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Mathematics — Cultural Product or Epistemic Exception?

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ABSTRACT. How does mathematics differ from the natural sciences and the arts? Many differences can be stated but one of the most important deals with the epistemic status of the disciplines: Whereas the arts and even the natural sciences are often seen to be invented by humans and culturally influenced in their developments, mathematics is still regarded as an epistemic exception, a culture-independent discipline without any contingency. This paper emphasizes a cultural but not relativist view on mathematics searching for an explanation for the high level of coherence of mathematical theories and concepts and the wide-spread consensus among mathematicians.

1 A duality in experiencing mathematics as a culture

It is characteristic for mathematics as a scientific domain that it has disconnected from everyday life and the socio-cultural foundation which it originally came from. Scarcely any other subject regards itself that definitively as being independent of time, values and culture. The exclusive reference on the formal and the abstractable [...] makes it difficult to discuss the relation between mathematics and cultural or social elements. Mathematics is [...] widely seen as the paradigm of the formal, the structural, or the algorithmical and contrasted to culture — *i.e.*, the historical, the dynamical, the informal, or the intuitive or social: thus mathematics and culture are conceived to be extremes, which are not reconcilable.

[Sch00, p.452]

This quote provides us with a concise characterization of mathematics as it is conceived in the minds of many non-mathematicians, but also mathematicians and mathematics teachers. However, in the disciplines that systematically reflect on mathematics, this characterization is questioned more

and more. Rejecting the old, absolutist image of mathematics, many philosophers of mathematics have established humanistic or social constructivist positions, in which mathematics is understood as a *cultural product* (cf. [Ern98, Tym85, ResvBeFis93] for the social-constructivist view, or [Whi93] for the humanistic position). Emphasis is put on both aspects: “product” accounts for the fact that mathematics cannot only be discovered but must be created by humans. These creations always take place in a specific cultural setting, thus it is a “cultural” product.

One of the most prominent proponents of this position is Reuben Hersh. In his book *What is mathematics, really?* [Her97], he describes mathematics as a human activity:

From the viewpoint of philosophy mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context. [Her97, p.11]

To understand mathematics as a cultural product means to acknowledge the human influence on mathematics. From this perspective there is no principal difference in the epistemic status between mathematics and the arts or natural sciences.

Nevertheless, Schroeder has pointed out an important difference in the way individuals experience mathematics: whereas in other disciplines, the human and cultural roots of the discipline are still visible, in mathematics every individual is confronted with an “objectively accessible” theory, an apparently unchangeable corpus of ideas, notions, and theorems. Mathematical realities seem to have an existence independent of the human mind.

This experience has led to long discussions about the ontological status of mathematical objects. A convincing explanation for this experience has been given by Leslie White. He explains this phenomenon by locating the mathematical reality in the intersubjective world of culture:

Mathematics does have objective reality. And this reality, as Hardy insists, is not the reality of the physical world. But there is no mystery about it. Its reality is cultural: the sort of reality possessed by a code of etiquette, traffic regulations, the rules of baseball, the English language or rules of grammar. [Whi47, p.302f]

The dual character of mathematical experience has recently been described by Jessica Carter also in the case of working mathematicians: In her historical case study about the development of K-theory, she puts emphasis on the phenomenon that on the one hand, mathematical objects are invented by individual mathematicians, but on the other hand, their objects gain an autonomous existence, a reality that is outside the individual’s range [Car02]. The term “constructivist realism”, by which she names her position, fits well with Whites cultural approach if we locate the “reality” of

this “realism” in the intersubjective world of the mathematician’s culture, *i.e.*, the culture.

The concept of culture clarifies the entire situation. Mathematical formulas, like other aspects of culture, do have in a sense an ‘independent existence and intelligence of their own’. The English language has, in a sense, ‘an independent existence of its own’. Not independent of the human species, of course, but independent of any individual or group of individuals, race or nation. [Whi47, p.295]

For White, mathematics is not only a cultural *product*, but a *living culture*. This shift of emphasis has various implications, especially concerning the dynamical character of mathematics, the importance of implicit knowledge, and the sociological dimensions of the discipline.

