Irreducible Holonomy Representations

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1 Introduction and history

An affine connection is one of the basic objects of interest in differential geometry. It provides a simple and invariant way of transferring information from one point of a connected manifold M to another and, not surprisingly, enjoys lots of applications in many branches of mathematics, physics and mechanics. Among the most informative characteristics of an affine connection is its (restricted) holonomy group which is defined, up to conjugacy, as the subgroup of $\mathrm{Aut}(T_pM)$ consisting of all automorphisms of the tangent space T_pM at $p \in M$ induced by parallel translations along p-based loops in M. The notion of the holonomy group was introduced by É. Cartan in 1923 [Car1, Car3].

Throughout this report, we let M denote a smooth connected n-manifold and let ∇ be an affine connection on M, i.e. a connection on the tangent bundle TM. Fix a point $p \in M$, let

$$\mathcal{L}_p = \{\gamma : [0,1] \to M \mid \gamma(0) = \gamma(1) = p\}$$

be the set of piecewise smooth loops based at p, and let $\mathcal{L}_p^0 \subset \mathcal{L}_p$ be those loops which are homotopic to the trivial loop.

For $\gamma \in \mathcal{L}_p$, denote by $P_\gamma : T_pM \longrightarrow T_pM$ the linear automorphism induced by ∇-parallel translations along γ. The *holonomy of* ∇ *at* $p \in M$ is defined as the subset

$$Hol_p := \{P_\gamma \mid \gamma \in \mathcal{L}_p\} \subset \mathrm{Aut}(T_pM),$$

and the *restricted holonomy* is given by

$$Hol_p^0 := \left\{P_\gamma \mid \gamma \in \mathcal{L}_p^0\right\} \subset Hol_p.$$

One of the first remarkable results which Cartan proved about the holonomy group is the following

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Theorem 1.1 *[Car2] Let (M, g) be a Riemannian symmetric space. Then the identity components of the holonomy group and the isotropy group of M coincide.*

Here, a symmetric space is a manifold such that the geodesic reflection at any point is globally defined and an isometry. Since the notion of a symmetric space is defined in terms of the connection only, one may similarly define an *affine symmetric space* as a manifold with a connection whose geodesic reflections at any point are globally defined and connection preserving. Cartan's theorem then generalizes to the class of affine symmetric spaces with irreducible isotropy group.

In the 1950's, the concept of holonomy became the subject of further investigation. Some of the basic properties of the holonomy group which were established during this time period are the following.

Proposition 1.2 *[BL, N1, N2] Let (M, ∇), $p \in M$ and $Hol_p^0 \subset Hol_p \subset Aut(T_pM)$ be defined as above. Then the following hold.*

1. *Hol_p^0 is the identity component of Hol_p.*
2. *If $\pi : \tilde{M} \to M$ is the universal cover and $\tilde{\nabla}$ is the lift of ∇ to $\tilde{M}$, then $Hol_{\tilde{p}} \cong Hol_p^0$, where $\pi(\tilde{p}) = p$. Thus, by lifting the connection to the universal cover, we may assume that the holonomy group is connected.*
3. *Hol_p^0 is a closed Lie subgroup of $Aut(T_pM)$; its Lie algebra $\mathfrak{hol}_p \subset End(T_pM)$ is called the holonomy algebra at p.*
4. *$Hol_p \cong Hol_q$ for all $p, q \in M$, with an isomorphism being induced by parallel translation along any path from p to q. Thus, if one fixes a linear isomorphism $\imath : T_pM \to V$, where V is a fixed vector space of the appropriate dimension, then the conjugacy class of $\imath(Hol_p) \subset Aut(V)$ does not depend on the choice of $p \in M$ or $\imath$.*

By a slight abuse of terminology, we refer to the conjugacy class of $Hol := \imath(Hol_p) \subset \mathrm{Aut}(V)$ (respectively, $Hol^0 := \imath(Hol_p^0) \subset \mathrm{Aut}(V)$) as the holonomy group (respectively, restricted holonomy group) of ∇. The Lie algebra $\mathfrak{hol} \subset \mathrm{End}(V)$ of $Hol \subset \mathrm{Aut}(V)$ is called the *holonomy algebra* of ∇.

To an affine connection ∇ we can associate two tensors, the *torsion* and the *curvature*, which are given by the formulae

$$Tor_p(x, y) = \nabla_X Y - \nabla_Y X - [X, Y], \text{ and}$$

$$R_p(x, y)z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Here, $x, y, z \in T_pM$, and X, Y, Z are vector fields with $X_p = x$, $Y_p = y$ and $Z_p = z$.

We call a connection ∇ *torsion free* if $Tor \equiv 0$. In this case, it is easy to show that the curvature satisfies the *first and second Bianchi identity*, i.e.

$$R(x, y)z + R(y, z)x + R(z, x)y = 0, \text{ and} \tag{1}$$

$$(\nabla_x R)(y,z) + (\nabla_y R)(z,x) + (\nabla_z R)(x,y) = 0 \tag{2}$$

for all $x, y, z \in T_pM$.

The *irreducible holonomy problem* which shall be of primary interest in this report is the following.

Given a finite dimensional vector space V, which are the irreducible (closed) connected Lie subgroups $\mathrm{H} \subset \mathrm{Aut}(V)$ that can occur as the holonomy group of a torsion free affine connection?

The condition of *torsion freeness* is necessary in order to make this problem non-trivial; namely, by a result of Hano and Ozeki [HO], *any* (closed) Lie subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ can be realized as the holonomy of an affine connection on some manifold M (with torsion, in general). Thus, throughout this report, we shall assume all connections to be *torsion free*. Also, by Proposition 1.2.2, the condition of *connectedness* can be achieved by passing to an appropriate cover of M and thus only disregards parts of the topological structure of M.

On the other hand, the assumption of *irreducibility* is a severe restriction which is necessary to make the holonomy problem more accessible. Indeed, the classification of reducible holonomy groups has been completed in very special cases only (e.g. [BI1, BI2]).

Since connected Lie subgroups are completely characterized by their Lie algebras, the following description of the holonomy algebra was an important step towards the treatment of the holonomy problem.

Ambrose-Singer Holonomy Theorem [AS] *Let ∇ be an affine connection on M and let $p \in M$. Then the holonomy algebra at p is given by*

$$\mathfrak{hol}_p = \langle \{ (P_\gamma R)(x,y) \mid x, y \in T_pM, \gamma \text{ a path with end point } p \} \rangle,$$

where $(P_\gamma R)(x,y) := P_\gamma \cdot R(P_\gamma^{-1}x, P_\gamma^{-1}y) \cdot P_\gamma^{-1}$.

It is obvious that $P_\gamma R$ also satisfies the first Bianchi identity (1). This motivated Berger [Ber1] to develop the following necessary condition for a Lie subalgebra to be the holonomy of a torsion free connection.

Let V be a vector space and $\mathfrak{h} \subset \mathrm{End}(V)$ a Lie subalgebra. We define the *space of formal curvature maps*

$$K(\mathfrak{h}) := \{ R \in \Lambda^2 V^* \otimes \mathfrak{h} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0 \text{ for all } x,y,z \in V \}, \tag{3}$$

and the *space of formal curvature derivatives*

$$K^1(\mathfrak{h}) := \{ \phi \in V^* \otimes K(\mathfrak{h}) \mid \phi(x)(y,z) + \phi(y)(z,x) + \phi(z)(x,y) = 0 \text{ for all } x,y,z \in V \}.$$

We also let $\underline{\mathfrak{h}} := \{ R(x,y) \mid R \in K(\mathfrak{h}),\ x, y \in V \} \subset \mathfrak{h}$. Evidently, $\underline{\mathfrak{h}} \lhd \mathfrak{h}$.

From (1) it follows that $P_\gamma R \in K(\mathfrak{hol}_p)$ for all path γ with end point p; hence the Ambrose-Singer Holonomy Theorem implies that $\underline{\mathfrak{hol}}_p = \mathfrak{hol}_p$. Moreover, from (2) it follows that the map $x \mapsto \nabla_x R$ lies in $K^1(\mathfrak{hol}_p)$. Thus, if $K^1(\mathfrak{hol}_p) = 0$ then $\nabla R \equiv 0$, i.e. the connection is locally symmetric. These facts motivate the following definition.

Definition 1.3 *A Lie subalgebra* $\mathfrak{h} \subset End(V)$ *is called a* Berger algebra *if* $\underline{\mathfrak{h}} = \mathfrak{h}$*. A Berger algebra* $\mathfrak{h} \subset End(V)$ *is called* symmetric *if* $K^1(\mathfrak{h}) = 0$ *and* non-symmetric *otherwise.*

A Lie subgroup $H \subset Aut(V)$ *is called a* (symmetric respectively non-symmetric) Berger group *if its Lie algebra* $\mathfrak{h} \subset End(V)$ *is a (symmetric respectively non-symmetric) Berger algebra.*

In the literature, the two criteria for a non-symmetric Berger algebra are usually referred to as *Berger's first and second criterion.* Our discussion from above now yields the following.

Proposition 1.4 *[Ber1] Let* $H \subset Aut(V)$ *be an irreducible Lie subgroup which occurs as the holonomy group of a torsion free affine connection on some manifold* M*. Then* H *must be a Berger group. If the connection is not locally symmetric, then* H *must be a non-symmetric Berger group.*

Therefore, the holonomy problem now splits into two parts:

1. Classify all irreducible connected Berger subgroups $\mathrm{H} \subset \mathrm{Aut}(V)$.
2. Decide for each of these Berger subgroups if it can occur as a holonomy group.

While the first problem is purely algebraic, the second is analytic in nature. By Theorem 1.1, the holonomy problem has the classification problem for irreducible symmetric spaces as a "sub-problem". This classification has been completed by Cartan in the Riemannian [Car2] and by Berger in the general irreducible case [Ber2]. In particular, this implies the classification of all *irreducible symmetric Berger subgroups.*

In [Ber1], Berger then proceeded to classify all (pseudo-)Riemannian Berger algebras, i.e. the holonomies of Levi-Civita connections of (pseudo-)Riemannian metrics. (In the non-definite case, there were some slight errors which were later corrected by Bryant [Br4].) We list the resulting entries in Tables A and B[1].

