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INTERCULTURAL PERSPECTIVES ON MATHEMATICS LEARNING - DEVELOPING A THEORETICAL FRAMEWORK

ABSTRACT. This paper explores Alan Bishop's assumption that all formal mathematics education produces cultural conflicts between the children's everyday culture and the culture of mathematics. Existing empirical studies support the assumption that mathematics learning indeed has some intercultural aspects which can be differentiated in language factors, effects of overlapping, divergent aims and purposes and moments of foreignness. Analysing mathematics learning in this way from an intercultural perspective allows us to integrate perspectives and results from psychological and pedagogical research on intercultural communication and intercultural learning into mathematics education research. Since these results are located on the descriptive and on the prescriptive level, they can be activated both for analysing and for arranging learning processes in an intercultural setting. Hence, in the third part, prescriptive consequences and didactical orientations are formulated for organising mathematics learning as intercultural learning.

I: Can you tell me what you think about the way your father did the sums, is it the same or is it different from the way you learned in school?
S: It is a different way, he does it in his head, mine is with the pen. I: Which do you think is the proper way?
S: School.
I: Which do you think gives a correct result?
S: My father.
I: Why?
S: Because I just think so. (De Abreu, Bishop & Pompeu, 1992, p. 27)

The interviewed Severina (14 year old, daughter of a sugar cane worker in Brazil) experiences a strong discrepancy between her everyday life and school mathematics. Although her way of calculating is the "proper way" in the sense that it is the way her teacher expects her to do it, she considers the father's way of calculating to give the "correct result" in the sense that it corresponds to their everyday experiences.

For De Abreu, Bishop and Pompeu, this interview gives an example of the general phenomenon, that children often experience conflicts "in terms of contradictory understandings generated through their participation in two different mathematics cultures, one outside school, linked to their everyday practices, and the other at school" (De Abreu, Bishop & Pompeu, 1992, p. 25). This observation of cultural conflicts has originally

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been made in ethnomathematical contexts, focusing on multi-cultural dissonances for children of indigenous cultures who were supposed to learn "Western Mathematics" (cf. Bishop, 1991; D'Ambrosio, 1999; Nunes et al., 1993). In a later paper, Bishop has widened this perspective from multicultural conflicts in the ethnomathematical sense to cultural conflicts in a more general sense (Bishop, 1994).

He points out "that many young people in the world are experiencing a dissonance between the cultural tradition represented outside school and that represented inside the school" (Bishop, 1994, p. 16). And this is not only the case for children in rural societies in Brazil, Africa or Asia, but for young people in many other societies as well. That is why Bishop asks for new conceptualisations of mathematics learning:

> For focusing on research ideas, however, I believe it is important to make a more radical assumption, namely that all formal mathematics education is a process of cultural interaction, and that every child experiences some degree of cultural conflict in that process. (Bishop, 1994, p. 16)

Bishop substantiates this assumption of cultural interaction in formal education by the argument that "schools are different social institutions from others such as homes" (Bishop, 1994, p. 16). He gives reference to various studies which have already documented conflict situations concerning very different aspects of mathematics learning, especially language, geometrical concepts, calculation procedures, symbolic representations, logical reasoning, attitudes, goals and cognitive preferences, values and beliefs. As a consequence, Bishop puts cultural conflicts on the "research agenda" (title of Bishop, 1994).

This new accentuation of cultural discontinuities is remarkable because it challenges Bishop's own philosophy of mathematics learning as a process of enculturation (Bishop, 1991) and the formerly underlying assumption of cultural continuity:

> In a sense, mathematical enculturation seemed a natural, educational evolutionary process, well interpretable within the already established frames of mathematics education. By contrast, in situations of dissonance between out-of-school and in-school cultural norms, it is very unclear what the educational task should be. Cultural continuity becomes a meaningless term, or should at least be treated problematically. The established theoretical constructs of mathematics education, developed through a research history which has failed to recognize cultural conflicts, are at best misleading, and at worst irrelevant and obstructive. (Bishop, 1994, p. 16)

Even if I do not share Bishop's position that the existing theoretical constructs were all irrelevant and obstructive, I found it very interesting to complement the existing theories of mathematics learning by a new perspective: Since most theories on mathematics learning adopt a *vertical view* by focusing on the processes of development *into* mathematics, a *ho-*

rizontal view of focusing on the *simultaneousness of different cultures* can enrich our understanding on mathematics learning. Whereas Bishop himself has recently returned to a more vertical view by conceptualising learning processes as acculturation (Bishop, 2002), this paper explores the horizontal view which has not been very present in mathematics education research up to now. This is very different in science education research where there has been a big discussion about the so-called "cross-cultural nature of learning" (cf., e.g., Hawkins & Pea, 1987; Duit, 1993).

This paper intends to explore and deepen Bishop's observation of cultural conflicts. For that, Bishop's assumption of cultural conflicts is embedded in a wider perspective: It follows the idea that mathematics learning often takes place in a situation of cultural interaction. However, this does not only produce difficulties but also chances. The phenomenon of appearing cultural conflicts can be differentiated in more detailed descriptive findings (Section 1*)*. In order to establish a suitable theoretical framework, the construct "culture" and its underlying philosophy of mathematics and the normative orientation is to be questioned (Section 2*)*. Section 3 asks for constructive consequences.

1. Intercultural Issues in Mathematics Learning - Differentiating the Descriptive Findings

1.1. Mathematical Language as a Foreign Language

Where in mathematical learning processes do we find cultural conflicts? The intercultural character of mathematics learning is most evident on the level of language: Being the technical language of mathematicians, for learners, the mathematical language is a foreign language (in this perspective, this paper follows Niederdrenk-Felgner, 1997).

Emphasising the character of a foreign language is not the perspective taken by most linguistic researchers on technical languages: In contrast, they underline the fact that technical languages and common language should not be considered as different languages since technical languages are only modified versions of the common language (e.g., Hoffmann, 1985). However, in the learner's perspective, mathematical language is an independent language which has to be acquired explicitly in order to be able to understand and actively use it. This can already be seen by the amount of vocabulary in mathematical textbooks. Ten year old German students are supposed to learn 170 mathematical expressions within one year (in comparison: in the foreign language English, they learn 800 expressions) (Maier & Schweiger, 1999, p. 117). Beyond its specific technical expressions, mathematical language has a separate grammar and separate linguistic elements, especially in the language of formulae, as Maier and Schweiger emphasise in their synopsis of differences between common language and mathematical language (Maier & Schweiger, 1999, Chapter 1).

