Homogeneous Connections with Special Symplectic Holonomy

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Abstract

We classify all homogeneous symplectic manifolds with a torsion free connection of special symplectic holonomy, i.e. a connection whose holonomy is an absolutely irreducible proper subgroup of the full symplectic group. Thereby, we obtain many new explicit descriptions of manifolds with special symplectic holonomies. We also show that manifolds with such a connection are homogeneous iff they contain no symmetric points and their symplectic scalar curvature is constant.

Contents

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1 Introduction

A connection ∇ on a manifold M provides a recipe for parallel translation of tangent vectors along curves. One of its basic invariants is the *holonomy group* of ∇ which is defined as the group of the automorphisms of the tangent space T_pM induced by parallel translation along p-based loops. Identifying the tangent space T_pM at a point $p \in M$ with a fixed vector space V of the appropriate dimension, we may regard the holonomy group as a subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ which is well-defined up to conjugacy, independent of the choice of $p \in M$.

There is a natural first-order integrability condition that can be posed on the connection, namely the vanishing of its torsion. In fact, throughout this paper we shall assume all connections to be torsion free.

Any H-invariant tensor on V induces a parallel tensor field on M in a canonical way. For example, if $\mathrm{H} \subset \mathrm{O}(n)$ then M carries a parallel Riemannian metric, whence ∇ is the Levi-Civita connection of this metric. Therefore, the holonomy groups $\mathrm{H} \subset \mathrm{O}(n)$ are called *Riemannian holonomy groups*. Moreover, if H is properly contained in $\mathrm{SO}(n)$ and acts irreducibly on V then H is called a *special Riemannian holonomy group*.

Analogously, if $\mathrm{H} \subset \mathrm{Sp}(V)$ where $\mathrm{Sp}(V)$ is the group of automorphisms of V which preserve a symplectic form, then there is a parallel symplectic form on M. Such an H is called a *symplectic holonomy group*, and we call H a *special symplectic holonomy group* if it is properly contained in $\mathrm{Sp}(V)$ and acts absolutely irreducibly on V.

While the possible special Riemannian holonomies were classified in 1955 by Berger [Be1], the existence of *special symplectic holonomy groups* was not known until the beginning of this decade. Namely, in [Br1] Bryant discovered two irreducible holonomy groups of torsion free connections in dimension four which he denoted by G_3 and H_3, respectively. While H_3 is special symplectic, its conformal extension G_3 preserves a symplectic form only up to a scale. (This is a four dimensional phenomenon; indeed, for all dimensions bigger than four, any holonomy group preserving a symplectic form up to a scale must preserve it properly.) Later, Chi, Merkulov and this author found several other examples of special symplectic holonomy groups [CS, CMS1, CMS2, MS1, MS2]. Moreover, in [MS1, S3] it was shown that these examples exhaust all possible special symplectic holonomy groups. Since Bryant had classified the holonomies which are neither Riemannian nor symplectic and showed existence of connections with these holonomies [Br2, Br3] , this finally completed the classification of irreducible holonomy groups.

A most remarkable feature of the special symplectic holonomy groups is the existence of a universal method for the construction of connections with these holonomies. This method is based on a certain quadratic deformation of a linear Poisson structure [CMS1]. As a consequence, it follows that, for a fixed special symplectic holonomy group H, the moduli space of (local) torsion free connections with holonomy contained in H is finite dimensional. This has some strong implications on the rigidity of these connections. For example, there is always a local symmetry group of positive dimension acting on M. Due to these local symmetries, there is an ambiguity when glueing together local neighborhoods of such a manifold, and this implies that there may be cohomological obstructions for the existence of maximal examples [Br1, S3].

There are several explicitly known classes of manifolds with the "conformal symplectic" holonomy group G_3. For example, moduli of rational curves in $\mathbb{CP}^2$ of fixed degree passing through a number of given points carry such a structure [Br1]; in [S2], all *homogeneous* G_3-connections are classified; finally, in [C] degenerate G_3-connections are described via the construction of their moduli space.