1.1 Holonomies of Riemannian manifolds

The list of possible *Riemannian* holonomies (Table A) received a tremendous amount of attention during the decades following Berger's classification. First, it turns out that the list of non-symmetric Riemannian holonomies is contained (in fact, is almost equal to) the list of transitive group actions on spheres [MoSa1, MoSa2, Bo1, Bo2]. This was later shown directly by Simons [Si].

Moreover, in this case, the assumption of irreducibility of the holonomy group is quite natural. Namely, the *deRham splitting theorem* [dR] states that the reducibility of the holonomy group of a Riemannian manifold (M, g) implies that the metric is locally a product metric; this holds even globally if the metric is complete and M is simply connected.

[1]Evidently, the entries in Table A reoccur in Table B, so we could have omitted Table A altogether. However, since we shall discuss the *Riemannian* holonomies in more detail, it seems useful to list them separately.

Table A

List of irreducible non-symmetric Riemannian Berger subgroups $H \subset \mathrm{SO}(n)$ [Ber1]

No.	H	V	restrictions
1	$\mathrm{SO}(n)$	$\mathbb{R}^n$	$n \geq 2$
2	$\mathrm{U}(m)$	$\mathbb{R}^n \cong \mathbb{C}^m$	$m \geq 2, n = 2m$
3	$\mathrm{SU}(m)$	$\mathbb{R}^n \cong \mathbb{C}^m$	$m \geq 2, n = 2m$
4	$\mathrm{Sp}(1) \cdot \mathrm{Sp}(m)$	$\mathbb{R}^n \cong \mathbb{H}^m$	$m \geq 2, n = 4m$
5	$\mathrm{Sp}(m)$	$\mathbb{R}^n \cong \mathbb{H}^m$	$m \geq 2, n = 4m$
6	G_2	$\mathbb{R}^7 \cong Im\mathbb{O}$	
7	$\mathrm{Spin}(7)$	$\mathbb{R}^8 \cong \mathbb{O}$	

Table B

List of irreducible non-symmetric pseudo-Riemannian Berger subgroups $H \subset \mathrm{SO}(n,m)$ [Ber1, Br4]

No.	H	V	restrictions
1.1 1.2	$\mathrm{SO}(p,q)$ $\mathrm{SO}(m,\mathbb{C})$	$\mathbb{R}^{p+q}$ $\mathbb{R}^{2m} \cong \mathbb{C}^m$	$p+q \geq 2$, $pq \neq 1$ $m \geq 3$
2	$\mathrm{U}(p,q)$	$\mathbb{R}^{2(p+q)} \cong \mathbb{C}^{p+q}$	$p+q \geq 2$
3	$\mathrm{SU}(p,q)$	$\mathbb{R}^{2(p+q)} \cong \mathbb{C}^{p+q}$	$p+q \geq 2$, $pq \neq 1$
4.1 4.2 4.3	$\mathrm{Sp}(n,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$ $\mathrm{Sp}(n,\mathbb{C}) \cdot \mathrm{SL}(2,\mathbb{C})$ $\mathrm{Sp}(p,q) \cdot \mathrm{Sp}(1)$	$\mathbb{R}^{4n} \cong \mathbb{R}^{2n} \otimes \mathbb{R}^2$ $\mathbb{R}^{8n} \cong \mathbb{C}^{2n} \otimes \mathbb{C}^2$ $\mathbb{R}^{4(p+q)} \cong \mathbb{H}^{p+q}$	$n \geq 2$ $n \geq 2$ $p+q \geq 2$
5	$\mathrm{Sp}(p,q)$	$\mathbb{R}^{4(p+q)} \cong \mathbb{H}^{p+q}$	$p+q \geq 2$
6.1 6.2 6.3	$\mathrm{G}_2 \subset \mathrm{SO}(7)$ $\mathrm{G}_2' \subset \mathrm{SO}(4,3)$ $\mathrm{G}_2^{\mathbb{C}}$	$\mathbb{R}^7$ $\mathbb{R}^7$ $\mathbb{R}^{14} \cong \mathbb{C}^7$	
7.1 7.2 7.3	$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ $\mathrm{Spin}(4,3) \subset \mathrm{SO}(4,4)$ $\mathrm{Spin}(7,\mathbb{C})$	$\mathbb{R}^8$ $\mathbb{R}^8$ $\mathbb{R}^{16} \cong \mathbb{C}^8$	

Table C

List of conformal pseudo-Riemannian Berger subgroups $H \subset \mathbb{R}^* \cdot \mathrm{SO}(n,m)$ [Ber1, Br4]

Notations: $T_{\mathbb{F}}$ denotes any connected subgroup of $\mathbb{F}^*$.

No.	H	V	restrictions
1.1 1.2	$T_{\mathbb{R}} \cdot \mathrm{SO}(p,q)$ $T_{\mathbb{C}} \cdot \mathrm{SO}(n,\mathbb{C})$	$\mathbb{R}^{p+q}$ $\mathbb{R}^{2n} \cong \mathbb{C}^n$	$p+q \geq 2$, $pq \neq 1$ $n \geq 3$
2	$\mathrm{GL}(1,\mathbb{H})$	$\mathbb{R}^4 \cong \mathbb{H}$	

Table D

LIST OF IRREDUCIBLE CONFORMALLY SYMPLECTICBERGER SUBGROUPS $H \subset \mathbb{F}^* \cdot \mathrm{Sp}(n)$

NOTATIONS: $Z_{\mathbb{F}}$ denotes either $\{1\}$ or $\mathbb{F}^*$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
$\odot^p V$ denotes the symmetric tensors of V of degree p.

No.	H	V	restrictions
1.1	$\mathrm{Sp}(n, \mathbb{R})$	$\mathbb{R}^{2n}$	$n \geq 3$
1.2	$\mathrm{Sp}(n, \mathbb{C})$	$\mathbb{C}^{2n}$	$n \geq 3$
1.3	$Z_{\mathbb{R}} \cdot \mathrm{Sp}(2, \mathbb{R})$	$\mathbb{R}^4$	
1.4	$Z_{\mathbb{C}} \cdot \mathrm{Sp}(2, \mathbb{C})$	$\mathbb{C}^4$	
2.1	$\mathrm{SL}(2, \mathbb{R}) \cdot \mathrm{SO}(p, q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q}$	$p + q \geq 3$
2.2	$\mathrm{Sp}(1) \cdot \mathrm{SO}(n, \mathbb{H})$	$\mathbb{H}^n$	$n \geq 2$
3.1	$Z_{\mathbb{R}} \cdot \mathrm{SL}(2, \mathbb{R})$	$\odot^3 \mathbb{R}^2$	
3.2	$Z_{\mathbb{C}} \cdot \mathrm{SL}(2, \mathbb{C})$	$\odot^3 \mathbb{C}^2$	
4.1	$\mathrm{Sp}(3, \mathbb{R})$	$\mathbb{R}^{14} \subset \Lambda^3 \mathbb{R}^6$	
4.2	$\mathrm{Sp}(3, \mathbb{C})$	$\mathbb{C}^{14} \subset \Lambda^3 \mathbb{C}^6$	

No.	H	V
5.1	$\mathrm{SL}(6, \mathbb{R})$	$\Lambda^3 \mathbb{R}^6$
5.2	$\mathrm{SU}(1, 5)$	$\{\omega \in \Lambda^3 \mathbb{C}^6 \mid *\omega = \omega\}$
5.3	$\mathrm{SU}(3, 3)$	$\{\omega \in \Lambda^3 \mathbb{C}^6 \mid *\omega = \omega\}$
5.4	$\mathrm{SL}(6, \mathbb{C})$	$\Lambda^3 \mathbb{C}^6$
6.1	$\mathrm{Spin}(2, 10)$	$\Delta^+_{(2,10)}$
6.2	$\mathrm{Spin}(6, 6)$	$\Delta^+_{(6,6)}$
6.3	$\mathrm{Spin}(6, \mathbb{H})$	$\Delta^{\mathbb{H}}_6$
6.4	$\mathrm{Spin}(6, \mathbb{C})$	$\Delta^{\mathbb{C}}_6$
7.1	E^5_7	$\mathbb{R}^{56}$
7.2	E^7_7	$\mathbb{R}^{56}$
7.3	$\mathrm{E}^{\mathbb{C}}_7$	$\mathbb{C}^{56}$

We shall now give a brief account of the geometric interpretation and the existence of these holonomies, focusing our attention to the question of existence of local, complete and compact Riemannian manifolds with these holonomies.

This survey is by no means complete. For a much more thorough treatment of Riemannian holonomies and their significance for the topology and geometry of the underlying manifold we refer the interested reader to the books by Besse [Bes] and Salamon [Sa] and the survey article [Br5].

1. *Hol* $= \mathrm{SO}(n)$. This is the "generic" case, i.e. a "generic" Riemannian metric on an oriented manifold will have this holonomy. Thus, the holonomy characterizes no geometric structure on M except the Riemannian metric itself.

2. *Hol* $= \mathrm{U}(m)$. In this case, there is a parallel orthogonal almost complex structure on M. The torsion freeness of the connection implies that this almost complex structure is *integrable*, thus we have the following result:

   Proposition 1.5 *Let (M, g) be a Riemannian manifold. Then Hol $\subset U(m)$ iff there is an orthogonal complex structure J on M such that (M, g, J) is Kähler.*

   Of course, Kähler metrics are known to exist, for example any smooth closed submanifold of $\mathbb{CP}_n$ will be compact Kähler. Moreover, the holonomy of a "generic" Kähler metric will be *all* of $\mathrm{U}(m)$ and will not be locally symmetric[2].

[2]The only symmetric spaces which are Kähler are $\mathbb{CP}_n$ and its non-compact dual $\mathbb{CP}_n^*$. Thus, a Kähler manifold is locally symmetric iff it is locally isometric to one of these two spaces.

Table E List of holonomy groups of general type

Notations: $T_{\mathbb{F}}$ denotes any connected subgroup of $\mathbb{F}^*$.
$\odot^p V$ denotes the symmetric tensors of V of degree p.

