cultural nature…

Depending on the available evidence, we could consider the possibility of dealing with the specific ‘teaching problem’ (that is, to decide whether the teacher would have to utilize some specific teaching strategies for this particular type of problems or s/he needs a more general re-consideration of his/her teaching practice).

B) HOW DO YOU TRANSFORM THE RAISED PROBLEM INTO A RESEARCH QUESTION STARTING FROM THE QUESTION ABOVE?

The transformation of the raised problem into a research question will also depend on the perspective taken with respect to issues 1, 2, 3, 4 and 5. That is, we need a hypothesis on why this happens, in order to be able to efficiently investigate what we can do and, especially, to decide whether we need to focus on students (cognition and ways of learning) or on the didactical approach (content knowledge and teaching practices).

C) WHAT IS YOUR RESEARCH DESIGN?

Concerning the research design, we will first have to clarify the extent to which this 'teaching problem' occurs. Also, we will need to check the relevance of the assumptions and the hypothesis made, by examining what happens in similar mathematical problems of another part of the curriculum and also in problems with the same mathematical content, but with different linguistic and semantic features. Then, we will be able to work with the teacher of the class to design well thought classroom interventions, which will deal with the features determining the 'teaching problem' (or the ones we have identify as such).

D) WHAT TYPE OF RESULTS WOULD YOU EXPECT?

Based on the above framework, the results should provide some insights into the complicated ways in which the classroom learning - teaching environment acts, shaping the mathematical knowledge negotiated and, thus, individual students' learning trajectories.