On the other hand, the only explicit examples of connections with the special symplectic holonomy H_3 are the space of conics passing through one fixed point [Br1], and a few more connections with a symmetry group of cohomogeneity one [S1]. In particular, no global examples of connections with any of the remaining special symplectic holonomies have been known so far.

It is one task of the present paper to provide such examples for almost all special symplectic holonomy groups. We call a triple (M, Ω, ∇) consisting of a symplectic manifold (M, Ω) and a symplectic connection ∇ *maximal* if it is not equivalent to a proper open subset of another manifold with a symplectic connection. Further, we introduce the notion of the *symplectic scalar curvature* of such a connection which is a quadratic scalar invariant of the curvature tensor. Then we obtain the following result.

Theorem 1.1 *Each of the total spaces of the flat homogeneous vector bundles over symplectic symmetric spaces $\pi : E \to G/L_0$ in Table 1 carries a G-invariant symplectic connection with special symplectic holonomy group H whose symplectic scalar curvature is constant non-zero. These connections are maximal and share the following properties.*

1. *The 0-section $E_0 \subset E$ is totally geodesic, and the restriction of the connection to $E_0 \cong G/L_0$ is equivalent to the symmetric connection on G/L_0.*

2. *All fibers $E_p = \pi^{-1}(p)$ are totally geodesic. Moreover, $\Omega|_{E_p}$ is the (unique) L_0-invariant symplectic form on E_p where Ω is the parallel symplectic form on E.*

3. *Let $\mathcal{H}$ be the horizontal distribution on E induced by the symmetric connection on G/L_0. Then $\mathcal{H}$ is Ω-orthogonal to the fibers, and $\Omega|_{\mathcal{H}} = \pi^*(\omega)$ where ω is the symplectic form on G/L_0.*

Moreover, every connection with special symplectic holonomy and with non-zero constant symplectic scalar curvature is locally equivalent to one of these connections.

To give further examples of (global) symplectic manifolds with special symplectic holonomy, we investigate connections which are (locally) homogeneous. Evidently, any homogeneous manifold must have constant symplectic scalar curvature. But the converse is true as well.

Theorem 1.2 *Let (M, Ω, ∇) be a symplectic manifold with a torsion free symplectic connection, i.e. $\nabla\Omega \equiv 0$. Moreover, assume that the holonomy group of ∇ is special symplectic. Call a point $p \in M$* symmetric *if $(\nabla R)_p = 0$, R being the curvature of ∇. Then the following are equivalent.*

Table 1: Flat homogeneous vector bundles $\mathbf{E} \to \mathbf{G}/\mathbf{L}_0$ with symplectic holonomy H
Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\mathrm{H} \subset \mathrm{End}(V)$	G/L_0	$\mathrm{L}_0 \subset \mathrm{Aut}(W)$ where $E = \mathrm{G} \times_{\mathrm{L}_0} W$
$\mathrm{Sp}(3, \mathbb{F})$ $V = (\Lambda^3\mathbb{F}^6)_0$	$\mathrm{Spin}(4, 3)/(\mathrm{Spin}(3, 2) \cdot \mathrm{Spin}(1, 1))$ or $\mathrm{Spin}(7, \mathbb{C})/(\mathrm{Spin}(5, \mathbb{C}) \cdot \mathrm{Spin}(2, \mathbb{C}))$	$\mathrm{L}_0 \cong \mathrm{Sp}(2, \mathbb{F}) \cdot \mathbb{F}^*$ $W = \mathbb{F}^4$
$\mathrm{SL}(2, \mathbb{R}) \cdot \mathrm{SO}(n + 1, n)$ or $\mathrm{SL}(2, \mathbb{C}) \cdot \mathrm{SO}(2n + 1, \mathbb{C})$ $V = \mathbb{F}^2 \otimes \mathbb{F}^{2n+1}$	$\mathrm{SL}(n + 2, \mathbb{F})/\mathrm{S}(\mathrm{GL}(2, \mathbb{F}) \cdot \mathrm{GL}(n, \mathbb{F}))$	$W = \mathbb{F}^2 \otimes \mathbb{F}$
$\mathrm{SL}(2, \mathbb{R}) \cdot \mathrm{SO}(2p + 1, 2q)$ $V = \mathbb{R}^2 \otimes \mathbb{R}^{2(p+q)+1}$	$\mathrm{SU}(p + 1, q + 1)/\mathrm{S}(\mathrm{U}(1, 1) \cdot \mathrm{U}(p, q))$	$\mathrm{L}_0 \cong \mathrm{S}(\mathrm{GL}(2, \mathbb{R}) \cdot \mathrm{U}(p, q))$ $W = \mathbb{R}^2 \otimes \mathbb{R}$