No.	H	V	restrictions
1.1	$T_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{C})$	$\{A \in M_n(\mathbb{C}) \mid A = A^*\}$	$n \geq 3$
1.2	$T_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{R}) \cdot \mathrm{SL}(m, \mathbb{R})$	$\mathbb{R}^n \otimes \mathbb{R}^m$	$n \geq m \geq 2, nm \neq 4$
1.3	$T_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{H}) \cdot \mathrm{SL}(m, \mathbb{H})$	$\mathbb{H}^n \otimes_{\mathbb{H}} \mathbb{H}^m$	$n \geq m \geq 1, nm \neq 1$
1.4	$T_{\mathbb{C}} \cdot \mathrm{SL}(n, \mathbb{C}) \cdot \mathrm{SL}(m, \mathbb{C})$	$\mathbb{C}^n \otimes \mathbb{C}^m$	$n \geq m \geq 2, nm \neq 4$
2.1	$T_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{R})$	$\mathbb{R}^n$	$n \geq 2$
2.2	$T_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{H})$	$\mathbb{H}^n$	$n \geq 2$
2.3	$T_{\mathbb{C}} \cdot \mathrm{SL}(n, \mathbb{C})$	$\mathbb{C}^n$	$n \geq 2$
2.4	$T_{\mathbb{C}} \cdot \mathrm{SU}(p, q)$	$\mathbb{C}^2$	$p + q = 2,\ T_{\mathbb{C}} \not\subset \mathbb{R}^*$
3.1	$T_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{R})$	$\Lambda^2 \mathbb{R}^n$	$n \geq 5$
3.2	$T_{\mathbb{C}} \cdot \mathrm{SL}(n, \mathbb{C})$	$\Lambda^2 \mathbb{C}^n$	$n \geq 5$
3.3	$T_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{H})$	$\{A \in M_n(\mathbb{H}) \mid A = A^*\}$	$n \geq 3$
4.1	$T_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{R})$	$\odot^2 \mathbb{R}^n$	$n \geq 3$
4.2	$T_{\mathbb{C}} \cdot \mathrm{SL}(n, \mathbb{C})$	$\odot^2 \mathbb{C}^n$	$n \geq 3$
4.3	$T_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{H})$	$\{A \in M_n(\mathbb{H}) \mid A = -A^*\}$	$n \geq 2$
5.1	$T_{\mathbb{R}} \cdot \mathrm{Spin}(5, 5)$	$\Delta^+_{(5,5)}$	
5.2	$T_{\mathbb{R}} \cdot \mathrm{Spin}(1, 9)$	$\Delta^+_{(1,9)}$	
5.3	$T_{\mathbb{C}} \cdot \mathrm{Spin}(10, \mathbb{C})$	$(\Delta^+_{10})^{\mathbb{C}}$	
6.1	$T_{\mathbb{R}} \cdot \mathrm{E}_6^1$	$\mathbb{R}^{27}$	
6.2	$T_{\mathbb{R}} \cdot \mathrm{E}_6^4$	$\mathbb{R}^{27}$	
6.3	$T_{\mathbb{C}} \cdot \mathrm{E}_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	

3. $Hol = \mathrm{SU}(m)$. By Proposition 1.5, a manifold with this holonomy must be Kähler, and further calculation shows that the holonomy is contained in $\mathrm{SU}(m)$ iff it is *Ricci flat* [L]. Such a Riemannian manifold (M, g) is called *special Kähler.*

   The first (incomplete) example of a special Kähler manifold was found by Calabi in 1960 [Cal]. In 1970, Calabi also obtained complete examples of such manifolds. Finally, in 1978 Yau produced a solution to the Calabi conjecture, stating the following.

   Theorem 1.6 *[Y] Let (M, g_0) be a compact Kähler manifold and let $c_1(M) \in H^2(M, \mathbb{R})$ be the first Chern class of M. Then for any closed $(1,1)$-form ρ which represents $2\pi c_1(M)$ in the deRham cohomology, there exists a unique Kähler metric g on M such that the Kähler forms of g and g_0 represent the same cohomology class, and the Ricci curvature of g is given by*

   $$Ric(x, y) = \rho(x, Jy)$$

   for all $x, y \in TM$. In particular, a compact Kähler manifold admits a special Kähler metric iff its first Chern class vanishes.

   With this result, many compact manifolds admitting a Riemannian metric with holonomy $\mathrm{SU}(m)$ can be constructed, e.g. algebraic hypersurfaces of degree $m + 2$ in $\mathbb{CP}_{m+1}$ [Bea].

4. $Hol = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$. A Riemannian manifold whose holonomy is contained in $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ is called *quaternion-Kähler.*

   Proposition 1.7 *[G] Let (M, g) be a $4m$-dimensional Riemannian manifold. Then the following are equivalent.*

   (a) The holonomy group of (M, g) is contained in $Sp(m) \cdot Sp(1)$.

   (b) M is covered by open sets on which there exist three orthogonal almost complex structures I_1, I_2, I_3 and 1-forms α, β, γ such that $I_i \cdot I_{i+1} = I_{i+2}$ (with indices mod 3) and such that for all $v \in TM$ the following relations hold:

   $$\begin{aligned} \nabla_v I_1 &= \quad \gamma(v) \cdot I_2 - \beta(v) \cdot I_3 \\ \nabla_v I_2 &= -\gamma(v) \cdot I_1 + \alpha(v) \cdot I_3 \\ \nabla_v I_3 &= \beta(v) \cdot I_1 - \alpha(v) \cdot I_2 \end{aligned}$$

   If these conditions are satisfied then there is a (globally defined) parallel S^2-subbundle $Q \subset End(TM)$, consisting of orthogonal almost complex structures. Locally, it is given by $Q = \{\sum c_i I_i \mid \sum c_i^2 = 1\}$.

   Examples of quaternion-Kähler manifolds are the quaternionic symmetric spaces $\mathbb{HP}^n$ and its non-compact dual. But there are also others, for example the homogeneous manifolds found by Alekseevskii [A] in 1968. Clearly, these metrics are complete.

   There are up to date no known examples of *compact* Riemannian manifolds with holonomy $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ which are *not* locally symmetric.

5. $Hol = \mathrm{Sp}(m)$. A Riemannian manifold whose holonomy is contained in $\mathrm{Sp}(m)$ is called *hyperkähler*. Since $\mathrm{Sp}(m) \subset \mathrm{SU}(m)$ it follows that a hyperkähler metric is a fortiori special Kähler. Indeed, a hyperkähler metric is characterized by the existence of three parallel orthogonal complex structures I_i satisfying $I_i \cdot I_{i+1} = I_{i+2}$ (with indices mod 3).

   The first examples of hyperkähler manifolds whose holonomy equals $\mathrm{Sp}(m)$ were discovered only in 1979 by Calabi [Cal]. The first *compact* examples of manifolds with holonomy $\mathrm{Sp}(m)$ were obtained in 1982 [Bea].

6. $Hol = \mathrm{G}_2$ and $Hol = \mathrm{Spin}(7)$. These two *exceptional holonomy groups* on Berger's list (Table A) were the last to be shown to exist; this was achieved by R. Bryant [Br1] in 1986. His method of proof turned out to be extremely useful for the investigation of local existence of many non-Riemannian holonomy groups as well. It is based on the translation of the structure equations for a given holonomy group into an *exterior differential system*, and then to use Cartan-Kähler theory [BCG^3] to conclude the existence of such metrics.

   Also, in 1996 D. Joyce has constructed a number of *compact* simply connected manifolds with these exceptional holonomies [J].

1.2 Non-Riemannian Holonomy groups

1.2.1 Pseudo-Riemannian Holonomy groups (cf. Table B)

As mentioned above, one of the most effective methods to solve the existence problem for connections with prescribed holonomy was Bryant's approach to describe torsion free connections with given holonomy as solutions to an Exterior Differential System [Br2], and then to use Cartan-Kähler theory [BCG^3] to prove the local existence of such connections. Using this method, Bryant was able to show the following.

Theorem 1.8 *[Br4] All entries of Table B occur as the holonomy of a (pseudo)-Riemannian manifold. Indeed, the local generality of metrics with these holonomies is given in Table F.*

Table F: LOCAL GENERALITY OF METRIC HOLONOMIES (MODULO DIFFEOMORPHISMS)
(Notation: "d of l" means "d functions of l variables")

n	H	local generality
$p+q \geq 2$ $2p$	$\mathrm{SO}(p,q)$ $\mathrm{SO}(p,\mathbb{C})$	$\frac{1}{2}n(n-1)$ of n $\frac{1}{2}p(p-1)^{\mathbb{C}}$ of $p^{\mathbb{C}}$
$2(p+q) \geq 4$ $2(p+q) \geq 4$	$\mathrm{U}(p,q)$ $\mathrm{SU}(p,q)$	1 of n 2 of $n-1$
$4(p+q) \geq 8$	$\mathrm{Sp}(p,q)$	$2(p+q)$ of $(2p+2q+1)$
$4(p+q) \geq 8$ $4p \geq 8$ $8p \geq 16$	$\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ $\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$ $\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$	$2(p+q)$ of $(2p+2q+1)$ $2p$ of $(2p+1)$ $2p^{\mathbb{C}}$ of $(2p+1)^{\mathbb{C}}$
7 7 14	G_2 G_2' $\mathrm{G}_2^{\mathbb{C}}$	6 of 6 6 of 6 $6^{\mathbb{C}}$ of $6^{\mathbb{C}}$
8 8 16	$\mathrm{Spin}(7)$ $\mathrm{Spin}(4,3)$ $\mathrm{Spin}(7,\mathbb{C})$	12 of 7 12 of 7 $12^{\mathbb{C}}$ of $7^{\mathbb{C}}$

1.2.2 Conformal holonomy groups (cf. Table C)

There are only very few possible holonomy groups which are contained in the conformal group $\mathrm{CO}(p,q) := \mathbb{R}^* \cdot \mathrm{SO}(p,q)$ but *not* in $\mathrm{SO}(p,q)$ (cf. Table C).

A connection whose holonomy equals $\mathrm{CO}(p,q)$ is called a *Weyl connection* of a conformal structure. Likewise, a connection with holonomy $\mathbb{C}^* \cdot \mathrm{SO}(n,\mathbb{C})$ is a *complex Weyl connection* of a complex conformal structure. Weyl connections are known to exist for any conformal structure.