Even if the duality of mathematical realities can partly be explained by locating them in the intersubjective world of culture, this explanation is not completely satisfying, because it cannot give any account for the *difference* between mathematics and other disciplines. As long as the similarities between mathematics and other disciplines concerning their cultural character are emphasized, we cannot explain why mathematics is in fact experienced in a different way than other disciplines, as it is described in the first quotation of this section.

2 Coherence and consensus in mathematics — evidences for the epistemic exception?

Why is mathematics experienced as being more disconnected from its cultural roots than other disciplines? Why do mathematical objects seem to have an existence being more autonomous than the products of other scientific disciplines?

The most obvious answer, appealing to the formalization of argumentation, is prominent and often given: the axiomatic-deductive constitution of mathematical reasoning eliminates human influences to the highest possible degree. It is the achievement of Bettina Heintz to have specified two other important phenomena: the high coherence of mathematical concepts and theories, and the wide consensus among mathematicians [Hei00]. Since they often serve as evidence for claiming a special epistemic status for mathematics, they shall be considered in detail here. Heintz describes *coherence* in mathematics as follows:

In contrast to other domains that decompose into separate and partly contradictory theories, mathematics is still a connected ensemble. In view of the enormous specialization [...] this coherence is not natural by any means. Mathematics is a collective product but not coordinated centrally. There is no authority which would ensure that the individual results match one

another. But although mathematicians operate relatively isolated and restrict themselves to a small domain of work, connections can be discovered again and again between areas which were developed independently.

[Hei00, p.19]

Coherence, in Heintz's sense, refers on the logical level to the fact that there are only very few (famous) contradictions within mathematical theories. On the conceptual level, it refers to unexpected connections between concepts from different branches of mathematics. The logical and the conceptual aspect have a counterpart on the discursive level, the high *consensus* among mathematicians. For Heintz, this aspect is even more important:

Ludwig Wittgenstein says in a famous passage that in mathematics, there is hardly a controversy, and if there is one, 'it is safe to decide' (Wittgenstein 1983: 571). In contrast to other sciences, mathematics does not provide any flexibility for interpretation. The conclusions of mathematics are mandatory. Whoever follows the rules of the mathematical method will inevitably arrive at the same result.

[Hei00, p.20]

Starting from this observation, even the sociologist Heintz (certainly not a supporter of an absolutist view of mathematics) comes to the conclusion that mathematics somehow is an epistemic exception:

Modern mathematics is characterized by features that hardly leave a scope for a sociological analysis. [...] A sociological perspective is legitimate and appropriate where it concerns the reconstruction of the development which led to that epistemic structure being typical for modern mathematics and singular in its coherence and argumentative rationality. [Hei00, p.274-275]

If Heintz were right, mathematics would not only be an inappropriate domain for a sociological analysis of social factors of influence on the scientific development, as she discusses, but also immune against human influence. Hence, it would be useless to consider mathematics as a cultural product. A human factor could only be detected in the historical development of mathematics, during the long phases in which humans made decisions, *e.g.*, about the style or the rigor of formal proofs. Due to its mandatory conclusions, contemporary mathematicians could only be creative in their ways of discovering theorems and proofs. In its consequence, Heintz's thesis of the "special epistemic status" claims that contingency in mathematics is only located in the *ways* to mathematical contents not in the *contents* themselves.

We do not need to follow David Bloor in his strong programme for the sociology of mathematics [Blo91] in order to reject the thesis of the epistemic exception that is outlined by Bettina Heintz. Emphasizing the character of the "proving discipline" (subtitle of [Hei00]), Heintz ignores crucial areas of mathematical activities in her theoretical conclusions. She neglects the entire process of mathematization (*i.e.*, the question of how initial non-mathematical problems are to be translated into mathematics), concept

formation, the development of theories, as well as the criteria of relevance for research questions are missing:

- How are mathematical concepts found?
- What influences the process of concept formation?
- How does the community decide whether a problem is adequately mathematized?
- Which factors affect the development of a theory?
- Who decides about the relevance of questions or theorems?