1. *M is locally homogeneous, i.e. there is a locally transitive group action via local diffeomorphisms preserving Ω and ∇.*

2. *M contains no symmetric points and has constant symplectic scalar curvature.*

3. *M contains no symmetric points, and there is a point $p \in M$ for which the function $scal - scal(p)$ vanishes at p of order at least three.*

Of course, the complement of the 0-section of each vector bundle in Table 1, i.e. the complement of the set of symmetric points, is G-homogeneous.

Every locally homogeneous space is modelled on a globally homogeneous space, and we completely classify these.

Theorem 1.3 *Let $M = G/L$ be a homogeneous space with a G-invariant symplectic form Ω and a G-invariant symplectic connection ∇ with special symplectic holonomy group. Then – up to coverings – M is the complement of the 0-section of one of the vector bundles in Table 1, or an entry of one of the Tables 2 or 3.*

Moreover, the homogeneous connections in Table 2 are maximal, while the homogeneous connections in Table 3 are not.

For the sake of simplicity of the presentation, we give only the Lie algebra of the symmetry group of the homogeneous spaces from Table 3. The explicit form of $\mathrm{L} \subset \mathrm{G}$ is given in section 5.2.2.

Table 2: Homogeneous Spaces with special symplectic holonomy of type 2
Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\mathrm{H} \subset \mathrm{End}(V)$	G	$M = \mathrm{G} \cdot \eta \subset \mathfrak{g}^* \cong \mathfrak{g}$ where $\eta \in \mathfrak{g}^* \cong \mathfrak{g}$ equals:
$\mathrm{H} = \mathrm{SL}(2,\mathbb{F})$ $V = \odot^3\mathbb{F}^2$	$\mathrm{SL}(2,\mathbb{F}) \rtimes \mathbb{F}^2$	$\eta = x + v$, where $0 \neq x \in \mathfrak{sl}(2,\mathbb{F})$ is nilpotent $0 \neq v \in \ker(x) \subset \mathbb{F}^2$
$\mathrm{Sp}(3,\mathbb{F})$ $V = (\Lambda^3\mathbb{F}^6)_0$	$\mathrm{G}_2^{4,3} \rtimes \mathbb{R}^7$ if $\mathbb{F} = \mathbb{R}$ $\mathrm{G}_2^{\mathbb{C}} \rtimes \mathbb{C}^7$ if $\mathbb{F} = \mathbb{C}$	$\eta = x_\alpha + v_\lambda$, where $0 \neq x_\alpha \in (\mathfrak{g}_2)_\alpha$, α a long root of $\mathfrak{g}_2$ $0 \neq v_\lambda \in (\mathbb{F}^7)_\lambda$, λ a weight of $\mathbb{F}^7$ $(\alpha, \lambda) = 0$

Recall that a homogeneous space $M = \mathrm{G}/\mathrm{L}$ is called *reductive* if there is a vector space decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ where $\mathfrak{g}$ and $\mathfrak{l}$ are the Lie algebras of G and L, respectively, such that $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$.