The thesis that the mathematical language is a foreign language from the learner's perspective can also be supported by empirical studies that found enormous deficits in language perception as well as in language production of many students (Maier & Schweiger, 1999, p. 117ff). Especially well known are the difficulties when using the language of formulae. Already the translation of simple relations into the algebraic language poses enormous problems for many pupils and adults (Rosnick & Clement, 1980).

And even for pupils with better language capabilities, the interaction of two different languages in mathematics classrooms causes many situations of communication difficulties and misunderstandings (cf. Maier & Schweiger, 1999, pp. 108-164; Maier, 2002).

1.2. Intercultural Misunderstandings

Although Bishop (1994) starts from cultural *conflicts,* this paper prefers to be more cautious and talks about intercultural misunderstandings. Considering misunderstandings in mathematics classrooms is an interesting starting point not only because language problems cause many misunderstandings, but also for systematic methodological reasons: In the international (linguistic, psychological and pedagogical) research on intercultural communication, starting from misunderstandings has proved to be an important approach for analysing and understanding intercultural situations. Hence, the intercultural misunderstanding has become a central category for research (cf. Ehlich, 1994, p. 925).

How can the importance of this approach be explained? In most intercultural situations in which language problems appear, there are also other reasons transcending the language level. Most frequently, they are located in other dimensions of intercultural interaction, for example in diverging underlying systems of values or interpretations. This phenomenon can also be found in intercultural interactions in mathematics learning situations as illustrated with the following little case example. It has been described and analysed by Heymann (1996) as a typical example of linguistic misunderstandings:

> While doing her homework, Katharina orderly followed the rules for operating with fractions and divided the number 2 by 1/4. Then, she came asking me since she was surprised by her result, 8. How could the quotient be greater than the dividend, although she had 'divided'?
>
> I tried to explain why it had to be like that (within the range of positive numbers) whenever we divide by numbers less than 1. As a counter-example, she confronted me with the fact that whenever she divides an apple 'in quarters,' the pieces are smaller than the whole apple. I tried to point out the difference between 'divide in quarters' and 'divide by quarters.'
>
> Finally, she said: 'Okay, now I know how to calculate it. But you cannot tell me that thinking in mathematics is logical!' (Heymann, 1996, p. 206)

Katharina's reproach that mathematicians do not think logically is based on her understanding of "unlogical" in the sense of missing connections to her individual reasoning. She feels a discrepancy between her thinking and the expected mathematical thinking.

Unlike Katharina, Heymann interprets this episode not as a matter of different thinking but as a misunderstanding that is mainly caused on the level of language (Heymann, 1996, p. 206ff). Her everyday understanding of the expression "divide" (as a sharing process, interpreted in the partitive model, cf. (Kirsch, 1970) and (Fischbein et al., 1985)) only covers a part of the mathematical meaning. In mathematics, the meaning comprises other aspects, for example "fits in" or "divide up" as a measurement division, interpreted in the quotative model ("How many quarters fit in two wholes?"). Thus for Heymann, it is a simple language problem rooted in different meanings of "divide" in mathematical language and common language.

A deeper analysis shows that it is more. Both conceptions, the partitive and the quotative model, are already relevant for natural numbers and for everyday situations. Common language phrases with "divide" can also be built in the quotative model: "Two apples are divided into portions of a quarter of apple." But although both aspects can appear, Katharina's understanding of "divide" is restricted to the first aspect. Apparently, this has not posed any problems as long as Katharina has calculated with natural numbers. But for the extension to fractions, her understanding is too narrow. For fractions, mainly the measurement division applies whereas the sharing division cannot be appropriately interpreted anymore (see Fischbein et al., 1985).

This problem is not evident for Katharina since she does not consider the expression "divide" as an element of a foreign language which has to be interpreted carefully. For her, it is a common language expression that she is familiar with. That is why she refuses a communication on the meaning of the expression in the mathematical language. This phenomenon of *interferences of meaning* shall be further explored as one example for *effects of overlapping* (in Section 1.3).

Beyond the interferences of meaning, there are two more important phenomena in this episode, the first of which is a typical reaction on the affective level. Katharina's refusal reaction is based on her *feeling of foreignness* for which she cannot find another way to cope ("But you cannot tell me that thinking in mathematics is logical!"). Psychological studies of intercultural conflicts have shown that this refusal is a typical pattern of behaviour in situations of foreignness (Breitenbach, 1979). As this kind of foreignness often appears in mathematics classrooms, it is worth drawing a further analysis (Section 1.4).

The (nonquoted) continuation shows that one reason for Katharina's feeling of foreignness is that she cannot understand the idea of why the meaning of the division must be shifted to the quotative model when the domain of numbers is extended to fractions. For her, it is an arbitrary convention that is irritating and hence, in her words, "not logical". If she had understood the underlying purposes, she would have perhaps been able to overcome her disapproval. Thus, it is not only the expression "divide" that causes misunderstandings but also the underlying diverging aims and purposes.

For a better understanding of this phenomenon *diverging aims and patterns of interpretation,* one can learn from the research on intercultural communication (Ehlich, 1994; Breitenbach, 1979). In their studies, diverging patterns of interpretation have turned out to be one major reason for barriers in conversation between members of different cultures (as in our example Katharina and the mathematically enculturated professor Heymann). Various empirical studies have shown how different cultural backgrounds impede the finding of common interpretations of situations due to missing common implicit background knowledge (more details in (Leenen & Grosch, 1998, p. 31f).

In mathematics education research on interactions, several barriers for communication in mathematics classrooms have been analysed in kindred theoretical frameworks. Some effects have been explained by means of the notions of "domains of subjective experiences" and "frames" without adopting an intercultural perspective. The underlying approaches of explanation start from individual learning biographies (see, e.g., Bauersfeld, 1983) or frames being situatively constructed within the interaction (e.g., Cobb, 1994). Although these approaches are useful and important, they should be extended by an intercultural perspective because some important barriers for communication are not only rooted in individual problems or in the specific interaction. Instead, they have their roots in features characteristic of the culture of mathematics. In Katharina's case, it is the mathematical way of extending and generalising operations to wider domains of numbers that enforces the restriction of meanings.

To sum up, when we analyse communication in mathematics classrooms from an intercultural perspective, we find that diverging cultural patterns are important causes of misunderstandings. The following sections describe the often accompanying effects of overlapping and feelings of foreignness.

1.3. Effects of Overlapping

As with every technical language, the mathematical language is based on common language. It borrows many expressions from common language but uses them in specifically defined meanings which sometimes differ completely from their original meaning. These differences can cause learning difficulties as Vollrath has described for geometrical concepts (Vollrath, 1978). His findings are quite similar to those of studies on intercultural communication situations. Problems of intercultural understanding do not only originate in the confrontation of different languages but mainly arise from their interpretation, categorisation and valuation according to the proper patterns of expectation. In linguistic research, this wrong transfer of meanings from the common language into the foreign language is called *interferences of meanings.* They are carefully considered since these interferences can heavily burden intercultural interactions (cf. Leenen & Grosch, 1998, p. 32).