In all these questions, the contingent character of mathematics is much more evident than when one restricts oneself to the process of proving theorems.

Following Hersh in his distinction between the “front” and the “back” of mathematics (*i.e.*, the way of presenting finished mathematics and the creating of mathematics, resp.), we can see that consensus is essentially restricted to the “front”:

“There’s amazing consensus in mathematics as to what’s correct or accepted. But just as important is what’s interesting, important, deep or elegant. Unlike correctness, these criteria vary from person to person, speciality to speciality, decade to decade. They’re no more objective than esthetic judgments in art or music.” [Her97, p.39]

Just as many philosophers of mathematics, the sociologist Heintz has taken the easy way out and concentrated her epistemological considerations exclusively on proofs. In contrast, in the empirical part of her study she describes that proofs only appear at the end of the mathematicians’ working process. And it is exactly in this “back” of mathematics, in the mathematics in the making, where the contingent, humanly influenced aspects of mathematics can be found.

Rejecting the thesis of the “special epistemic status” of mathematics, a further analysis of the phenomena of coherence and consensus is needed. If we insist on the cultural character of mathematics and the contingency of parts of mathematical knowledge, then the coherence and the absence of real conflicts or revolutions in mathematics cannot be explained easily. In order to find an account for it in the cultural framework, Fleck’s philosophy of science and a suited theory of mathematical development are presented in the following section.

3 Approaches to an alternative explanation for coherence and consensus

In order to explain the phenomena of coherence and consensus, let us first take a look at Ludwig Fleck's philosophy of science, developed in his book *Entstehung und Entwicklung einer wissenschaftlichen Tatsache: Einführung in die Lehre vom Denkstil und vom Denkkollektiv* (English title: 'Genesis and development of a scientific fact') [Fle35]. Today Fleck is widely recognized as a pioneer of the constructivist-relativist tendencies in philosophy of science and of the sociologically-oriented approach to the study of the evolution of scientific and medical knowledge. He deserves this recognition and respect all the more as during his lifetime his philosophical achievement passed completely unnoticed, until the well known philosopher of science Thomas Kuhn recognized Fleck's main work as a source of inspiration of his *The Structure of Scientific Revolutions* [Kuh70]. As Fleck is not well known in philosophy of mathematics, his work shall be described in some detail.

3.1 Fleck's theory of the thought-collectives and thought-styles

Fleck may be called a pioneer of cultural epistemologies, because he did not only consider the subject and the object of perception, but he added the conditions of perception as a third important component of epistemology. According to his theory, the conditions of perception are determined by the existing standards of knowledge, which are not located in the individual, but in the collective. He describes an

interaction between the perceived and the perception: the already perceived influences the way of new perceiving; perception enhances, regenerates, reinvents the perceived. Thus, perception is not an individual process of theoretical consciousness, it is the result of a social activity, because the particular standard of perception exceeds the individual's limits.

[Fle35, p.54]

Fleck outlines this idea in a case study of changing concepts of syphilis in medical history. He demonstrates, how the development of a scientific fact is influenced by the culturally determined ways of thinking.

In order to describe the intersubjective character of perception and science in general, Fleck developed the notions of thought-style (*Denkstil*) and thought-collective (*Denkkollektiv*). The thought-collective is defined as a community possessing a common thought-style. This style develops successively, and is at every stage connected to its own history. It creates a certain definite readiness and dictates what and how the members of the thought-collective can observe. Thought-style is defined as directed perceiving. It is characterized by

- common attributes of the problems of interest for the collective,
- common judgments of what is considered to be evident;
- common methods as media of perceiving.
- Eventually, it is accompanied by a technical and literary style of a system of knowledge. [Fle35, p.130]

The thought-styles in which individuals think are the result of their theoretical and practical education. Passing from teachers to students, they contain certain traditional values, which are subjected to a specific historical development and specific sociological laws.