After passing to an appropriate cover of M if necessary, we may assume that there is a momentum map $\mu : M \to \mathfrak{g}^*$ where $\mathfrak{g}$ is the Lie algebra of G. The homogeneity implies that μ is an immersion – in fact a covering map – hence we may identify M with its image $\mu(M) \subset \mathfrak{g}^*$. Recall that the coadjoint orbit of any element in $\mathfrak{g}^*$ carries a canonical symplectic structure. We determine which homogeneous spaces $\mu(M) \subset \mathfrak{g}^*$ are coadjoint orbits.

Theorem 1.4 *Let $\pi : E \to G/L_0$ be a homogeneous vector bundle from Table 1. Then the momentum map $\mu : E\backslash 0 \to \mathfrak{g}^*$ is the double cover of a coadjoint orbit and thus, $\mu : E \to \mathfrak{g}^*$ is a branched double cover of its image.*

The homogeneous spaces in Table 2 are equivalent to coadjoint orbits while the homogeneous spaces in Table 3 are not.

The two homogeneous spaces in Table 2 with holonomy $H = SL(2,\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, are reductive; the remaining homogeneous spaces are not reductive.

Since every holonomy irreducible symmetric space must be pseudo-Riemannian, there cannot be any locally symmetric connections with special symplectic holonomy. By our classification, there are also some special symplectic holonomy groups which do not even admit any *locally homogeneous* connections or, equivalently, no connections of constant symplectic scalar curvature. We list the number of possibilities of non-isomorphic homogeneous connections for the various special symplectic holonomy groups in Table 4.

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Table 3: Homogeneous Spaces with special symplectic holonomy of type 3

notations/conventions: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$.