Next, a connection on a 4-manifold whose holonomy equals $\mathrm{GL}(1,\mathbb{H})$ corresponds to a (generic) Yang-Mills connection on a half-conformally flat 4-manifold.

The only holonomy groups in Table B which were not classically known are the groups $\mathrm{H}_\lambda \cdot \mathrm{SO}(n,\mathbb{C})$ where $\mathrm{H}_\lambda := \{exp(t(\lambda + i)) \mid t \in \mathbb{R}\} \subset \mathbb{C}^*$ with $\lambda \geq 0$. These were detected as possible holonomy groups by Bryant [Br4].

1.2.3 Holonomy groups of general type (cf. Table E)

In [Br4], there is also a classification of those irreducible Berger algebras $\mathrm{H} \subset \mathrm{Aut}(V)$ for which do not preserve any (symmetric or skew-symmetric) bilinear form up to a scale. Again, a treatment of these holonomies with the Cartan-Kähler machinery yields the following result.

Theorem 1.9 *[Br4, Br5] The irreducible subgroups $H \subset Aut(V)$ listed in Table E are Berger subgroups. Moreover, all of them occur as holonomies of torsion free connections on some manifold.*

1.2.4 Conformal symplectic holonomy groups (cf. Table D)

Historically, these holonomies which are listed in Table D were the last to be discovered. Of course, the full symplectic groups $\mathrm{Sp}(n, \mathbb{R})$ and $\mathrm{Sp}(n, \mathbb{C})$, respectively were known to occur: indeed, given a (holomorphically) symplectic manifold (M, Ω) then the "generic" (holomorphic) torsion free connection ∇ with the property that $\nabla\Omega \equiv 0$ will have holonomy $\mathrm{Sp}(n, \mathbb{R})$ ($\mathrm{Sp}(n, \mathbb{C})$, respectively). This can be easily seen by considering Darboux coordinates.

Also, it is well known that only in dimensions $n = 2, 4$ there may be torsion free connections on a (holomorphically) symplectic manifold (M^n, Ω) which preserve Ω only up to a scale, i.e. $\nabla_X \Omega = \alpha(X)\Omega$ for some non-vanishing 1-form α. Again, it is easy to write down such a connection in Darboux coordinates explicitly. This means that the holonomy groups 1.1 – 1.4 in Table D can easily be realized as holonomies.

Definition 1.10 *Let (M, Ω, ∇) be a symplectic manifold with a torsion free connection ∇ such that $\nabla\Omega \equiv 0$. If the holonomy group of ∇ is absolutely irreducible and properly contained in Sp$(n, \mathbb{R})$ (Sp$(n, \mathbb{C})$, respectively) then ∇ is called a* special symplectic connection*, and its holonomy group is called a* special symplectic holonomy group*.*

The first examples of special symplectic holonomy groups were found by Bryant [Br3] in 1991. These were examples on (real or complex) symplectic manifolds of (real or complex) dimension 4. Thus, their conformal extensions also occurs as a holonomy group. These are the entries 3.1 and 3.2 in Table D.

The remaining entries of Table D were discovered by Chi, Merkulov and this author [CS, CMS1, CMS2, MeSc1, MeSc2]. We shall discuss the construction of connections with these holonomies and some of their geometric properties in section 3.

Finally, in [MeSc1, MeSc2], a complete classification of irreducible Berger groups was given.

Theorem 1.11 *[MeSc1, MeSc2] The entries of Tables B,C,D,E yield a complete list of irreducible Berger subgroups. All of them can occur as holonomies of a torsion free connection on some manifold.*

Let V be a complex vector space with $\dim V =: n$, and let $\mathrm{H} \subset \mathrm{Aut}(V)$ be a connected irreducible Lie subgroup. We let $\tilde{\mathcal{C}} \subset V^* \backslash \{0\}$ be the H-orbit of a highest weight vector of the dual representation, and let $\mathcal{C} \subset \mathbb{P}(V^*)$ be its projectivization. $\mathcal{C}$ is called the *sky* of H.

The list of special symplectic holonomy groups is quite special indeed. There are several equivalent descriptions available which we collect in the following proposition.

Proposition 1.12 *Let $H \subset$ Sp(V, ω) be an absolutely irreducible connected closed subgroup and let $\mathfrak{h} \subset \mathfrak{sp}(V)$ be its Lie algebra. Then the following are equivalent.*

1. *H is the holonomy group of a symplectic connection.*

2. *There is a* Ad_H*-invariant symmetric bilinear form* $(\ ,\ )$ *on* $\mathfrak{h}$ *and a bilinear map* $\circ : \odot^2 V \to \mathfrak{h}$ *such that for all* $x, y, z \in V$ *and* $A \in \mathfrak{h}$*, the following equations are satisfied:*

$$\begin{aligned} (A, x \circ y) &= \omega(Ax, y) \\ (x \circ y)z - (x \circ z)y &= 2\omega(y,z)x - \omega(x,y)z + \omega(x,z)y \end{aligned} \tag{4}$$

3. *There is an irreducible symmetric space of the form* $G/(SL(2,\mathbb{F}) \cdot H)$ *with* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*.*

Furthermore, if V *and* H *are both complex and* $H \subsetneq Sp(V)$*, then the above conditions hold iff the sky* $\mathcal{C} \subset \mathbb{P}(V)$ *of* H *is a Legendre submanifold and* $H = Aut(\mathcal{C})$*.*

While the first of these equivalences are shown e.g. in [MeSc1, Sc1], the characterization of the sky of the representation follows from the work of Alekseevskii and Cortés [AC].

2 The Merkulov Twistor space

In this section, we shall give a brief exposition of a twistor theory which can be associated to a holomorphic torsion free connection on a complex manifold M. This twistor theory has been developed by Merkulov in [M1, M2, M3, M4]. Throughout this section, we shall work in the complex category. That is, all manifolds, functions, vector fields, forms etc. are understood to be *holomorphic*. Also, TM and T^*M stand for the *holomorphic* (co-)tangent bundle of the manifold M.

Definition 2.1 *Let* Y *be a manifold, let* $\mathcal{D}$ *be a codimension-*1 *distribution on* Y*, and define the line bundle* L *by the exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow TY \longrightarrow L \longrightarrow 0.$$

If the L*-valued* 2*-form* θ *on* $\mathcal{D}$ *given by* $\theta(x,y) := [x,y] \bmod \mathcal{D}$ *is non-degenerate, then* $\mathcal{D}$ *is called a* contact structure *on* Y*, and* L *is called the* contact line bundle *of* Y*.*

A submanifold $X \subset Y$ *is called a* contact submanifold *if* $TX \subset \mathcal{D}$*. If* X *is a contact submanifold with* $\dim X = (\dim Y - 1)/2$ *then* X *is called a* Legendre submanifold*.*

Note that from the maximal non-integrability of $\mathcal{D}$ it follows that Legendre submanifolds are contact submanifolds of maximal dimension.

Given a contact manifold Y and a compact Legendre submanifold $X_0 \subset Y$, a natural question is when the moduli space of "close-by" Legendre submanifolds carries the structure of a manifold. To make this more precise, we need the following definition.

Definition 2.2 *Let* Y *be a contact manifold. An* analytic family of compact Legendre manifolds *is a submanifold* $S \hookrightarrow M \times Y$ *with some manifold* M *such that the restriction* $\pi_1 : S \to M$ *is a submersion, and* $X_p := \pi_2(\pi_1^{-1}(p)) \subset Y$ *is a compact Legendre submanifold for all* $p \in M$*. Here,* π_i *is the projection of* $M \times Y$ *onto the* i*-th factor. In this case, we call*

M *a* moduli space of Legendre submanifolds*, and say that the submanifolds* X_p*,* $p \in M$*, are contained in the analytic family.*

S *is called* maximal (locally maximal, respectively) *if for every analytic family* $S' \subset M' \times Y$ *with* $M \subset M'$ *and* $S \subset S'$*, it follows that* $S = S'$ *and* $M = M'$ *(*S *open in* S' *and* M *open in* M'*, respectively).*

Then one can show the following deformation result.

Theorem 2.3 *[M1] Let* Y *be a contact manifold with contact line bundle* $L \to Y$*, and let* $X_0 \subset Y$ *be a compact Legendre submanifold. If* $H^1(X_0, L_{X_0}) = 0$ *then there exists a maximal analytic family* $S \hookrightarrow Y \times M$ *containing* X_0*. Moreover, there is a canonical isomorphism* $T_pM \cong H^0(X_p, L_{X_p})$*, and hence,* $\dim M = \dim H^0(X_0, L_{X_0})$*.*

Now, let Y be a contact manifold, $X \subset Y$ compact Legendre, and assume that X is homogeneous, i.e. $X = \mathrm{G}/P$ where G is a semi-simple Lie group and $P \subset \mathrm{G}$ a parabolic subgroup. W.l.o.g. we assume that $\mathrm{G} = \mathrm{Aut}(X)$ is the biholomorphism group of X. Furthermore, suppose that the restriction L_X is very ample. It is well-known that in this case $H^1(X, L_X) = 0$, whence the moduli space M from Theorem 2.3 is a manifold.

Remarkably, there are cases when M comes equipped with a torsion free connection. Namely, the following holds.

Theorem 2.4 *[M1] Let* Y *be a contact manifold with contact line bundle* $L \to Y$*, and let* $X_0 \subset Y$ *be a compact Legendre submanifold. Suppose that* X_0 *is a generalized flag variety* $X_0 = G/P$*, such that the restriction* $L \to X_0$ *is a homogeneous very ample line bundle.*

Let $N := J^1L$ *denote the first jet bundle of* L*. If* $H^1(X_0, L \otimes \odot^2 N) = 0$ *then there exists a holomorphic torsion free connection on* M *whose holonomy is contained in* G*.*

Further, if $H^0(X_0, L \otimes \odot^2 N) = 0$ *then this connection is unique.*

The remarkable fact in this theory is that this process can be reverted in the sense that every torsion free holomorphic connection can be realized locally as the canonical connection on an associated moduli space, and we shall now explain this process in some more detail.