If a certain thought-style is sufficiently elaborated, it does not only determine the perception, but also what is considered to be true. Therefore, truth is located in the intersubjective dimension:

The notion of truth in its classical significance, as a value independent of the subject of cognition and of social forces, compels one to accept truth as an unattainable ideal. Besides, the history of science teaches us that we do not approach that ideal, even asymptotically, for the development of science is not unidirectional and does not consist only in accumulating new pieces of information, but also in overthrowing the old ones. Thus, classical theories of cognition ought to distinguish between:

- (1) the ideal, unattainable truth,
- (2) the official “truths” which “should” somehow approach it,
- (3) illusions and mistakes. At the same time they have to admit that there is no general criterion of truth. [...]

The epistemology which is the science of thought-styles, of their historic and sociological development, considers truth as the up-to-date stage of changes of thought-styles. [Fle36, p.111]

We can learn a lot from Fleck’s theory of thought-collectives and thought-styles for mathematics: In Fleck’s view, the sciences are specific thought-collectives that are especially stable. If we consider mathematics to be such a thought-style, Fleck gives us interesting answers to our question why there is this wide consensus and this high coherence. According to Fleck, these phenomena give no evidence for a “special epistemic status,” but they are to be understood in correlation to the standard of a discipline:

The more a field of knowledge is elaborated, the more it is developed, the smaller are the differences [...]. It is, as if the scope for development was shortened with the growth of nodes, as if more resistance appeared, as if the room for free thinking was restricted. [Fle35, p.110]

This idea is followed by the notion of “active and passive linking” (*Kopplung*): “every active part of knowledge corresponds to a passive linking,

which results mandatorily” [Fle35, p.110]. The more active parts of knowledge belong to a thought-style, the more passive linkings evolve as more or less mechanical consequences. Thus, according to Fleck, we can understand mathematics as a field of knowledge that is elaborated to a high degree. His short thesis

The deeper we go into a field of knowledge, the stronger it is bound to a thought-style (*Denkstilgebundenheit*). [Fle35, p.109]

gives a good explanation for the high consensus among mathematicians: Mathematics is a thought-style that is well elaborated and has a long tradition. This enforces constraints on thought (*Denkzwang*). This is most obvious in the field of deductive reasoning, a very important aspect of the mathematical way of generating and justifying knowledge. (Although the discussions about proofs have shown that the myth of strict deductivist criteria for proofs must be questioned since proofs always have logical gaps, see, *e.g.*, [Her97]).

But constraints of thought in Mathematics do not only concern the logical reasoning. Fleck’s conception of constraints of thought also comprises common attributes of the problems that are interesting to mathematicians, common assessment of values, and common methods used for mathematical cognition.

To sum this up, we do not need the “special epistemic status” as an explanation for consensus of mathematicians. The degree of elaboration of the mathematical thought-style supplies an alternative and more convincing explanation. More than in other fields of knowledge, the active elements of mathematical knowledge produce passive linkings (mainly but not only due to deductive reasoning). Thus, the evolution of the mathematical thought-style has indeed superseded contingency to a high degree. Nevertheless, this process can never end in the complete elimination of contingency in mathematics.

In addition to these explanations given by Fleck’s theory, historical investigations can help us understanding the phenomena of coherence and consensus in mathematics. Although the authors cited in the following passages do not all have homogenous philosophical positions, some of their ideas can serve for the paper’s argumentation.

3.2 Changes of thought-styles in mathematics: historical investigations

An important contribution to our discussion has been made by Philip Kitcher in specifying some interesting characteristics of the development of mathematics. In his book *The nature of mathematical knowledge* [Kit84], Philip Kitcher compares mathematical change and scientific change. In analogy

to Fleck's concept of thought-style, Kitcher defines the notion *mathematical practice*:

We view a mathematical practice as consisting of five components: a language, a set of accepted statements, a set of accepted reasonings, a set of questions selected as important, and a set of meta-mathematical views (including standards for proof and definition and claims about the scope and structure of mathematics). [Kit84, p.229]

Similarly to Fleck's investigation of scientific progress as transitions of thought-styles, Kitcher describes mathematical change as transition from one mathematical practice to the next:

The problem of accounting for the growth of mathematical knowledge becomes that of understanding what makes a transition from a practice $\langle L, M, Q, R, S \rangle$ to an immediately succeeding practice $\langle L', M', Q', R', S' \rangle$ a rational transition. [Kit84, p.229]

He shows in various historical examples that these transitions are often initiated by discrepancies between the components of the mathematical practices. By changing one or more components, they can be re-equilibrated. For example, theorems are retained valid by changing the language: Instead of rejecting a theorem when counter-examples are found, mathematicians often restrict the concerned notions in such a way that the theorem becomes again valid (this mechanism has been described in detail in Lakatos' book *Proofs and Refutations* [Lak76]).

So, where in the case of science we find the replacement of one theory by another [...], in the mathematical case there is the adjustment of language and a distinction of questions, so that the erstwhile "rivals" can coexist with each other. Mathematical change is cumulative in a way that scientific change is not, because of the existence of a special kind of interpractice transition. [Kit84, p.229]

Kitcher considers this mechanism for producing consensus and coherence to be characteristic for mathematics. It helps to avoid explicit discontinuities. Although singular components of the mathematical practices must be revised in order to face inconsistencies, mathematical practices are rarely abandoned completely.

In short: coherence in mathematics emerges, because mathematicians immediately search for solutions to level inconsistencies whenever they appear. In other words, inconsistencies do not exist in mathematics, because they are not tolerated.

This thesis is supported by the work of Raymond Wilder who has analyzed mathematics as a developing cultural system [Wil81]. He emphasizes the cultural relativity of mathematics:

Because of its cultural basis, there is no such thing as the absolute in mathematics; there is only the relative. [Wil81, p.148]

Anyhow, mathematics is not arbitrary and real discontinuities can only be found on the meta-level, as he postulates in agreement with Crowe [Cro75]:

Revolutions may occur in the metaphysics, symbolism and methodology of mathematics, but not in the core of mathematics. [Wil81, p.142]

On this matter, Heinz's and Wilder's positions coincide. But for Wilder, the absence of revolutions does not imply that mathematical knowledge grows cumulatively, since the patterns of development are more complicated. When he describes these patterns, he does not focus on standards of rigor for proofs nor on other aspects on the meta-level, but he concentrates on central elements within mathematics: mathematical objects, concepts, and theories. Over the centuries, mathematical objects and concepts undergo radical changes in their meaning and their role within the theories. On the basis of historical case studies, Wilder tries to specify "laws" of this evolutionary process and figures out different characteristic mechanisms.

Besides the mechanisms *abstraction* and *generalization* which have often been described, he attaches importance to *consolidation* by which he means the unification of theories or concepts [Wil81, p.87]. By connecting mathematical concepts, these processes of consolidation play an important role for establishing coherence not only on the logical, but also on the conceptual level. I will come back to this aspect in section 3.3.

Using the notion *hereditary stress*, Wilder characterizes the culturally determined phenomena that initiate the evolution of theory and concepts, such as mathematical or non-mathematical problems, a changing conception of nature, discovered inconsistencies or paradoxes, growing demands for rigor etc. In addition, there is the mechanism of *diffusion* of ideas and methods, by which mathematical thoughts are transferred from one domain to another. This is an important condition for processes of consolidation.

On the whole, Wilder considers these evolutionary processes to be embedded into their cultural background. Therefore, his patterns of change put the singular achievements of individual mathematicians into a cultural perspective (instead of celebrating single mathematicians as genius discoverers, as it was usual in the former historiography of mathematics). Starting from the observation of multiple discoveries and the "before his time" — phenomenon (*i.e.*, concepts or ideas which fail to attract attention at their time but are rediscovered and appreciated later), Wilder describes to what degree individual thinkers rely on their cultural environment. He concludes the following:

The individual mathematician cannot do otherwise than preserve his contact with the mathematical culture stream; he is not only limited by the state of its development and the tools which it has devised, but he must accommodate to those concepts which have reached a state where they are ready for synthesis. [Wil81, p.145]

Thus, according to Wilder, the cultural influence on every individual thinker provides another explanation for the phenomena of consensus and coherence: If all further developments in mathematics are based on the same cultural background, they coincide significantly in most cases. And when inconsistencies appear, they initiate processes of consolidation, which ensure consistency again.