$\mathrm{H} \subset \mathrm{End}(V)$	$\mathfrak{g} = \sum_{i=0}^{3} \mathfrak{g}_i$	restrictions
$\mathrm{H} = \mathrm{Sp}(3,\mathbb{R})$ $V = (\Lambda^3\mathbb{R}^6)_0$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(p,q)$ $\mathfrak{g}_1 = \mathbb{R}^2 \otimes (\odot^2\mathbb{R}^{p,q})_0$ $\mathfrak{g}_2 = \mathbb{R} \otimes (\odot^2\mathbb{R}^{p,q})_0$ $\mathfrak{g}_3 = \mathbb{R}^2 \otimes \mathbb{R}$	$(p,q) = (3,0)$ or $(p,q) = (2,1)$
$\mathrm{H} = \mathrm{Sp}(3,\mathbb{C})$ $V = (\Lambda^3\mathbb{C}^6)_0$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$ $\mathfrak{g}_1 = \mathbb{C}^2 \otimes (\odot^2\mathbb{C}^3)_0$ $\mathfrak{g}_2 = \mathbb{C} \otimes (\odot^2\mathbb{C}^3)_0$ $\mathfrak{g}_3 = \mathbb{C}^2 \otimes \mathbb{C}$	
$\mathrm{H} = \mathrm{SU}(3,3)$ $V = \{\alpha \in \Lambda^3\mathbb{C}^6 \mid *\alpha = \alpha\}$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(p,q)$ $\mathfrak{g}_1 = \mathbb{R}^2 \otimes \mathfrak{su}(p,q)$ $\mathfrak{g}_2 = \mathbb{R} \otimes \mathfrak{su}(p,q)$ $\mathfrak{g}_3 = \mathbb{R}^2 \otimes \mathbb{R}$	$(p,q) = (3,0)$ or $(p,q) = (2,1)$
$\mathrm{H} = \mathrm{SL}(6,\mathbb{F})$ $V = \Lambda^3\mathbb{F}^6$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{F}) \oplus \mathfrak{sl}(3,\mathbb{F})$ $\mathfrak{g}_1 = \mathbb{F}^2 \otimes \mathfrak{sl}(3,\mathbb{F})$ $\mathfrak{g}_2 = \mathbb{F} \otimes \mathfrak{sl}(3,\mathbb{F})$ $\mathfrak{g}_3 = \mathbb{F}^2 \otimes \mathbb{F}$	
$\mathrm{H} = \begin{cases} \mathrm{Spin}(6,6) & \text{for } \mathbb{F} = \mathbb{R}, \\ \mathrm{Spin}(12,\mathbb{C}) & \text{for } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \Delta_{12} \cong \mathbb{F}^{32}$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{F}) \oplus \mathfrak{sp}(3,\mathbb{F})$ $\mathfrak{g}_1 = \mathbb{F}^2 \otimes (\Lambda^2\mathbb{F}^6)_0$ $\mathfrak{g}_2 = \mathbb{F} \otimes (\Lambda^2\mathbb{F}^6)_0$ $\mathfrak{g}_3 = \mathbb{F}^2 \otimes \mathbb{F}$	
$\mathrm{H} = \mathrm{Spin}(6,\mathbb{H})$ $V = \Delta_6^{\mathbb{H}} \cong \mathbb{R}^{32}$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sp}(p,q)$ $\mathfrak{g}_1 = \mathbb{R}^2 \otimes (\Lambda^2\mathbb{H}^3)_0$ $\mathfrak{g}_2 = \mathbb{R} \otimes (\Lambda^2\mathbb{H}^3)_0$ $\mathfrak{g}_3 = \mathbb{R}^2 \otimes \mathbb{F}$	$(p,q) = (3,0)$ or $(p,q) = (2,1)$
$\mathrm{H} = \begin{cases} \mathrm{Spin}(6,6) & \text{for } \mathbb{F} = \mathbb{R}, \\ \mathrm{Spin}(12,\mathbb{C}) & \text{for } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \Delta_{12} \cong \mathbb{F}^{32}$	$\mathfrak{g}_0 = \mathfrak{sp}(3,\mathbb{F})$ $\mathfrak{g}_1 = (\Lambda^3\mathbb{F}^6)_0 \oplus \mathbb{F}^6$ $\mathfrak{g}_2 = (\Lambda^2\mathbb{F}^6)_0 \oplus \mathbb{F}^6$ $\mathfrak{g}_3 = \mathbb{F}^6$	2 non-equivalent connections for $\mathbb{F} = \mathbb{R}$
$\mathrm{H} = \begin{cases} \mathrm{E}_7^{(5)} & \text{with } \mathbb{F} = \mathbb{R}, \\ \mathrm{E}_7^{(7)} & \text{with } \mathbb{F} = \mathbb{R}, \\ \mathrm{E}_7^{\mathbb{C}} & \text{with } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \mathbb{F}^{56}$	$\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{F}) \oplus (\mathfrak{f}_4^{(a)})$ $\mathfrak{g}_1 = \mathbb{F}^2 \otimes \mathbb{F}^{26}$ $\mathfrak{g}_2 = \mathbb{F} \otimes \mathbb{F}^{26}$ $\mathfrak{g}_3 = \mathbb{F}^2 \otimes \mathbb{F}$	$\mathfrak{f}_4^{(a)} = \begin{cases} \mathfrak{f}_4^{(1)} & \text{for } \mathrm{H} = \mathrm{E}_7^{(5)}, \\ \mathfrak{f}_4^{(2)} & \text{for } \mathrm{H} = \mathrm{E}_7^{(7)}, \\ \mathfrak{f}_4 & \text{for } \mathrm{H} = \mathrm{E}_7^{(7)}, \\ \mathfrak{f}_4^{\mathbb{C}} & \text{for } \mathrm{H} = \mathrm{E}_7^{\mathbb{C}} \end{cases}$

Table 3: Homogeneous Spaces with special symplectic holonomy of type 3 (cont.)