Interferences of meanings have also been analysed for expressions of the mathematical language (Maier & Schweiger, 1999). It turned out that nearly all mathematical expressions have at least a slightly different meaning in common language. As a consequence, “students already possess a certain understanding of the introduced mathematical terms, but one that differs from the term’s mathematical meaning” (Maier & Schweiger, 1999, p. 121). Sometimes, the everyday meaning is wider or more general than the mathematical meaning (like for “similar” which is restricted to figures of the same form in mathematics). Sometimes they are more specific than the mathematical meaning (like for “quadrangle” which in German common language is used synonymously for rectangles or squares). Some follow other systematics (like the use of furniture being some inches “wide, high and deep” instead of “long, high and wide”, which mathematicians use for parallelepipeds). These differences may produce an interference where the students' everyday understanding of an expression disturbs the mathematical understanding (Maier & Schweiger, 1999, p. 122).

Even more important than these kind of *linguistic interferences* are those on the *level of conceptions* and *ways of thinking.* Students do not only activate their everyday understanding of technical terms but their prior conceptions on all levels. That is why mathematical classrooms sometimes operate in a tension between everyday culture and the mathematical culture of thinking.

In mathematics education research, individual constructions of meanings have been studied, for example by means of the notion of tacit model (Fischbein, 1989), by confronting basic mathematical ideas and individual images (vom Hofe, 1998) or by the notion of domains of subjective experience (Bauersfeld, 1983). Whereas these studies focus on the effects of overlapping which are produced by *individual* preconditions (partly grown within mathematics classrooms as for vom Hofe's individual images), increasingly papers focus on *collective* preconditions being rooted in everyday culture. Authors of these studies refer to very different aspects of the mathematical culture:

- lingustic expressions like similarity (Vollrath, 1978); calculating strategies (Nunes et al., 1993; Saxe, 1991; De Abreu, Bishop & Pompeu (1992); and others);
- mathematical conceptions like chance and probability (Fischbein, 1975); symmetry (Hartmann, 2002), or functional dependency (Lengnink, 2002);
- general strategies like ordering or the idea of condensation (Lengnink & Peschek, 2001), activities like generating, collecting and comparing (Lengnink & Prediger, 2000);
- general considerations about the relation between mathematical thinking and common sense (Keitel et al., 1995).

We can interpret these different studies as empirical evidences for the perspective that mathematics learning sometimes takes place in a situation of intercultural overlapping. While encountering mathematics, students activate their everyday experiences which are then confronted with mathematical conceptions. Sometimes, they are changed or replaced, sometimes everyday and mathematical conceptions are both maintained and activated due to the specific situative contexts. These situations of overlapping can cause difficulties as has been described many times (Nunes et al., 1993; Bishop, 1994). But they also offer chances as Bishop (2002) has pointed out later. The chances of reflecting on the effects of overlapping for developing mathematical literacy have been described by Lengnink and Peschek (2001). For the cultural dimension of such kinds of overlapping effects, there is still an enormous need for careful research in which the chances should be especially emphasised (see Section 3.3).

1.4. Foreignness as a Category of Experience

Just like Katharina in the case example above, many students encounter mathematics as a different, sometimes foreign culture. For some of them, the experience of foreignness is so strong that they struggle with mathematics for their whole life. Extreme forms of mathematics phobia or mathematics anxiety have been described and analysed by mathematics educators (like Baruk, 1985) and psychologists (Ashcraft et al., 1998). Mostly, these extreme forms are combined with a deficient self-attribution with respect to mathematics.

For finding a conceptualisation of this phenomenon, it is worth considering intercultural psychology research again. In their theoretical frameworks, foreignness has always been an important category since the encounter of another culture often produces experiences of foreignness. But foreignness is not considered to be an attribute of a culture, it is instead conceptualised as a *mode of relation* of members of one culture towards another culture (see Auernheimer, 1998, p. 20). Hence, foreignness is constructed in processes of attribution. Analogically, it is instructive to think of mathematical foreignness as a category of experience.

As in other intercultural processes, the experience of foreignness cannot only produce disapproval or rejection but also fascination. It has often been described that mathematics is a polarising subject, being the favorite subject for many students and the least favorite subject for others. Both phenomena can be understood from an intercultural perspective. Learners activate different strategies for interculturality, i.e. they develop different ways of coping with differences and foreignness (between the everyday culture and the culture of mathematics). Rejection is a pattern of reaction which is as typical as fascination (Thomas, 1988, p. 86f). Adopting this perspective can help to find an adequate way of coping with moments of foreignness when mathematical learning processes are arranged.

To sum up: Bishops observation of “cultural conflicts” must be differentiated by several features of mathematical learning processes. Language and non-language misunderstandings, effects of overlapping and moments of foreignness are characteristics we can find when we analyse learning processes from an intercultural perspective. For a better understanding of these phenomena, conceptualisations of psychological and pedagogical research on intercultural situations proved to be instructive. In the following sections, this descriptive parallel offers the frame for discussing prescriptive questions, especially on the adequate arrangements of these intercultural situations.

2. Conceptualising Mathematics Learning as an intercultural process - theoretical consolidations

2.1. Mathematics as a Culture -Aspects of the Underlying Philosophy of Mathematics

Stating that mathematics learning has intercultural features implicitly assumes that mathematics is a culture. For my work, Bishop's assumption has proved to be useful since it offers a base for explaining some important phenomena in learning processes (see Section 1*)*. Nevertheless, the cultural perspective on mathematics is not that clear as it first appears within Bishop's metaphoric use. In what sense mathematics is a culture? And on which part of mathematics does this perspective focus, on the scientific discipline, on school mathematics or on everyday mathematics? These questions are worth treating extensively (see Prediger, 2002, 2004, 2005 for more details). Here, at least some aspects of the philosophical background shall be made explicit.

The category of culture has entered the philosophical discussion about mathematics mainly with the emerging position that mathematics should be considered as a cultural product (cf. Tymoczko, 1985; Hersh, 1997; Ernest, 1998; Bishop, 1991). Especially in humanistic or social constructivist positions, mathematics nowadays is understood as a socio-cultural phenomenon. Emphasis is put on both parts. "Product" accounts for the fact that mathematics cannot only be discovered but must be created by humans. This creation always takes place in a specific cultural setting, thus it is a "cultural" product. The idea of mathematics being a cultural product is linked with a number of important developments in the history, philosophy and social studies of mathematics, especially the insight of the fallibility of mathematics, and the emerging view that the social context and professional communities of mathematicians play a central role in the creation and justification of mathematical knowledge (cf. Ernest, 1999, p. 67ff, for a more detailed description).