Paul Ernest has described these patterns of mathematical change in his “generalized logic of discovery” (built on Lakatos’ “logic of discovery” [Ern98]). He considers the process of discoveries to be a dialectical cyclic process in which definitions, proposals, and relations are discussed in the community. Along this social process, the proposals are accepted or rejected. Rejecting them initiates modifications of the original proposal [Ern98, p.149-160]. The community always acts in a scientific and epistemic cultural context, “including problems, concepts, methods, informal theories, proof criteria and paradigms, language, and metamathematical views” [Ern98, p.151].

Just as Kitcher, also Wilder and Ernest emphasize the important role of well-working mechanisms that re-establish coherence in mathematics. Thus, these authors do not consider coherence to be a surprising phenomenon that legitimizes the hypothesis of the “special epistemic status”, but to be an aim for which mathematicians consequently strike again and again.

3.3 Changing the question: Importance and Conditions for Coherence

Against the background of these historical investigations, we must pose the question of coherence in a different way: From a cultural perspective, we do not need to ask why mathematical theories are coherent, but why they are always (re)made coherent and how this is possible. Following this question we find the major difference between mathematics and other disciplines: it is the role of the value coherence that makes mathematics exceptional.

Values in the mathematical culture.

The most important answer for the question why mathematical theories are always (re)made coherent can be found in the prevalent view of mathematics. When a community (of platonists or others) is convinced that inconsistencies *cannot* appear, the participants will make great efforts to remove them whenever they *do* appear.

It is similar on the conceptual level: Sociological studies have shown that integrating different mathematical theories, *i.e.*, establishing coherence between theories, is highly valued among working mathematicians. Leone Burton describes the “drive to establish connectivities” by “tremendous satisfaction when two apparently different areas are found to connect” [Bur99, p.137]. She illustrates this thesis by various quotations of interviewed mathematicians, *e.g.*, “I am certainly impressed by links and in my own work I feel very happy if I can tie things up.” (cited in [Bur99, p.137]). A typical example for the value which is attached to such connections is Euler’s formula $e^{i\pi} = -1$: It has won the competition of being the “most beautiful” theorem in an election among readers of the mathematical journal *Mathematical Intelligencer* [Wel90]. Euler’s formula is said to be the “most beautiful” theorem because it shows unexpected connections between originally unconnected mathematical domains [Hei00, p.145-150].

Following this line, the prevalent view of mathematics and its values has proved to be a “self-fulfilling prophecy” again and again: When inconsistencies are not tolerated and coherence is highly estimated within the community, mathematician’s efforts are oriented at these values. Many things were changed in order to leave this central value unchanged. It would be an interesting question for historical investigations whether the value of coherence itself has ever been seriously questioned.

Ontological status of mathematical objects as precondition.

Why can coherence be re-established more easily in mathematics than in other sciences?

Obviously, one important reason is the ontological status of mathematics. Whenever necessary, mathematical theories have been detached from the physical reality. In this way, refutations of theorems can be answered by changing (mostly restricting) the concerned mathematical concepts. Hence, one reason for the possibility of coherence is the convertibility of mathematical concepts which is the opposite of their often presumed *a priori* status. Whereas scientific concepts must correspond to reality, mathematical concepts can be built in an explicitly constructed mathematical reality in order to avoid complete refutations.

This has been made possible by an important decision about the conception of truth in mathematics: Mathematicians prefer consistency within the theory to conformity with reality. This preference has even been formalized in Hilbert’s notion of truth as freedom from contradiction, in short: consistency. To sum up, it is the special ontological status which makes the difference between mathematics and other sciences or arts possible.

Outsourcing as an organisational mechanism.