$\mathrm{H} \subset \mathrm{End}(V)$	$\mathfrak{g} = \sum_{i=0}^{3} \mathfrak{g}_i$	restrictions/remarks
$\mathrm{H} = \begin{cases} \mathrm{E}_7^{(5)} & \text{for } \mathbb{F} = \mathbb{R}, \\ \mathrm{E}_7^{\mathbb{C}} & \text{for } \mathbb{F} = \mathbb{C} \end{cases}$ $V = \mathbb{F}^{56}$	$\mathfrak{g}_0 = \mathfrak{sp}(4, \mathbb{F})$ $\mathfrak{g}_1 = (\Lambda^3 \mathbb{F}^8)_0$ $\mathfrak{g}_2 = (\Lambda^2 \mathbb{F}^8)_0$ $\mathfrak{g}_3 = \mathbb{F}^6$	
$\mathrm{SL}(2, \mathbb{R}) \cdot \mathrm{SO}(p, q)$ $V = \mathbb{R}^2 \otimes \mathbb{R}^{p+q}$	$\mathfrak{g}_0 = \mathfrak{sp}(k+1, \mathbb{R}) \oplus \mathfrak{so}(p', q')$ $\mathfrak{g}_1 = \mathbb{R}^{2(k+1)} \otimes (\mathbb{R}^{p',q'} \oplus \mathbb{R})$ $\mathfrak{g}_2 = \Lambda^2 \mathbb{R}^{2(k+1)} \otimes \mathbb{R}$ $\oplus\ \mathbb{R} \otimes (\mathbb{R}^{p',q'} \oplus \mathbb{R})$ $\mathfrak{g}_3 = \mathbb{R}^{2(k+1)} \otimes \mathbb{R}$	$p \geq 2, \quad q \geq 1$ $p' := p - 2k - 2$ $q' := q - 2k - 1$ $0 \leq k \leq min(\frac{p-2}{2}, \frac{q-1}{2})$
$\mathrm{SL}(2, \mathbb{C}) \cdot \mathrm{SO}(n, \mathbb{C})$ $V = \mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathfrak{g}_0 = \mathfrak{sp}(k+1, \mathbb{C}) \oplus \mathfrak{so}(m, \mathbb{C})$ $\mathfrak{g}_1 = \mathbb{C}^{2(k+1)} \otimes (\mathbb{C}^m \oplus \mathbb{C})$ $\mathfrak{g}_2 = \Lambda^2 \mathbb{C}^{2(k+1)} \otimes \mathbb{C}$ $\oplus\ \mathbb{C} \otimes (\mathbb{C}^m \oplus \mathbb{C})$ $\mathfrak{g}_3 = \mathbb{C}^{2(k+1)} \otimes \mathbb{C}$	$n \geq 3$ $m := n - 4k - 3$ $0 \leq k \leq \frac{n-3}{4}$

2 Symplectic manifolds and homogeneous connections

Let V be a vector space over $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We call a pair $(V, \langle\ ,\ \rangle)$ a *symplectic vector space* if $\langle\ ,\ \rangle$ is a non-degenerate skew-symmetric bilinear form on the vector space V. The Lie group of symplectic automorphisms is then defined as

$$\mathrm{Sp}(V, \langle\ ,\ \rangle) = \{x \in \mathrm{Aut}(V) \mid \langle xv, xw \rangle = \langle v, w \rangle \text{ for all } v, w \in V\},$$

and the Lie algebra of symplectic endomorphisms of V is

$$\mathfrak{sp}(V, \langle\ ,\ \rangle) = \{x \in \mathrm{End}(V) \mid \langle xv, w \rangle + \langle v, xw \rangle = 0 \text{ for all } v, w \in V\}.$$

We shall frequently omit the explicit reference to $\langle\ ,\ \rangle$ and thus write $\mathrm{Sp}(V)$ and $\mathfrak{sp}(V)$, respectively. It is known that $\mathfrak{sp}(V)$ is the Lie algebra of $\mathrm{Sp}(V)$, and that both are simple. Moreover, $\mathfrak{sp}(V) \cong \odot^2 V$, with an isomorphism given by

$$(vw) \cdot u := \langle v, u \rangle\, w + \langle w, u \rangle\, v. \tag{1}$$

Definition 2.1 *Let (M, Ω, ∇) be a triple consisting of a connected manifold M with a symplectic form Ω and a torsion free affine connection ∇ such that $\nabla\Omega \equiv 0$. Then ∇ is called a* symplectic connection *on M.*