Leslie White first formulated the position that mathematics is not only a cultural *product* but a *living culture* (White, 1947). This shift of stress has various implications concerning especially the processual character of mathematics, the importance of implicit knowledge, mathematicians' practices and the sociological dimensions of the discipline (cf. Tymoczko, 1985).

White's conception can be deepened by well founded theories on *sciences as cultures* developed in higher education research and cultural studies (cf. Arnold, 2001). They emphasise that the identity of a discipline, e.g., mathematics, is formed not only by its corpus of knowledge, but by many other elements having often been neglected in considering scientific disciplines:

> their traditions, customs and practices, transmitted knowledge, beliefs, morals and rules of conduct, as well as their linguistic and symbolic forms of communication and the meanings they share. (Becher, 1993, p. 24)

These elements form what is called in sum a "scientific culture." Or, more illustratively: "The notion comprises everything a student must acquire in order to become an accepted member of the community" (Arnold, 2001, p. 2). Various historical and sociological studies have shown that the cited components of a scientific culture have great influence on the growth of mathematical knowledge (Wilder, 1981; Ernest, 1998; Heintz, 2000) and that development in mathematics can be understood as transitions of selected components of the prevailing mathematical culture (Kitcher, 1984). These studies give reason for adopting a cultural perspective on mathematics, hence considering mathematics to be a *scientific culture.*

This cultural perspective on mathematics must be widened in several aspects: First of all, as far as mathematics learning in *school* is concerned, pupils are obviously only partly confronted with the culture of the *scientific* discipline as it exists in universities. Taking into account important differences (that cannot be discussed here, they are one main subject for example in (Brenner & Moschkovich, 2002); see also (De Abreu, Bishop & Presmeg, 2002)), we can nevertheless consider school mathematics as a culture itself. It is formed by the same components (with different specifications) as the scientific culture, including all parts of implicit knowledge (like language, shared understandings, norms, questions accepted to be relevant), roles, forms of communication, habitus etc. Although the cultures of school mathematics immensely vary all over the world and even within one country, this article keeps relating to "the" culture of mathematics because in the focus of this study are not the international differences but the fact that every *individual* pupil is confronted with *one* culture of mathematics in his classroom. It is an important research agenda to study the different characteristics of school mathematical cultures and their impacts in details. Although empirical research has already informed us about the importance of some more or less isolatedly considered aspects (e.g., socio-mathematical norms (Yackel & Cobb, 1996) or ways of negotiation of meanings (Voigt, 1994)), we are far away from understanding in depth what characterises the different school mathematical cultures. Nevertheless, this research agenda is not part of that paper.

In order to be more precise about the *function* of the mathematical culture for humans, it is instructive to rely on Alexander Thomas' definition of culture as a "group-specific *system for orientation"* which influences perceiving, thinking, valuing and acting of all members. Thomas describes its function as follows: "The system for orientation allows a specific way of managing life and environment for the members of the group." (Thomas, 1988, pp. 82-83). This description not only applies to the scientific culture or the culture of school mathematics, but also to everyday mathematics, being a part of the main culture we live in.

The underlying idea of considering mathematics as a specific approach to the world and as a tool for problem-solving is common in mathematics education. Here, it is meant in the broad sense of (school and everyday) mathematics as a "system of codification" which allows describing, dealing, understanding and managing reality (D'Ambrosio, 1999, p. 52; cf. Bishop, 1991).

Unlike the culture of the scientific discipline with its quite clear institutional locus, everyday mathematical culture as a cultural system for orientation cannot be delimited in absolute terms. What counts as mathematical thinking and what does not? Where is the borderline? Since there are no absolute limits between the mathematical (sub)culture and the everyday culture (as they are also not for any culture!), every comparison or confrontation between elements of the mathematical culture with elements of everyday culture can only be done for analytical reasons. This difficulty is well known in cultural studies (see Bolten, 2001, p. 21); nevertheless, the absolute delimitation is rarely needed for the analysis of intercultural interactions.

The sketched cultural perspective on mathematics is general in the sense that the described aspects also apply to every other school subject. The section concludes with some hints on the characteristics which are *specific* to mathematics: How does mathematics differ from other sciences and subjects in school?

To understand mathematics as a culture means to acknowledge the human origins of mathematics. From this perspective, there is no principal difference in the epistemic status between mathematics and other sciences or arts. Nevertheless, there is a difference in the ontological status of objects and especially in the way individuals *experience* mathematics. Whereas in other disciplines the human and cultural roots of the discipline are still visible, in mathematics, the individual is confronted with an depersonalised, objectively accessible theory, an apparently unchangeable corpus of ideas, notions, and theorems (cf. Burton, 1995). It seems to be characteristic for mathematics that it has apparently been disconnected from its human and cultural roots.

The duality between the human inventions of mathematical objects and their apparent objective reality was explained by White who located the mathematical reality in the inter subjective world of culture. Mathematical formulas, like other aspects of culture, do have an "independent existence and intelligence of their own" (White, 1947, p. 295), although they are human inventions. In his view, "mathematics does have objective reality" but "this reality [...] is *not* the reality of the physical world, [...] its reality is cultural" (White, 1947, p. 302f).

Even if the cultural perspective clarifies the situation, it cannot account for the phenomenon that mathematics is perceived in a way different from other scientific disciplines. Although there are different approaches for explaining this phenomenon (cf. Prediger, 2005), it remains an important characteristic whenever individuals are engaged with mathematics.

2.2. Intercultural Competence as an Important Aim for Mathematics Education - the Normative Orientation

Although the intercultural perspective on mathematics learning has been formulated in a descriptive mode so far, it also gains prescriptive power since it changes the importance of well known phenomena. For example, moments of foreignness, misunderstandings and effects of overlapping are not only perceived to be side issues which disturb the "normal" learning process, but natural, integral parts of every encounter with mathematics. For science education, Hawkins and Pea have emphasised the prescriptive power of similar descriptive theories:

> A major hypothesis of this framework is that explicit recognition of the cross-cultural nature of science education has rich and unexpected implications for improving its theories and practices. The framework makes sensible new orientations to students' difficulties, because it leads to examination of new contributory factors and different pathways out of the problem of science difficulties. (Hawkins & Pea, 1987, p. 298)

The prescriptive dimension of this conceptualisation of learning is not only an automatic outcome but also an explicitly intended aspect of this paper's approach. In contrast to some other epistemologies that only focus on the descriptive understanding of the complex phenomenon "mathematics learning", this perspective aims at being activated in the prescriptive mode for decisions on how to design mathematics learning arrangements appropriately.