To begin with, let M be a complex manifold, and let $\pi : T^*M \to M$ be its holomorphic cotangent bundle. We let λ denote the *Liouville form* on T^*M which is given by the equation

$$\lambda(v_\theta) := \theta(\pi_*(v_\theta))$$

for all $v_\theta \in T_\theta(T^*M)$. The 2-form

$$\omega := d\lambda$$

is non-degenerate and is called the *canonical symplectic form* on T^*M. It is also easy to verify that

$$m_t^*\lambda = t\lambda \quad \text{and} \quad m_t^*\omega = t\omega,$$

where $m_t : T^*M \to T^*M$ denotes the scalar multiplication by $t \in \mathbb{C}^*$.

Consider a G-structure $F \subset \mathfrak{F}$ on M. Clearly, the cotangent bundle of M and its projectivization can be expressed as $T^*M = F \times_G V^*$ and $\mathbb{P}T^*M = F \times_G \mathbb{P}(V^*)$. Let

$$\tilde{S} := F \times_G \tilde{\mathcal{C}} \subset T^*M \backslash \{0\},$$

and

$$S := F \times_G \mathcal{C} \subset \mathbb{P}T^*M.$$

Obviously, S is the quotient of $\tilde{S}$ by the natural $\mathbb{C}^*$-action. The restriction $\omega_{\tilde{S}}$ of ω is no longer non-degenerate, and we let $\mathcal{N} \subset T\tilde{S}$ be its annihilator, i.e.

$$\mathcal{N} := \{v \in T\tilde{S} \mid v \lrcorner \omega_{\tilde{S}} = 0\}.$$

If we denote the canonical projection by $\pi : \tilde{S} \to M$, then it is easy to see that for all $p \in M$,

$$\mathcal{N} \cap T\pi^{-1}(p) = 0.$$

We make the simplifying assumption that $\dim \mathcal{N}$ is constant. Since $\omega_{\tilde{S}}$ is closed, it follows that $\mathcal{N}$ is integrable. Thus, restricting to a sufficiently small open subset of M, we may assume that the set of integral leafs of $\mathcal{N}$ is a *manifold* $\tilde{Y}$, i.e. we have a submersion

$$\tilde{\mu} : \tilde{S} \longrightarrow \tilde{Y}$$

such that $\mathcal{N}$ is precisely the tangent space of the fibers of $\tilde{\mu}$.

Let v be a vector field on $\tilde{S}$ with $v_s \subset \mathcal{N}$ for all $s \in \tilde{S}$. Then $\mathfrak{L}_v \omega_{\tilde{S}} = v \lrcorner d\omega_{\tilde{S}} + d(v \lrcorner \omega_{\tilde{S}}) = 0$, and therefore $\omega_{\tilde{S}}$ can be pushed down to $\tilde{Y}$ via $\tilde{\mu}$; in other words, there is a 2-form $\tilde{\omega}$ on $\tilde{Y}$ with

$$\omega_{\tilde{S}} = \tilde{\mu}^*(\tilde{\omega}).$$

It is obvious that $\tilde{\omega}$ is nondegenerate. Moreover, $0 = d\omega_{\tilde{S}} = \tilde{\mu}^*(d\tilde{\omega})$, and since $\tilde{\mu}$ is a submersion, it follows that $d\tilde{\omega} = 0$, i.e. $(\tilde{Y}, \tilde{\omega})$ is a symplectic manifold.

Since the distribution $\mathcal{N}$ is invariant under the natural $\mathbb{C}^*$-action on $\tilde{S}$, there is an induced $\mathbb{C}^*$-action on $\tilde{Y}$ for which

$$m_t^* \tilde{\omega} = t\tilde{\omega} \text{ for all } t \in \mathbb{C}^*. \tag{5}$$

Also, $\mathcal{N}$ factors through to an integrable distribution on $S = \tilde{S}/\mathbb{C}^*$, and if we denote the leaf space of this distribution by Y then we get a submersion $\mu : S \to Y$, and Y is the quotient of $\tilde{Y}$ by the $\mathbb{C}^*$-action. We denote the canonical projection by $p : \tilde{Y} \to Y$.

Let ∂_t denote the vector field on $\tilde{Y}$ whose flow induces this $\mathbb{C}^*$-action. Then by (5), $\mathfrak{L}_{\partial_t} \tilde{\omega} = \tilde{\omega}$, and since $\tilde{\omega}$ is closed, this implies that

$$\tilde{\omega} = d\tilde{\lambda}, \quad \text{where} \quad \tilde{\lambda} = \partial_t \lrcorner \tilde{\omega}.$$

Evidently, $\tilde{\lambda}(\partial_t) = 0$, and $\tilde{\lambda}$ is nowhere vanishing. Thus, for each $\tilde{y} \in \tilde{Y}$, there is a unique 1-form $0 \neq \underline{\lambda}_{\tilde{y}} \in T_y^*Y$ where $y = p(\tilde{y})$, such that $p^*(\underline{\lambda}_{\tilde{y}}) = \tilde{\lambda}_{\tilde{y}}$. Hence, the map $\imath : \tilde{Y} \hookrightarrow T^*Y \backslash \{0\}$ with $\imath(\tilde{y}) := \underline{\lambda}_{\tilde{y}}$ is well-defined and, by (5), a $\mathbb{C}^*$-equivariant embedding whose image is a $\mathbb{C}^*$-subbundle. It is now evident that $\tilde{\lambda} = \imath^*\lambda_Y$ where λ_Y denotes the Liouville 1-form on T^*Y, and thus $\tilde{\omega} = \imath^*\omega_Y$ where ω_Y is the canonical symplectic form on T^*Y. But since $\tilde{\omega}$ is non-degenerate on $\tilde{Y}$, it follows that the distribution $\mathcal{D}$ on Y which is annihilated by $\imath(\tilde{Y})$ defines a *contact structure on* Y, and $\imath(\tilde{Y}) \subset T^*Y\backslash\{0\}$ is precisely the dual of the contact line bundle $L \to Y$. Thus, identifying $\tilde{Y}$ with its image under this inclusion, we get the following commutative diagram:

$$\begin{array}{ccccc} & & \tilde{S} & \xrightarrow{\tilde{\mu}} & L^*\backslash\{0\} \\ & \swarrow & \downarrow{\scriptstyle \mathbb{C}^*} & & \downarrow{\scriptstyle \mathbb{C}^*} \\ M & \leftarrow & S & \xrightarrow{\mu} & Y \end{array}$$

For $p \in M$, we let $S_p := \pi^{-1}(p) \subset S$. Since $\mathcal{N} \cap TS_p = 0$, it follows that the map $\pi \times \mu : S \to M \times Y$ is an embedding. Moreover, it follows easily from the construction that $Y_p := \mu(S_p) \subset Y$ is a *contact submanifold*, and hence, S determines an analytic family of compact contact submanifolds.

Let us now address the question under which circumstances the contact submanifolds $Y_p \subset Y$ are *Legendre*. A dimension count yields that this is the case iff $\dim \mathcal{N} = \operatorname{codim}(S \subset \mathbb{P}T^*M) = \operatorname{codim}(\tilde{S} \subset T^*M)$. Evidently, we have the inequality $\dim \mathcal{N} \leq \operatorname{codim}(\tilde{S} \subset T^*M)$, as ω is non-degenerate on T^*M. Thus, $Y_p \subset Y$ is Legendre iff the dimension of $\mathcal{N}$ is maximal. If this is the case at some point, then by semi-continuity of the rank, this holds for a neighborhood of that point as well. If $\dim \mathcal{N}$ is maximal *everywhere* then we call the G-structure F *non-degenerate.*

Proposition 2.5 *Let M be a manifold, and let $F \subset \mathfrak{F}$ be a non-degenerate G-structure with irreducible $G \subset Aut(V)$, and let $S \subset \mathbb{P}T^*M$ be as before. Then the inclusion $\pi \times \mu : S \hookrightarrow M \times Y$ is a locally maximal analytic family of Legendre submanifolds of Y.*

The reason why we are particularly interested in this twistor description of non-degenerate G-structures is the following result whose proof is merely a straightforward calculation.

Theorem 2.6 *Every torsion free G-structure F on M with irreducible $G \subset GL(n, \mathbb{C})$ is non-degenerate, and thus M can be realized as a locally maximal analytic family of compact homogeneous Legendre submanifolds of a contact manifold Y.*

We call a submanifold $\Sigma \subset M$ *conic* if for each $p \in \Sigma$, $(T_p\Sigma)^\perp \subset T_p^*M$ is tangent to $\mathcal{C}_p \subset T_p^*M$. Then by a dimension count, we get as an immediate but striking consequence of Proposition 1.12 the following result:

Corollary 2.7 *[Sc2] Let (M, Ω, ∇) be a symplectic manifold with a special symplectic connection. Then every conic submanifold of M is Lagrangian. In fact, the Merkulov twistor space is the moduli space of all maximal conic totally geodesic Lagrangian submanifolds $\Sigma \subset M$.*

This double duality (points in M) $\leftrightarrow$ (Legendre submanifolds of Y) and (points in Y) $\leftrightarrow$ (conic Lagrangian submanifolds of M) makes the twistor theory for the special symplectic holonomy groups particularly intriguing.

3 Construction of Special Symplectic Connections

We shall now describe the method to construct torsion free connections with special symplectic holonomy. For this, we need the following definitions.

Definition 3.1 *Let* $(P, \{\ ,\ \})$ *be a Poisson manifold. A* symplectic realization *of* P *is a symplectic manifold* (S, Ω) *and a submersion*

$$\pi : S \longrightarrow P$$

which is compatible with the Poisson structures, i.e.

$$\{\pi^*(f), \pi^*(g)\}_S = \pi^*(\{f, g\}) \quad \text{for all } f, g \in C^\infty(P, \mathbb{R}), \tag{6}$$

where the Poisson bracket $\{\ ,\ \}_S$ *on* S *is induced by the symplectic structure.*

The following result ensures the existence of such realizations, at least locally.