A torsion free connection ∇ on a manifold M is symplectic w.r.t. some symplectic form Ω iff the holonomy group of the connection is conjugate to a subgroup of $\mathrm{Sp}(V)$. Also, since

Table 4: Number of homogeneous connections for all special symplectic holonomies

$\mathrm{H} \subset \mathrm{End}(V)$	V	$\#$(homog. conn. with Hol = H)
$\mathrm{SL}(2,\mathbb{R})$	$\odot^3\mathbb{R}^2 \cong \mathbb{R}^4$	1
$\mathrm{SL}(2,\mathbb{C})$	$\odot^3\mathbb{C}^2 \cong \mathbb{C}^4$	1
$\mathrm{Sp}(3,\mathbb{R})$	$(\Lambda^3\mathbb{R}^6)_0 \cong \mathbb{R}^{14}$	4
$\mathrm{Sp}(3,\mathbb{C})$	$(\Lambda^3\mathbb{C}^6)_0 \cong \mathbb{C}^{14}$	3
$\mathrm{SU}(1,5)$	$\{\alpha \in \Lambda^3\mathbb{C}^6 \mid *\alpha = \alpha\} \cong \mathbb{R}^{20}$	0
$\mathrm{SU}(3,3)$	$\{\alpha \in \Lambda^3\mathbb{C}^6 \mid *\alpha = \alpha\} \cong \mathbb{R}^{20}$	2
$\mathrm{SL}(6,\mathbb{R})$	$\Lambda^3\mathbb{R}^6 \cong \mathbb{R}^{20}$	1
$\mathrm{SL}(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6 \cong \mathbb{C}^{20}$	1
$\mathrm{Spin}(2,10)$	$\Delta_{2,10} \cong \mathbb{R}^{32}$	0
$\mathrm{Spin}(6,6)$	$\Delta_{6,6} \cong \mathbb{R}^{32}$	3
$\mathrm{Spin}(6,\mathbb{H})$	$\Delta_6^{\mathbb{H}} \cong \mathbb{R}^{32}$	2
$\mathrm{Spin}(12,\mathbb{C})$	$\Delta_{12}^{\mathbb{C}} \cong \mathbb{C}^{32}$	2
$\mathrm{E}_7^{(5)}$	$\mathbb{R}^{56}$	2
$\mathrm{E}_7^{(7)}$	$\mathbb{R}^{56}$	2
$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$	2
$\mathrm{Sp}(1)\cdot\mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n \cong \mathbb{R}^{4n}$, $n \geq 2$	0
$\mathrm{SL}(2,\mathbb{R})\cdot\mathrm{SO}(p,q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p,q}$, $p \geq q$, $p+q \geq 3$	$q+\varepsilon, \quad \varepsilon = \begin{cases} 1 & \text{if } p=q \text{ and } q \text{ odd} \\ 1 & \text{if } p+q \text{ odd}, p \geq q+2 \\ 2 & \text{if } p=q+1 \\ 0 & \text{otherwise} \end{cases}$
$\mathrm{SL}(2,\mathbb{C})\cdot\mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$, $n \geq 3$	$[\frac{n+1}{4}]+\varepsilon, \quad \varepsilon = \begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd} \end{cases}$

Ω^n is a parallel volume form for $n = \frac{1}{2}\dim M$, all curvature endomorphisms are trace free, i.e. $tr(R_p(v,w)) = 0$ for all $v, w \in T_pM$ and all $p \in M$. Thus, the first Bianchi identity for R_p implies that the *Ricci curvature* which is given by

$$Ric_p(v,w) := tr(R_p(v,\_)w)$$

is *symmetric*, i.e. $Ric_p(v,w) = Ric_p(w,v)$. We define the the section of the endomorphism bundle $\underline{Ric} \in \Gamma(\mathrm{End}(TM))$ by

$$Ric_p(v,w) = \Omega(\underline{Ric}_p v, w) \quad \text{for all } v, w \in T_pM,\ p \in M.$$

From the symmetry of Ric_p it follows that $\underline{Ric}_p \in \mathfrak{sp}(T_pM, \Omega_p)$ and hence, $tr\underline{Ric}_p = 0$. Therefore, the definition of scalar curvature which would be analogous to the one from Riemannian geometry contains no information at all. Instead, we introduce the following notion.