But prescriptive consequences cannot be drawn only from descriptions of learning processes. Instead, they should be canalised by normative orientations about the aims of mathematics education. So, which normative orientation is compatible with the descriptive perspective on mathematics learning as an intercultural process? What are the general long-terms aims of students encountering mathematics beyond the elementary level?

This paper's starting point on the normative level is to question the big goal of a mathematical enculturation for all (Bishop, 1991). When students go to France on an intercultural voyage for two or three weeks, nobody would claim that they should become French locals in this time. Analogically, higher mathematics classrooms should not claim that they expect all students to become mathematicians within the limited time and subjects. Although the metaphor of mathematical enculturation is very instructive to describe certain aspects of the *process* of mathematics learning, it is not suitable for describing the *aim* whenever mathematics beyond the elementary level is considered. Instead of aiming for educating future experts in mathematics, this paper concentrates on the education of mature, responsible citizens and the mathematical literacy (in the sense of Jablonka, 2003) they need outside the mathematics classroom. For this reason, the normative category "intercultural competence towards mathematics" shall be explored.

In general, intercultural competence is specified as the competence to communicate successfully with members of other cultures (cf. Hammer, 1989; Leenen & Grosch, 1998, p. 39). In this sense, intercultural competence towards mathematics means the competence to communicate with persons that activate the mathematical system for orientation such as engineers or economists.

The cognitive aspects of this normative orientation have been formulated and justified within the conception of general mathematics education developed by Roland Fischer (2001). Fischer postulates that the "ability to communicate with experts and with the general public" must be the main aim for secondary mathematics education. This necessity is legitimatised by socio-philosophical reasons: Our highly differentiated, democratic society is based upon the division of labor (Heymann, 1996, p. 113) and depends on an emancipated contact with highly specialised expert knowledge. Hence, every citizen should be able to communicate with experts whenever s/he is confronted with expertises. Usually, non-experts can rely on the professional correctness of the experts' statements, but not on the relevance for their specific individual purpose. Thus lay-persons should be able to evaluate the relevance and importance of a given expertise for specific individual purposes (Fischer, 2001, p. 153). Fischer's position about the importance of evaluating mathematics instead of developing mathematisations is well rooted in the actual international discussion about mathematical literacy (see (Jablonka, 2003) for a survey on the discussion about mathematical literacy).

For developing this competence, two areas of knowledge must mainly be acquired: basic knowledge and abilities about mathematical terms, concepts and forms of representation as a prerequisite for communicating with experts, and reflective knowledge concerning possibilities, limits and the meaning of terms, concepts and methods which is necessary for judging their expertises. More concrete is the simple example of Katharina's equation "2 : 1/4 = 8" (see Section 1.2*):* Developing the ability to communicate with mathematicians about this equation would include the basic knowledge about different semantic interpretations of division (the partitive and the quotative model). Reflective competence would be expressed for example by the following questions: "Can you give me an example which makes sense of the quotient being greater than the dividend?", "what is the difference to what I have thought?", "why do you use division in this context, what is the advantage of it?".

Although the orientation "ability to communicate with experts" still needs to be made specific for all mathematical contents, it already offers an instructive orientation on the cognitive level. Beyond that, the research on intercultural psychology has found further important aspects of intercultural competence. These further aspects mainly concern personal attributes and attitudes (Luchtenberg, 1999, p. 215f).

The first prerequisite for competent communication with experts is the *will* to get involved in mathematical argumentations. This includes the courage to communicate with mathematical experts and with it the stable self-confidence with respect to mathematics. This is the opposite of naive belief in mathematical objectivity or a self-attribution of incompetence whenever mathematical elements appear.

Another important aspect of intercultural competence has been described as *cultural awareness* (Tomalin & Stempleski, 1993). Cultural awareness with respect to mathematics means consciousness of the fact that the mathematical orientation-system is one specific approach to the world among others. For achieving this consciousness, students should be able to reflect on differences and parallels between mathematical ways of describing phenomena and others. They should understand mathematics as a system for orientation which influences perceiving, thinking, valuing, and acting in a specific way, with its specific benefits and characteristic limits (this consciousness and this knowledge have been defined to be an important part of the mathematical literacy construct in the PISA-framework, cf. (OECD, 1999, p. 43)).

These multi-layered aspects (which are explained in more detail in (Prediger, 2004)) briefly show how the category of intercultural competence mirrors the complexity of the underlying understanding of mathematical literacy. Besides the basic and reflective knowledge in Fischer's framework, the focus is on cultural awareness and the attitudes which are necessary for activating the acquired reflective knowledge.

3. Orientations for Arranging Mathematics Learning as intercultural learning - some prescriptive consequences

Even if we presume that mathematics learning mostly takes place in a situation of intercultural overlapping between everyday culture and mathematical culture, we still know that even then, intercultural competence does not develop automatically. Processes of intercultural learning mostly have to be initialised explicitly by adequate learning arrangements. This insight has been one important factor for the development of research on intercultural learning in the international exchange movement (Müller, 1987). When this section asks for didactical orientations for arranging mathematics learning in a way that it can contribute to the development of intercultural competence towards mathematics, it can profit from many ideas and results that intercultural pedagogy has found after the mentioned disillusioning insight.

3.1. An Illustrating Episode: The Center of a Triangle

For explaining these ideas, this section starts with revisiting an instructive example that has been presented by Hefendehl-Hebeker under a slightly different perspective (Hefendehl-Hebeker, 1995, p. 87-89).

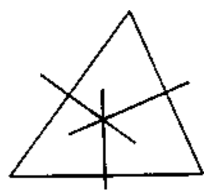
Geometry lesson held by a teacher student (called Hans here) in a grade 7 of a German Gymnasium. The lesson starts with a presentation of the homework, in which the midperpendiculars of a given triangle were to be constructed by compass and ruler. The phase of construction on the blackboard ends with teacher's question:

Hans: What is interesting in this figure?

Jörg: The midperpendiculars all meet approximately in the center!

Hans: Okay. *[Writes on the blackboard]:* Assumption: The midperpendiculars of the triangle ABC all meet at one point.

Jörg: But this is no assumption, it is true. The assumption is only that they meet in the center!