Proposition 3.2 *[W] Let* $(P, \{\ ,\ \})$ *be a Poisson manifold. Then for every point* $p_0 \in P$*, there is an open neighborhood* U *of* p_0 *and a symplectic realization* $\pi : S \longrightarrow U$*.*

If $\pi : S \to P$ is a symplectic realization then, in order to avoid confusion, we denote the Hamiltonian vector fields on S by ξ_h where $h \in C^\infty(S)$, while we denote the Hamiltonian vector fields on P by η_f for $f \in C^\infty(P)$. With this, we have for all $f, g \in C^\infty(P)$

$$\pi_*(\xi_{\pi^*(f)}) = \eta_f \quad \text{and} \quad \begin{array}{rcll} [\xi_{\pi^*(f)}, \xi_{\pi^*(g)}] & = & \xi_{\{\pi^*(f), \pi^*(g)\}_S} & \\ & = & \xi_{\pi^*(\{f,g\})} & \text{by (6).} \end{array} \tag{7}$$

This implies that the distribution Ξ on S given by

$$\Xi_s = \{(\xi_{\pi^*(f)})_s \mid f \in C^\infty(P)\} \quad \text{for all } s \in S$$

is integrable. Evidently, $(\xi_{\pi^*(f)})_s$ only depends on $df_{\pi(s)}$, and since π is a submersion, the map $\pi^* : T^*_{\pi(s)}P \to T^*_s S$ is injective. Thus, the canonical map

$$\begin{array}{rccc} \Theta : & \Xi_s & \longrightarrow & T^*_{\pi(s)}P \\ & (\xi_{\pi^* f})_s & \longmapsto & df_{\pi(s)} \end{array}$$

is a linear isomorphism and hence, Ξ has constant rank equal to the dimension of P. Moreover, if $F \subset S$ is an integral leaf of Ξ then by (7), there is a symplectic leaf $\Sigma \subset P$ such that $\pi : F \to \Sigma$ is a submersion.

Let $\mathrm{H} \subset \mathrm{Aut}(V)$ be a Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathrm{End}(V)$. If there is a torsion free connection on some manifold M with holonomy H, then there is a principal H-bundle $F \to M$ and on F with an $\mathfrak{h} \oplus V$-valued coframing $\omega + \theta$ where θ and ω are the tautological respectively the connection 1-form [KN]. For each $w \in \mathfrak{h} \oplus V$, we let $\xi_w \in \mathfrak{X}(F)$ be the vector field characterized by $(\omega+\theta)(\xi_w) = w$. Then the *structure equations* for the connection [KN] yield the following.

$$\begin{array}{lcl} [\xi_A, \xi_B] & = & \xi_{[A,B]} \\ [\xi_A, \xi_x] & = & \xi_{A\cdot x} \\ [\xi_x, \xi_y](s) & = & \xi_{R_s(x,y)}, \end{array} \tag{8}$$

where $A, B \in \mathfrak{h}$, $x, y \in V$ and $R_s : \Lambda^2 V \to \mathfrak{h}$ is the curvature map at s. Conversely, if there are vector fields ξ_w satisfying (8) then F can be embedded into a principal H-bundle over $M := F/\mathrm{H}$ which is – at least locally – a manifold, such that the distribution on F given by $\{\xi_x \mid x \in V\}$ yields a connection on M [KN].

The basic idea of our construction is now to consider a Poisson structure on the vector space $P := (\mathfrak{h} \oplus V)^*$ and a symplectic realization $\pi : S \to U \subset P$ of some open neighborhood U. If we let $F \subset S$ be a leaf of the distribution Ξ, then $T_s F \cong T^*_{\pi(s)} P \cong \mathfrak{h} \oplus V$ since P is a vector space. Thus, we get again an $\mathfrak{h} \oplus V$-valued coframe on F. Now from (7) it is evident that this coframe satisfies the structure equations (8) iff the Poisson structure on P is of the form

$$\{f, g\}(p) := p([A + x, B + y]) + \phi(p_{\mathfrak{h}^*})(x, y), \tag{9}$$

where, $df_p = A + x$ and $dg_p = B + y$ are the decompositions of $df_p, dg_p \in T^*_p P^* \cong P = \mathfrak{h} \oplus V$, $p_{\mathfrak{h}^*}$ denotes the $\mathfrak{h}^*$-component of p and $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ is a smooth map. Now it is straightforward to verify the following

Lemma 3.3 *The equation (9) defines a Poisson structure on P iff ϕ satisfies the following properties:*

(i) *ϕ is H-equivariant,*

(ii) *for every $p \in \mathfrak{h}^*$, the dual map $(d\phi_p)^* : \Lambda^2 V \to \mathfrak{h}$ is contained in $K(\mathfrak{h})$ (cf. (3)).*

Of course, it is a priori not evident that such functions ϕ exist for a given $\mathfrak{h} \subset \mathrm{End}(V)$. However, in the case of special symplectic holonomies, such a ϕ can be explicitly given. Namely, we have the following result.

Theorem 3.4 *[CMS1, CMS2, MeSc1] Let $H \subset Sp(n, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ be a symplectic holonomy group, i.e. an entry of Table D. Then the map $\phi : \mathfrak{h} \cong \mathfrak{h}^* \to \Lambda^2 V^*$ given by*

$$\phi(A)(x, y) := (2(A, A) + c)\omega(x, y) - 2\omega(Ax, Ay)$$

satisfies the conditions in Lemma 3.3 for any constant $c \in \mathbb{F}$. In particular, there are torsion free connections whose holonomy equals H.

This can be verified immediately from (4), where we use the identification $\mathfrak{h} \cong \mathfrak{h}^*$ via the inner product $(\ ,\ )$. Surprisingly, the converse of Theorem 3.4 is true as well.

Theorem 3.5 *[CMS1, CMS2, MeSc1] Let $H \subsetneq Sp(n, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ be a special symplectic holonomy group, i.e. an entry of Table D, except 1.1–1.4 and 3.1, 3.2 with $Z_{\mathbb{F}} = \mathbb{F}^*$. Then every torsion free connection is locally equivalent to a connection of the form described in Theorem 3.4.*

From this complete characterization, we can now deduce the following properties.

Corollary 3.6 *Let M be a manifold which carries a torsion free connection whose holonomy is contained in a special symplectic holonomy group $H \subsetneq Sp(V, \Omega)$. Then the following hold.*

(1) *The connection is analytic.*

(2) *The map $\pi : F \to P$ has constant even rank $2k$ which we shall call the rank of the connection. $k = 0$ iff the connection is flat.*

(3) *$\pi(F)$ is contained in a $2k$-dimensional symplectic leaf Σ of the Poisson structure on P induced by ϕ. In particular, $\pi : F \to \Sigma$ is a submersion onto its image.*

(4) *Conversely, every symplectic leaf $\Sigma \subset P$ can be covered by open neighborhoods $\{U_\alpha\}$ such that there is a special symplectic connection with $\pi(F_\alpha) = U_\alpha$.*

(5) *The moduli space of torsion free connections with any of the above holonomies is finite dimensional. Indeed, the 2nd derivative of the curvature at a single point in M completely determines the connection on all of M.*

(6) *Let $\mathfrak{g}$ be the Lie algebra of infinitesimal symmetries, i.e. of vector fields X on F with the property that $\mathcal{L}_X(\omega + \theta) = 0$. Then $\dim \mathfrak{g} = \dim P - 2k = \dim \mathfrak{h} + \dim V - 2k$ where k is the rank of the connection. Moreover, $\dim \mathfrak{g} \geq rk(\mathfrak{h}) > 0$.*

Of course, (4) is not an optimal statement. One would like to show that there are connections such that $\pi(F)$ is an *maximal symplectic leaf*. The difficulty is that, in general, one cannot expect to have a *global* symplectic realization $\pi : S \to P$. Moreover, due to the existence of infinitesimal symmetries there is no canonical way to glue germs of these connections together since they can always be "slided" along the infinitesimal symmetries.

Definition 3.7 *Let (M, Ω, ∇) be a triple consisting of a symplectic manifold (M, Ω) and a connection with special symplectic holonomy. We call ∇* maximal *if M is not equivalent to a proper open subset of another manifold with a symplectic connection, i.e. if there is no connection preserving immersion $\imath : M \hookrightarrow M'$ whose image is a proper open subset of M'.*

From the construction and Theorem 3.5, the following is now evident.

Proposition 3.8 *Let (M, Ω, ∇) be a symplectic manifold with a connection with special symplectic holonomy, let $F \to M$ be the holonomy bundle and $\pi : F \to \Sigma \subset P$ be the symplectic realization where $\Sigma \subset P$ is a maximal symplectic leaf of the Poisson structure on $P := (\mathfrak{h} \oplus V)^*$. Moreover, let $\mathfrak{g}$ be the Lie algebra of infinitesimal symmetries.*

Table G: Flat homogeneous vector bundles $\mathbf{E} \to \mathbf{G}/\mathbf{L}_0$ with symplectic holonomy $\mathbf{H}$
Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\mathrm{H} \subset \mathrm{End}(V)$	G/L_0	$\mathrm{L}_0 \subset \mathrm{Aut}(W)$ where $E = \mathrm{G} \times_{\mathrm{L}_0} W$
$\mathrm{Sp}(3, \mathbb{F})$ $V = (\Lambda^3 \mathbb{F}^6)_0$	$\mathrm{Spin}(4,3)/(\mathrm{Spin}(3,2) \cdot \mathrm{Spin}(1,1))$ or $\mathrm{Spin}(7,\mathbb{C})/(\mathrm{Spin}(5,\mathbb{C}) \cdot \mathrm{Spin}(2,\mathbb{C}))$	$\mathrm{L}_0 \cong \mathrm{Sp}(2,\mathbb{F}) \cdot \mathbb{F}^*$ $W = \mathbb{F}^4$
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(n+1,n)$ or $\mathrm{SL}(2,\mathbb{C}) \cdot \mathrm{SO}(2n+1,\mathbb{C})$ $V = \mathbb{F}^2 \otimes \mathbb{F}^{2n+1}$	$\mathrm{SL}(n+2,\mathbb{F})/\mathrm{S}(\mathrm{GL}(2,\mathbb{F}) \cdot \mathrm{GL}(n,\mathbb{F}))$	$W = \mathbb{F}^2 \otimes \mathbb{F}$
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(2p+1,2q)$ $V = \mathbb{R}^2 \otimes \mathbb{R}^{2(p+q)+1}$	$\mathrm{SU}(p+1,q+1)/\mathrm{S}(\mathrm{U}(1,1) \cdot \mathrm{U}(p,q))$	$\mathrm{L}_0 \cong \mathrm{S}(\mathrm{GL}(2,\mathbb{R}) \cdot \mathrm{U}(p,q))$ $W = \mathbb{R}^2 \otimes \mathbb{R}$