Last but not least, the mechanism of outsourcing should be mentioned as an organisational mechanism to assure coherence. Over the centuries, mathematics has outsourced many (usually applied) sub-domains when they developed their own ways of thinking and working [Lau72]. By considering them not to be a part of mathematics anymore, inconsistencies or conflicts could be removed in an easy way. Even today, there are disciplines of mathematics (like scientific computing and other parts of experimental mathematics) whose standards have removed from the widely accepted mathematical standards. Again, lively discussions have been raised, for example about the computer-generated proof of the Four-Color Theorem, whether these approaches still belong to mathematics or whether the mathematical community must begin to accept such differences [Hei00].

3.4 Conclusion

It turned out to be characteristic for mathematics that its cultural relativity can be hidden more easily than those of other cultural achievements (like language). By means of high coherence and wide consensus, the human dimension and cultural origin of mathematics is obscured more successfully than in other sciences. But for explaining these phenomena, the thesis of the special epistemic status of mathematics is not necessary. In historical investigations instead, various factors have been found within the cultural system of mathematics, which can give accounts for the fact that mathematics is more coherent and less contradictory than other disciplines:

- Mathematics is a highly developed thought-style in which all active parts of knowledge produce a lot of passive linkings.
- Historical investigations show that coherence has been re-established whenever it was questioned. For that, different mechanisms have evolved, especially interpractice transitions and consolidations.
- Establishing connectivities is a highly-valued activity for mathematical research, hence, coherence is also a result of a special value system in mathematics. This is the case also with consistency which has even become the inner-mathematical criterion for truth (replacing conformity with reality).
- An important precondition for these decisions lies in the ontological status of mathematical objects, namely in their disconnection from physical reality.

Although the phenomena of consensus and coherence mainly refer to the “front” of mathematics, the most important reasons for their existence concern mathematics in the making, *i.e.*, the “back” of mathematics. This observation agrees with new tendencies in the philosophy of mathematics, namely to focus on mathematical practices and not only on the products (*cf.* [Tym85], Schlimm in this volume). This focus can be deepened by taking a cultural perspective.

4 Outlook: Analysing mathematics as a culture

How does mathematics differ from the natural sciences and the arts? We started with this question, went through some epistemological and ontological considerations, and have seen in Section 3 that the most important differences between mathematics and other scientific disciplines lie in the scientific practices and the scientific culture in which practices are embedded. This shift of focus suggests adopting a cultural perspective on the sciences.

The idea to consider sciences as cultures raised in the 1960s in different disciplines, among them anthropology and sociology of knowledge, and has been elaborated in the last decades in higher education research. It is currently adopted in the Austrian project “Science as Culture” [Fis+98], [Arn01] in which several scientific disciplines are compared under a cultural perspective (namely physics, biology, liberal arts, and history). The project starts from the assumption that the culture of a discipline “consists of all elements which are characteristic for every culture” [Fis+98, p.5], *i.e.*,

their traditions, customs and practices, transmitted knowledge, beliefs, morals and rules of conduct, as well as their linguistic and symbolic forms of communication and the meanings they share. To be admitted to membership of a particular sector of the academic profession involves not only sufficient level of technical proficiency in one’s intellectual trade but also a proper measure of loyalty to one’s collegial group and adherence to its norms.
Becher (1984), cited in [Fis+98, p.7]

All these elements together form a scientific culture. The notion comprises “everything a student must acquire in order to become an accepted member of the community” [Arn01, p.2]. Fischer and Arnold show that this wider perspective is very instructive for comparing scientific communities, since it does not only include the body of explicit knowledge and the ontological and epistemological characteristics, but also important parts of implicit knowledge and sociological aspects. Although there are already a lot of interesting partial investigations upon the mathematical culture (*e.g.*, [Kit84], [ResvBeFis93], [Hei00]), there is (up to now) no *comprehensive* study on mathematics from the cultural perspective. In [Pre02], this work was started from an educational point of view by collecting a list of central

elements of the mathematical culture that affect students when they are engaged with the culture of mathematics in learning processes. But if we really want to understand the differences between mathematics and other sciences, we should put deeper and more systematic investigations on the research agenda.

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