For an answer, Hans explains carefully that the lines of the drawn figure are quite thick and are thus not sufficient for deciding whether the midperpendiculars really all meet at one point. Then he continues searching for a proof in a teacher-guided frontal teaching sequence. Jörg does not really pay attention to it and prefers playing with his compass. (Hefendehl-Hebeker, 1995, p. 87, shortened)

In her analysis of this scene, Hefendehl-Hebeker points out different didactical problems appearing in this scene. Adopting an intercultural perspective here, most interesting is the intercultural background of the misunderstanding between Hans and Jörg: Jörg is not concerned with the *existence* of a common meeting point, but with its position "in the center". However, for the teacher student Hans, Jörg's answer gives a reason to continue his program by formulating the prepared assumption without taking up Jörg's remark about the center. When Jörg contradicts, Hans tries to convince him of the necessity of proof with the hint that one should never simply trust drawn figures. This scene has typical features of an intercultural misunderstanding. It follows general patterns described by the science education researchers Hawkins and Pea:

> As in cross-cultural processes of understanding, challenges arise for making sense of why people of a foreign culture do what they do. [...] The cross-cultural nature of what is taken to be 'problematic' and what is considered 'explanatory' among teachers and students is often observed as important source of educational problems. (Hawkins & Pea, 1987, p. 297f)

In Jörg's view, evidence and the need for proof are accentuated in a different way from that of the teacher. He focuses on the phenomenon "center". Hans, in contrast, cannot value the discovery of the phenomenon "center" since from his mathematical perspective, the center of a triangle is a problematical conception (as he knows for example that there exist triangles with midperpendiculars meeting outside the triangle). The proof for the existence of a common meeting point is important for Hans since in mathematics, proofs not only give evidence for the correctness of an assertion but they also draw connections to prior theorems and fundamental axioms. Thus, the discussion of whether the assumption needs to be proved or not is rooted in diverging interpretations of the proof's *function* (see (de Villiers, 1990) for different functions of proof in mathematics). Jörg activates an everyday understanding of reasoning for verification purposes. For him, a proof is basically supposed to convince, and thus, it is not necessary if one is already convinced. Hans operates within the mathematical culture in which a proof is mainly supposed to draw logical connections between different parts of the mathematical theory. Instead of discussing these divergent backgrounds, Hans tries to question Jörg's evidence by pointing out that Jörg should not put his trust in a single drawn figure.

3.2. Handling Misunderstandings: The Conflict Approach

Adopting an intercultural perspective changes the view on conflicts like the one between Hans and Jörg. Misunderstandings and irritations are often perceived as disturbances of a process that is supposed to be - ideally

- without any difficulties. In contrast, the intercultural perspective suggests that misunderstandings, conflicts and the experiences of foreignness are normal, integral parts of every situation of intercultural interaction (cf. Haumersen & Liebe, 1990, p. 36f).

This understanding is based on the assumption that there exist differences between the learner's everyday culture of thinking, arguing and perceiving and the mathematical culture, and that these differences influence the learner's encounter with mathematics. The assumption of differences is also fundamental for many pedagogical approaches to intercultural education (Auernheimer, 1998, p. 19f). With respect to mathematics, the assumption of differences is adopted here not as a statement about the nature of mathematics in a philosophical sense (discussed in (Lengnink & Peschek, 2001)), but as an empirically grounded starting point on the level of *individual experience* (see Sections 1.4 and 2.1*)*. The individual encounter with mathematics is often characterised by the experience of differences, no matter where they come from.

Teachers can try to reduce these experiences of difference by an adequate organisation of the learning environment and by overcoming misunderstandings. Nevertheless, it is impossible to smooth all discontinuities and differences in order to eliminate conflicts. In the discussion between Hans and Jörg for example, Hans could have considered more than one triangle and could have challenged Jörg's idea of the center. It might have even been possible to convince Jörg that the existence of a meeting point is not evident for all triangles. But although the conflict might not have been apparent then, the underlying divergent functions attributed to proofs would not have been overcome.

For a constructive way of coping with such situations, intercultural educators have developed the so-called conflict approach (Haumersen & Liebe, 1990). This approach considers intercultural learning to be effective when intercultural conflicts can be exploited for learning instead of smoothing them. Thus:

> Irritation is the starting point for a situation with the potential for intercultural learning. The dynamics of an international interaction is rooted in the cultural differences which cause a break with normality. [...] When intercultural conflicts are admitted and the intercultural backgrounds are explored, than intercultural learning can start. (Haumersen & Liebe, 1990, p. 25f)

This idea can be transferred to mathematics education where tendencies of harmonising differences and conflicts are also strong. Guided by the goal of making mathematical contents more accessible, many didactical approaches try to ignore or to deny differences between mathematical thinking and everyday thinking. This article is a plea for facing them instead of smoothing them.

If Hans had not ignored the diverging understanding of reasoning in Jörg's and his own argumentation, he could have said, "okay, let us not call it 'assumption' but 'observation.' Although you are already convinced by it, I would like to know *why* the midperpendiculars all meet in one point, even if you see *that* they do". This could have been the starting point for an interesting comparison of verificative and explicative functions for reasoning in everyday thinking and in mathematics (see de Villiers, 1990). In pointing out the differences and similarities between both, Hans could have contributed to Jörg's development of cultural awareness towards mathematics and to his acquisition of important reflective knowledge.

In a similar way we can consider Katharina in the example in Section 1.2. When Katharina wonders about the fact that 2 divided by 1/4 is greater than 2, then she should know that this is not only her individual misunderstanding, but rooted in different interpretations of division. Her individual misunderstanding is rooted in the extension of the known operation of division to the new numbers (fractions) that required a shift in interpretation. What first appeared to be her personal problem turns out to be a necessary break that is directly connected with the transition from naturals to fractions since Katharina's everyday understanding of division cannot apply for fractions. This clarification might have prevented her from experiencing foreignness in this negative way and might have given her the chance to get to know general ideas about the extension of mathematical operations and the entailed loss of interpretation.

In both examples, reflecting on the background of the conflict gives rise to learning chances on a meta-level that are very important for the development of reflective competences. Similarly, facing irritations on the affective level (like negative experiences of foreignness) can offer opportunities for intercultural learning. For example, one of the author's Grade 10 students, called Anna here, once asked a question full of frustation about solving algebraic equations: "What has this solution-machinery got to do with *me?"* Although Anna mastered solving equations on the technical level, she was searching for *individual access* to it. My first answer ("Algebraic equations help us to solve real-world problems we could not solve without them") was too easy. The reference to the equations' usefulness was not what Anna was looking for. Getting engaged in a discussion why she could not find any personal access to the technique of solving equations, it became interesting: "When I do that, I feel like a machine. My thinking is not demanded here!" By discussing this phenomenon she could realise that this is indeed one important characteristic of algebraic transformations. Rule-guided manipulation of algebraic expressions can be done without "thinking", i.e. without any reference to a context. Krämer has put this in strong words when she described the historic development of the idea of formalisation as "the history in which we learned to act like machines when we operate with symbols" (Krämer, 1988, p. 4). And it is exactly the lack of interpretation that makes a mechanical calculus possible. The mechanical calculus is independent of individual thinking, but this is its function and a typical feature of mathematics. Along these lines of discussion, the self-reflection ("Why don't I like solving equations?") opened a way to a big idea of mathematics: the idea of algorithm, more generally the idea of formal operations without reference to contexts. Again, the conflict approach offered the possibility for acquiring reflective knowledge about the culture of mathematics.