Table H: Homogeneous Spaces with special symplectic holonomy of type 2
Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\mathrm{H} \subset \mathrm{End}(V)$	G	$M = \mathrm{G} \cdot \eta \subset \mathfrak{g}^* \cong \mathfrak{g}$ where $\eta \in \mathfrak{g}^* \cong \mathfrak{g}$ equals:
$\mathrm{H} = \mathrm{SL}(2,\mathbb{F})$ $V = \odot^3 \mathbb{F}^2$	$\mathrm{SL}(2,\mathbb{F}) \rtimes \mathbb{F}^2$	$\eta =\ x + v$, where $0 \neq x \in \mathfrak{sl}(2,\mathbb{F})$ is nilpotent $0 \neq v \in ker(x) \subset \mathbb{F}^2$
$\mathrm{Sp}(3,\mathbb{F})$ $V = (\Lambda^3 \mathbb{F}^6)_0$	$\mathrm{G}_2^{4,3} \rtimes \mathbb{R}^7$ if $\mathbb{F} = \mathbb{R}$ $\mathrm{G}_2^{\mathbb{C}} \rtimes \mathbb{C}^7$ if $\mathbb{F} = \mathbb{C}$	$\eta =\ x_\alpha + v_\lambda$, where $0 \neq x_\alpha \in (\mathfrak{g}_2)_\alpha$, α a long root of $\mathfrak{g}_2$ $0 \neq v_\lambda \in (\mathbb{F}^7)_\lambda$, λ a weight of $\mathbb{F}^7$ $(\alpha, \lambda) = 0$

If $\pi : F \to \Sigma$ is surjective and a principal G-bundle where G is a Lie group with Lie algebra $\mathfrak{g}$ such that the flow of the infinitesimal symmetries generates the principal action of G on F, then (M, Ω, ∇) is maximal.

We shall now focus our attention to *homogeneous* examples, i.e. to such examples for which the (local) action of the symmetry group G on M is locally transitive. This is equivalent to saying that the (local) action of $\mathrm{G} \times \mathrm{H}$ on F is transitive and thus equivalent to say that $\pi(F) \subset \Sigma \subset (\mathfrak{h} \oplus V)^*$ consists of a single H-orbit. In other words, we wish to determine those symplectic leafs $\Sigma \subset (\mathfrak{h} \oplus V)^*$ which contain an open H-orbit.

Let (M, Ω, ∇) be a symplectic manifold with a symplectic connection, i.e. a torsion free connection s.th. $\nabla \Omega \equiv 0$. Since the volume form Ω^n with $n = \frac{1}{2} \dim M$ is then also parallel, all curvature endomorphisms are trace free, i.e. $tr(R_p(v, w)) = 0$ for all $v, w \in T_pM$ and all $p \in M$. Thus, the first Bianchi identity for R_p implies that the *Ricci curvature* which is

Table I: Homogeneous Spaces with special symplectic holonomy of type 3

notations/conventions: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$.

$\mathrm{H} \subset \mathrm{End}(V)$	$\mathfrak{g} = \sum_{i=0}^{3} \mathfrak{g}_i$	restrictions
$\mathrm{H} = \mathrm{Sp}(3, \mathbb{R})$ $V = (\Lambda^3 \mathbb{R}^6)_0$	$\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(p, q)$ $\mathfrak{g}_1 = \mathbb{R}^2 \otimes (\odot^2 \mathbb{R}^{p,q})_0$ $\mathfrak{g}_2 = \mathbb{R} \otimes (\odot^2 \mathbb{R}^{p,q})_0$ $\mathfrak{g}_3 = \mathbb{R}^2 \otimes \mathbb{R}$	$(p, q) = (3, 0)$ or $(p, q) = (2, 1)$
$\mathrm{H} = \mathrm{Sp}(3, \mathbb{C})$ $V = (\Lambda^3 \mathbb{C}^6)_0$	$\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$ $\mathfrak{g}_1 = \mathbb{C}^2 \otimes (\odot^2 \mathbb{C}^3)_0$ $\mathfrak{g}_2 = \mathbb{C} \otimes (\odot^2 \mathbb{C}^3)_0$ $\mathfrak{g}_3 = \mathbb{C}^2 \otimes \mathbb{C}$	
$\mathrm{H} = \mathrm{SU}(3, 3)$ $V = \{\alpha \in \Lambda^3 \mathbb{C}^6 \mid *\alpha = \alpha\}$	$\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(p, q)$ $\mathfrak{g}_1 = \mathbb{R}^2 \otimes \mathfrak{su}(p, q)$ $\mathfrak{g}_2 = \mathbb{R} \otimes \mathfrak{su}(p, q)$ $\mathfrak{g}_3 = \mathbb{R}^2 \otimes \mathbb{R}$	$(p, q) = (3, 0)$ or $(p, q) = (2, 1)$
$\mathrm{H} = \mathrm{SL}(6, \mathbb{F})$ $V = \Lambda^3 \mathbb{F}^6$	$\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{sl}(3, \mathbb{F})$ $\mathfrak{g}_1 = \mathbb{F}^2 \otimes \mathfrak{sl}(3, \mathbb{F})$ $\mathfrak{g}_2 = \mathbb{F} \otimes \mathfrak{sl}(3, \mathbb{F})$ $\mathfrak{g}_3 = \mathbb{F}^2 \otimes \mathbb{F}$	
$\mathrm{H} = \begin{cases} \mathrm{Spin}(6,6) & \text{for } \mathbb{F} = \mathbb{R}, \\ \mathrm{Spin}(12, \mathbb{C}) & \text{for } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \Delta_{12} \cong \mathbb{F}^{32}$	$\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{sp}(3, \mathbb{F})$ $\mathfrak{g}_1 = \mathbb{F}^2 \otimes (\Lambda^2 \mathbb{F}^6)_0$ $\mathfrak{g}_2 = \mathbb{F} \otimes (\Lambda^2 \mathbb{F}^6)_0$ $\mathfrak{g}_3 = \mathbb{F}^2 \otimes \mathbb{F}$	
$\mathrm{H} = \mathrm{Spin}(6, \mathbb{H})$ $V = \Delta_6^{\mathbb{H}} \cong \mathbb{R}^{32}$	$\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sp}(p, q)$ $\mathfrak{g}_1 = \mathbb{R}^2 \otimes (\Lambda^2 \mathbb{H}^3)_0$ $\mathfrak{g}_2 = \mathbb{R} \otimes (\Lambda^2 \mathbb{H}^3)_0$ $\mathfrak{g}_3 = \mathbb{R}^2 \otimes \mathbb{F}$	$(p, q) = (3, 0)$ or $(p, q) = (2, 1)$
$\mathrm{H} = \begin{cases} \mathrm{Spin}(6,6) & \text{for } \mathbb{F} = \mathbb{R}, \\ \mathrm{Spin}(12, \mathbb{C}) & \text{for } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \Delta_{12} \cong \mathbb{F}^{32}$	$\mathfrak{g}_0 = \mathfrak{sp}(3, \mathbb{F})$ $\mathfrak{g}_1 = (\Lambda^3 \mathbb{F}^6)_0 \oplus \mathbb{F}^6$ $\mathfrak{g}_2 = (\Lambda^2 \mathbb{F}^6)_0 \oplus \mathbb{F}^6$ $\mathfrak{g}_3 = \mathbb{F}^6$	2 non-equivalent connections for $\mathbb{F} = \mathbb{R}$
$\mathrm{H} = \begin{cases} \mathrm{E}_7^{(5)} & \text{with } \mathbb{F} = \mathbb{R}, \\ \mathrm{E}_7^{(7)} & \text{with } \mathbb{F} = \mathbb{R}, \\ \mathrm{E}_7^{\mathbb{C}} & \text{with } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \mathbb{F}^{56}$	$\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{F}) \oplus (\mathfrak{f}_4^{(a)})$ $\mathfrak{g}_1 = \mathbb{F}^2 \otimes \mathbb{F}^{26}$ $\mathfrak{g}_2 = \mathbb{F} \otimes \mathbb{F}^{26}$ $\mathfrak{g}_3 = \mathbb{F}^2 \otimes \mathbb{F}$	$\mathfrak{f}_4^{(a)} = \begin{cases} \mathfrak{f}_4^{(1)} & \text{for } \mathrm{H} = \mathrm{E}_7^{(5)}, \\ \mathfrak{f}_4^{(2)} & \text{for } \mathrm{H} = \mathrm{E}_7^{(7)}, \\ \mathfrak{f}_4 & \text{for } \mathrm{H} = \mathrm{E}_7^{(7)}, \\ \mathfrak{f}_4^{\mathbb{C}} & \text{for } \mathrm{H} = \mathrm{E}_7^{\mathbb{C}} \end{cases}$

Table I: Homogeneous Spaces with special symplectic holonomy of type 3 (cont.)