3.3. Exploiting Overlappings: Building Bridges

Hans and Jörg's discussion offers an example of a situation of intercultural overlapping in which the student argues on the basis of his everyday understanding. This phenomenon has been raised in the theory of *conceptual change* (a wide, and partly contradictory field of research. Here, we mainly refer to (Strike & Posner, 1982)). The notion "conceptual change" stands for the empirically grounded fact that "learning of scientific conceptions usually means *relearning* since everyday conceptions and scientific conceptions are often opposed in central aspects" (Duit & von Rhöneck, 1996, p. 158). In order to facilitate the process of conceptual change, science educators search for pathways of learning that start from the learner's everyday conceptions, pass over intermediate conceptions and finally reach the scientific conceptions (Hawkins & Pea, 1987; Duit, Goldberg & Niedderer, 1992). Conceptual change approaches do not intend to replace everyday conceptions by scientific conceptions completely, but they intend to initiate a shift of contexts in which each of them are activated.

How can the idea of finding pathways with intermediate conceptions be transferred to mathematical subjects? A speculative continuation of the episode with Jörg and his idea about the center of a triangle can illustrate it:

Let us have a look at what happens when we engage with the issue of the center and set out its phenomenology. What is the center of a (two-dimensional geometric) figure? Can we directly say about any figure that they have a center?

- The circle has a 'total' center. It possesses one point with equal distance to all points on its boundary.

- Every regular polygon 'inherits' the center of its circumcircle. This mid-point does not have equal distance to all boundary points but at least equal distance to all vertices of the polygon.
- Also the rectangle has such a mid-point. It can be constructed as the meeting point of the diagonals (that bisects the diagonals).
- Diagonals bisect each other also in parallelograms. The meeting point of the diagonals is not equidistant to all vertices. But it has another nice center attribute, it is the center of symmetry.

In this phase of exploration, intuition and conceptual specification interplay. As an outcome, we gain different aspects as to what the center of a figure might be: center of symmetry, a point with equal distance to all vertices (if the figure is a polygon), a point with equal distance to all boundary points. Additionally for polygons, we can ask for the point with equal distance to all sides.

In which respect can we talk about the 'center' of a triangle? The most radical center-criterion only applies to the circle. The second criterion of equal distance to the vertices seem to make sense. It can be explored by searching for this point in a given arbitrary triangle ABC. That is not trivial for eight-graders. Instead, it demands a systematic treatment, for example by considering only A and B first (then, all appropriate points are on their midperpendicular). The next step considers B and C and finds the suitable point as the meeting point of the two midperpendiculars. This meeting point turns out to be on the third midperpendicular as well.

The adopted strategy first focuses on one part of the criterion. Then it restricts its solution by the second part of the criterion. In this way, the search for a suitable point resulted in the theorem of meeting midperpendiculars and generated the arguments for its proof.

Testing further triangles can increase the confidence in the generality of the statement. Simultaneously, the exploration of further examples destroys an illusion. The point that we were ready to accept as the center of a triangle can lie outside the triangle! (Hefendehl-Hebeker, 1995, pp. 88-89, here shortened and liberally translated)

With these considerations, Hefendehl-Hebeker gives a good example about possible pathways from Jörg’s idea of the center towards the theorem of meeting midperpendiculars. The constructed intermediate conception is “The meeting point of the midperpendiculars is the center of the triangle in the sense that it is the same distance from all vertices.” The examination of this conception with further triangles produces the cognitive conflict that can motivate Jörg for a further conceptual change.

The existing situation of overlapping can be exploited constructively only by the explicit bridge being built between the idea of the center and the theorem to be proved. Jörg can understand the connection and differences between his interest in the center and Hans' interest in the meeting of midperpendiculars. By tracing this pathway, he can experience why his idea is not sustainable from a mathematical perspective. Beyond that, he gains important cultural knowledge about mathematics. In mathematics, phenomena are not only described intuitively but these intuitions are specified in a precise manner. The precise mathematical definition allows a further exploration that sometimes brings up unforeseen consequences. As far as the development of intercultural competence towards mathematics is concerned, the process in which he can experience the development of a mathematical concept with its specification and losses is even more important than the resulting theorem of meeting midperpendiculars.

Unlike Hefendehl-Hebeker who analysed this episode as an example of the vertical process of mathematical knowledge development, this situation can be considered horizontally as an intercultural interaction. In this perspective, the emphasis is put on the possible simultaneity of different perspectives. Jörg's interest in the phenomenon of center is not necessarily *replaced* by the knowledge that midperpendiculars do not always meet in the center. Instead, it is *enriched* by the knowledge about how and with what consequences mathematicians try to formalise criteria for the center. As Hefendehl-Hebeker mentions briefly (ibid.), Jörg could also decide to continue the search for other criteria that fit better with his initial idea of the center (like the center of gravity as the meeting point of the side-bisectors).

Jörg would have developed intercultural competence towards geometry if he could conclude like the following: "I am interested in the phenomenon 'center,' and this can be mathematically formalised in different ways. None of these ways exactly fit to what I originally wanted to say since in the end, 'center' is an aesthetic question that cannot be forced into simple rules for all triangles." In this way, Jörg could have encountered the different interesting points in the triangle as different mathematical models for his idea and would have decided that none of them is really adequate.

All these insights can only be found when the bridge between Jörg's thinking about the center and mathematical ways of describing this phenomenon is explicitly constructed. This idea has been described by intercultural pedagogues with the sentence that intercultural learning mostly takes place "in between" the cultures: Intercultural learning is a "learning in intermediate spaces" (Holzbrecher, 1997, pp. 169-244). Although situations of intercultural overlapping always exist, these instructive intermediate spaces must explicitly be constructed in the learning process.

Hefendehl-Hebeker's thought experiment on the center of a triangle shows one possibility of how to transfer the conceptual change approach of learning with intermediate conceptions to mathematics education. For many other mathematical concepts, the transfer is less direct due to its more abstract character. Scientific concepts usually explain real-world phenomena for which everyday conceptions exist (as for example "energy" that can be "wasted" in everyday conceptions but not in physical models).