$H \subset End(V)$	$\mathfrak{g} = \sum_{i=0}^{3} \mathfrak{g}_i$	restrictions/remarks
$H = \begin{cases} E_7^{(5)} & \text{for } \mathbb{F} = \mathbb{R}, \\ E_7^{\mathbb{C}} & \text{for } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \mathbb{F}^{56}$	$\mathfrak{g}_0 = \mathfrak{sp}(4,\mathbb{F})$ $\mathfrak{g}_1 = (\Lambda^3\mathbb{F}^8)_0$ $\mathfrak{g}_2 = (\Lambda^2\mathbb{F}^8)_0$ $\mathfrak{g}_3 = \mathbb{F}^6$	
$SL(2,\mathbb{R}) \cdot SO(p,q)$ $V = \mathbb{R}^2 \otimes \mathbb{R}^{p+q}$	$\mathfrak{g}_0 = \mathfrak{sp}(k+1,\mathbb{R}) \oplus \mathfrak{so}(p',q')$ $\mathfrak{g}_1 = \mathbb{R}^{2(k+1)} \otimes (\mathbb{R}^{p',q'} \oplus \mathbb{R})$ $\mathfrak{g}_2 = \Lambda^2\mathbb{R}^{2(k+1)} \otimes \mathbb{R}$ $\oplus\ \mathbb{R} \otimes (\mathbb{R}^{p',q'} \oplus \mathbb{R})$ $\mathfrak{g}_3 = \mathbb{R}^{2(k+1)} \otimes \mathbb{R}$	$p \geq 2, \quad q \geq 1$ $p' := p - 2k - 2$ $q' := q - 2k - 1$ $0 \leq k \leq min(\frac{p-2}{2}, \frac{q-1}{2})$
$SL(2,\mathbb{C}) \cdot SO(n,\mathbb{C})$ $V = \mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathfrak{g}_0 = \mathfrak{sp}(k+1,\mathbb{C}) \oplus \mathfrak{so}(m,\mathbb{C})$ $\mathfrak{g}_1 = \mathbb{C}^{2(k+1)} \otimes (\mathbb{C}^m \oplus \mathbb{C})$ $\mathfrak{g}_2 = \Lambda^2\mathbb{C}^{2(k+1)} \otimes \mathbb{C}$ $\oplus\ \mathbb{C} \otimes (\mathbb{C}^m \oplus \mathbb{C})$ $\mathfrak{g}_3 = \mathbb{C}^{2(k+1)} \otimes \mathbb{C}$	$n \geq 3$ $m := n - 4k - 3$ $0 \leq k \leq \frac{n-3}{4}$

given by

$$Ric_p(v,w) := tr(R_p(v,\_)w)$$

is *symmetric*, i.e. $Ric_p(v,w) = Ric_p(w,v)$. We define the section of the endomorphism bundle $\underline{Ric} \in \Gamma(\mathrm{End}(TM))$ by

$$Ric_p(v,w) = \Omega(\underline{Ric}_p v, w) \quad \text{for all } v,w \in T_pM,\ p \in M.$$

From the symmetry of Ric_p it follows that $\underline{Ric}_p \in \mathfrak{sp}(T_pM, \Omega_p)$ and hence, $tr\underline{Ric}_p = 0$. Therefore, the definition of scalar curvature which would be analogous to the one from Riemannian geometry contains no information at all. Instead, we introduce the following notion.

Definition 3.9 *Let (M,Ω,∇) be a symplectic manifold with a symplectic connection ∇, and define $\underline{Ric} \in \Gamma(End(TM))$ as above. Then the* symplectic scalar curvature of ∇ *is the function $scal : M \to \mathbb{F}$ given by $scal := tr(\underline{Ric}^2)$.*

Evidently, the action of the symmetry group preserves the symplectic scalar curvature, whence any homogeneous connection must have constant symplectic scalar curvature. However, as it turns out, the converse of this is almost true as well. More precisely, we have the following result.

Theorem 3.10 *[Sc4] Let (M,Ω,∇) be a symplectic manifold with a special symplectic connection. Call a point $p \in M$* symmetric *if $(\nabla R)_p = 0$, R being the curvature of ∇. Then the following are equivalent.*

Table J: Number of homogeneous connections for all special symplectic holonomies

$\mathrm{H} \subset \mathrm{End}(V)$	V	#(homog. conn. with Hol = H)
$\mathrm{SL}(2,\mathbb{R})$	$\odot^3\mathbb{R}^2 \cong \mathbb{R}^4$	1
$\mathrm{SL}(2,\mathbb{C})$	$\odot^3\mathbb{C}^2 \cong \mathbb{C}^4$	1
$\mathrm{Sp}(3,\mathbb{R})$	$(\Lambda^3\mathbb{R}^6)_0 \cong \mathbb{R}^{14}$	4
$\mathrm{Sp}(3,\mathbb{C})$	$(\Lambda^3\mathbb{C}^6)_0 \cong \mathbb{C}^{14}$	3
$\mathrm{SU}(1,5)$	$\{\alpha \in \Lambda^3\mathbb{C}^6 \mid *\alpha = \alpha\} \cong \mathbb{R}^{20}$	0
$\mathrm{SU}(3,3)$	$\{\alpha \in \Lambda^3\mathbb{C}^6 \mid *\alpha = \alpha\} \cong \mathbb{R}^{20}$	2
$\mathrm{SL}(6,\mathbb{R})$	$\Lambda^3\mathbb{R}^6 \cong \mathbb{R}^{20}$	1
$\mathrm{SL}(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6 \cong \mathbb{C}^{20}$	1
$\mathrm{Spin}(2,10)$	$\Delta_{2,10} \cong \mathbb{R}^{32}$	0
$\mathrm{Spin}(6,6)$	$\Delta_{6,6} \cong \mathbb{R}^{32}$	3
$\mathrm{Spin}(6,\mathbb{H})$	$\Delta_6^{\mathbb{H}} \cong \mathbb{R}^{32}$	2
$\mathrm{Spin}(12,\mathbb{C})$	$\Delta_{12}^{\mathbb{C}} \cong \mathbb{C}^{32}$	2
$\mathrm{E}_7^{(5)}$	$\mathbb{R}^{56}$	2
$\mathrm{E}_7^{(7)}$	$\mathbb{R}^{56}$	2
$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$	2
$\mathrm{Sp}(1)\cdot\mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n \cong \mathbb{R}^{4n}$, $n \geq 2$	0
$\mathrm{SL}(2,\mathbb{R})\cdot\mathrm{SO}(p,q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p,q}$, $p \geq q$, $p+q \geq 3$	$q+\varepsilon, \quad \varepsilon = \begin{cases} 1 & \text{if } p = q \text{ and } q \text{ odd} \\ 1 & \text{if } p+q \text{ odd}, \ p \geq q+2 \\ 2 & \text{if } p = q+1 \\ 0 & \text{otherwise} \end{cases}$
$\mathrm{SL}(2,\mathbb{C})\cdot\mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$, $n \geq 3$	$[\frac{n+1}{4}]+\varepsilon, \quad \varepsilon = \begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd} \end{cases}$

1. *M is locally homogeneous, i.e. there is a locally transitive group action via local diffeomorphisms preserving Ω and ∇.*

2. *M contains no symmetric points and has constant symplectic scalar curvature.*

3. *M contains no symmetric points, and there is a point $p \in M$ for which the function $scal - scal(p)$ vanishes at p of order at least three.*

Since the Poisson structures on $(\mathfrak{h} \oplus V)^*$ are explicitly known, it is possible to classify those symplectic leafs Σ which have an open H-orbit. The resulting classification can be described as follows.

Theorem 3.11 *[Sc4] Each of the total spaces of the flat homogeneous vector bundles over symplectic symmetric spaces $\pi : E \to G/L_0$ in Table G carries a G-invariant symplectic connection with special symplectic holonomy group H whose symplectic scalar curvature is constant non-zero. These connections are maximal and share the following properties.*

1. *The 0-section $E_0 \subset E$ is totally geodesic, and the restriction of the connection to $E_0 \cong G/L_0$ is equivalent to the symmetric connection on G/L_0.*

2. *All fibers $E_p = \pi^{-1}(p)$ are totally geodesic. Moreover, $\Omega|_{E_p}$ is the (unique) L_0-invariant symplectic form on E_p where Ω is the parallel symplectic form on E.*

3. *Let $\mathcal{H}$ be the horizontal distribution on E induced by the symmetric connection on G/L_0. Then $\mathcal{H}$ is Ω-orthogonal to the fibers, and $\Omega|_{\mathcal{H}} = \pi^*(\omega)$ where ω is the symplectic form on G/L_0.*

Moreover, every connection with special symplectic holonomy and with non-zero constant symplectic scalar curvature is locally equivalent to one of these connections.

Of course, the complement of the 0-section of each vector bundle in Table G, i.e. the complement of the set of symmetric points, is G-homogeneous.

Every locally homogeneous space is modelled on a globally homogeneous space, and these are completely classified.

Theorem 3.12 *[Sc4] Let $M = G/L$ be a homogeneous space with a G-invariant symplectic form Ω and a G-invariant special symplectic connection ∇. Then – up to coverings – M is the complement of the 0-section of one of the vector bundles in Table G, or an entry of one of Table H or I.*

Moreover, the homogeneous connections in Table H are maximal, while the homogeneous connections in Table I are not.

For the sake of simplicity of the presentation, we give only the Lie algebra of the symmetry group of the homogeneous spaces from Table I. The explicit form of L $\subset$ G is given in [Sc4].

Recall that a homogeneous space $M = \mathrm{G}/\mathrm{L}$ is called *reductive* if there is a vector space decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ where $\mathfrak{g}$ and $\mathfrak{l}$ are the Lie algebras of G and L, respectively, such that $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$.

After passing to an appropriate cover of M if necessary, we may assume that there is a momentum map $\mu : M \to \mathfrak{g}^*$ where $\mathfrak{g}$ is the Lie algebra of G. The homogeneity implies that μ is an immersion – in fact a covering map – hence we may identify M with its image $\mu(M) \subset \mathfrak{g}^*$. Recall that the coadjoint orbit of any element in $\mathfrak{g}^*$ carries a canonical symplectic structure. We determine which homogeneous spaces $\mu(M) \subset \mathfrak{g}^*$ are coadjoint orbits.

Theorem 3.13 *[Sc4] Let $\pi : E \to G/L_0$ be a homogeneous vector bundle from Table G. Then the momentum map $\mu : E\backslash 0 \to \mathfrak{g}^*$ is the double cover of a coadjoint orbit and thus, $\mu : E \to \mathfrak{g}^*$ is a branched double cover of its image.*

The homogeneous spaces in Table H are equivalent to coadjoint orbits while the homogeneous spaces in Table I are not.

The two homogeneous spaces in Table H with holonomy $H = SL(2, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, are reductive; the remaining homogeneous spaces are not reductive.

Since every holonomy irreducible symmetric space must be pseudo-Riemannian, there cannot be any locally symmetric connections with special symplectic holonomy. By our classification, there are also some special symplectic holonomy groups which do not even admit any *locally homogeneous* connections or, equivalently, no connections of constant symplectic scalar curvature. The number of possibilities of non-isomorphic homogeneous connections for the various special symplectic holonomy groups are listed in Table J.

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