Although there exist some mathematical concepts of this ontological quality (for example “chance”, cf. (Fischbein, 1975)), the phenomenological dimension of most concepts of higher mathematics is more hidden, thus the connection to the everyday thinking first has to be reconstructed.

This has been done for example for the concept of function (Lengnink, 2002). In order to find connections between the mathematical concept of “function” and everyday thinking, Lengnink first reconstructed the concept’s meaning and the underlying basic ideas. She decided that the conception of mapping is less suited for building bridges than the conception of functional dependency. Her learning sequence then started with a general consideration of dependencies between quantities. In this phase, students were stimulated to make explicit their prior conceptions on dependency. The mathematical concept of functional dependency was described as one conception of dependency among others. The students had the opportunity to think about which aspects of their thinking can be grasped by means of this mathematical concept and which cannot (for example causal dependency).

The claim of picking out similarities and differences between mathematical and non-mathematical concepts and strategies as a central theme by building bridges has a quadruple background: Building bridges can help to decrease difficulties in overlapping situations because implicit divergent conceptions can be made explicit. This can help to clarify or avoid misunderstandings (see Section 3.2).

The pathway from everyday conceptions to mathematics can help to “anchor” mathematical concepts and thinking strategies within individual thinking. This effect for learning mathematics has often been emphasised (cf. Civil, 1998; Bonotto, 2001).

Reciprocally, the bridge can facilitate the transfer back. Students can acquire the ability to activate mathematical concept in out-of-school situations only if mathematical concepts are contextualised in suitable non-mathematical contexts (cf., e.g., Heymann, 1996, p. 241; Evans, 1999, p. 40f for more details). In the language of intercultural learning, this aspect has been discussed as the aim of a “synthesis between systems for orientation” (Thomas, 1988, p. 46).

For the development of reflective knowledge and cultural awareness about mathematics, the explicit comparison of similarities and differences can enhance the awareness about specific advantages and limits of mathematical concepts and thinking styles. This aspect is an important contribution to the development of intercultural competence towards mathematics (cf. Lengnink & Peschek, 2001, and Section 2.2 of this article).

Building bridges is not only a methodical matter of organising learning processes. Instead, it mostly necessitates a reconstruction of the *content* that changes its character: Hefendehl-Hebeker (1995) approached the theorem of the midperpendiculars by means of the question of the center, Lengnink (2002) builds the bridge from the mathematical concept of function to everyday thinking by shifting the stress away from the mathematical concept towards the everyday phenomenon that can be described with it: functional dependency instead of mapping.

As far as methodical questions are concerned, two variants are possible: Whereas Lengnink's approach is the result of a carefully planned process, bridges can also be built situatively as in Hefendehl-Hebeker's thought experiment. This is the connection between the claim to build bridges and the conflict approach presented above. Conflicts and irritations can be good starting points, also for unforeseen bridges to be built (see Prediger, 2004 for details).

4. Discussion and Outlook

Starting from Bishop's claim to put cultural conflicts on the research agenda, this paper has described mathematics learning from an intercultural perspective. The considerations have given some evidence to consider mathematics classrooms as a place of intercultural interaction between the everyday culture and the culture of mathematics.

Such a theoretical framework can never claim to have found "the true account" for the reality of mathematics learning (since there is never a single true account for reality), nor can it generate completely new and unforeseen orientations in the prescriptive mode (since nearly every isolated didactical idea has already been formulated and realised anywhere in some context). Instead, the major value of such a theoretical framework is that it interconnects the different modes of discussion in mathematics education research in a coherent manner: Philosophical assumptions about the nature of mathematics should correspond to the basic descriptive ideas about mathematics learning, and both of them should not contradict the normative orientation. All three of these aspects together can form a coherent base for the core of mathematics education research: giving prescriptive orientations for arranging learning processes.

In this article, mathematics is understood as an autonomous culture that is formed not only by its corpus of knowledge, but by all issues that typically characterise cultures: their traditions, customs and practices, language, transmitted knowledge, beliefs, norms and values, purposes, shared meanings etc. When mathematics is specific in all these issues, this aspect should not be neglected in the conceptualisation of mathematics learning processes. Whereas most epistemologies of mathematics learning do not make a major difference between mathematics learning and primary everyday learning, Bishop claimed to focus on the fact that learning in school always takes place in a situation of cultural interaction between the mathematical culture and the everyday culture. Thus, this article tries to complement the vertically oriented accounts for mathematical knowledge development by a horizontal orientation and focuses on the simultaneous presence of more than one culture that influences perceiving, thinking, valuing and acting. In this way, typical intercultural issues in mathematics learning become visible: overlapping, differences, foreignness and misunderstandings. This intercultural perspective allows the activating of concepts and results of pedagogical and psychological research on intercultural interactions for analysing mathematics learning processes.

Prescriptive orientations for arranging learning processes should be guided by a normative orientation concerning the aims of mathematics education which must be compatible with the philosophical and descriptive base. In Section 2.2*,* this is realised by the notion of intercultural competence towards mathematics. This normative orientation can start from the intercultural perspective in the descriptive mode and can be connected with Fischer's conception of higher general education. In this way, the formulated aims of mathematics education are well located in the actual discourse on mathematical literacy.

These three issues together (philosophical account of the nature of mathematics, descriptive perspective on mathematics learning and the normative orientation) offer a framework for prescriptive orientations as they have been sketched in Section 3. Even if the prescriptive side could only be presented in selected aspects, it demonstrated how the developed framework can bring existing principles and ideas into a new context of argumentation. For this, experiences and results from research about intercultural learning in general have proved to be very useful (this is further elaborated in (Prediger, 2004)).

For the realisation of the developed prescriptive orientations, there already exist some promising examples (for more, see Prediger, 2004). Nevertheless, much more effort is needed for a wider implementation in practice. Since the approach implies a changed role for teachers (from an instructor towards an intercultural mediator), one starting point for its implementation should be teacher education in which prospective teachers can develop intercultural sensitiveness and the necessary intercultural competence towards mathematics. Beyond the acquisition of teacher's individual competences and sensitiveness, the realisation of the developed framework demands systematic didactical developmental research in which many researchers and practitionners should analyse mathematical contents, their relations to everyday conceptions, their underlying purposes, benefits and limits. Especially for the challenge of building bridges, a careful analysis of mathematical contents and their meanings is needed.

One possible pathway for this kind of devlopmental research is given by the science education research program of educational reconstruction (Kattmann, Duit & Gropengießer, 1998). Based on conceptual change theories, the researchers have formulated a model for educational research that comprehends three interplaying components of research: investigation of students' conceptions, scientific clarification, and construction of instruction (Kattmann et al., 1998). This model gives a promising methodological background for systematic efforts in building bridges in mathematics education that have only appeared as examples so far.

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