Definition 2.2 *Let (M, Ω, ∇) be a symplectic manifold with a symplectic connection ∇, and define $\underline{Ric} \in \Gamma(End(TM))$ as above. Then the* symplectic scalar curvature of ∇ *is the function $scal : M \to \mathbb{F}$ given by $scal := tr(\underline{Ric}^2)$.*

If a manifold M carries a torsion free connection whose holonomy is contained in H then there is a principal H-bundle $\pi : F \to M$, called the *holonomy bundle*, and a $V \oplus \mathfrak{h}$-valued coframe $\theta + \omega$ where θ and ω are called the *tautological one-form* and the *connection one-form,* respectively [KN]. Throughout, we shall assume that H is a connected Lie group. Since $\theta + \omega : T_pF \to V \oplus \mathfrak{h}$ is an isomorphism, we call $\theta + \omega$ the *connection coframe on F.*

Definition 2.3 *Let $\pi : F \to M$ be the holonomy bundle of a torsion free connection and let $\theta + \omega$ be the connection coframe. A vector field X on F is called an* infinitesimal connection symmetry on F *if $\mathcal{L}_X(\theta + \omega) = 0$.*

A vector field X_0 on M is called an infinitesimal connection symmetry on M *if there exists an infinitesimal connection symmetry X on F with $\pi_*(X) = X_0$.*

The connection is called locally homogeneous *if the infinitesimal connection symmetries on M act locally transitive on M, i.e. if for each $v_p \in TM$ there is a connection symmetry X_0 on M such that $(X_0)_p = v_p$.*

It is evident that the infinitesimal symmetries form a Lie algebra which we shall denote by $\mathfrak{g}$. Note that $\mathcal{L}_X\theta = 0$ implies that $\pi_*(X)$ is well-defined. Also, if X_0 is an infinitesimal connection symmetry on M then the infinitesimal connection symmetry X on F satisfying $\pi_*(X) = X_0$ is unique. Thus, the Lie algebras of infinitesimal connection symmetries on M and on F are canonically isomorphic, justifying the ambiguous use of the term.

If ∇ is a symplectic connection on M, i.e. if $\mathrm{H} \subset \mathrm{Sp}(V)$, and if $\pi : F \to M$ is the holonomy bundle then the parallel symplectic form Ω on M is determined by

$$\pi^*(\Omega) = \langle \theta, \theta \rangle .$$

Thus, the action of $\mathfrak{g}$ on M is *symplectic*, i.e. $\mathfrak{L}_X\Omega = 0$ for all $X \in \mathfrak{g}$. If we assume that M is simply connected then there is a momentum map $\mu_0 : M \to \mathfrak{g}^*$ which is – up to adding a constant – uniquely determined by

$$\langle d(\mu_0)_x, X_0\rangle = -(X_0 \lrcorner \Omega_x) \quad \text{for all infinitesimal symmetries } X_0 \in \mathfrak{g} \text{ on } M.$$

Evidently, $\ker(d(\mu_0)_x) = \{v \in T_xM \mid \Omega(v, (X_0)_x) = 0 \text{ for all } X_0 \in \mathfrak{g}\}$. Thus, $d(\mu_0)_x$ is injective iff x has a locally homogeneous neighborhood, and μ_0 is an immersion iff the connection is locally homogeneous.

For $p \in F$ we let

$$\mathfrak{g}_p := \{(\theta + \omega)(X_p) \mid X \text{ an infinitesimal symmetry on } F\} \subset V \oplus \mathfrak{h}.$$

Since the evaluation map map $\mathfrak{g} \to T_pF$ is injective, $\mathfrak{g}_p \cong \mathfrak{g}$ as a vector space. Moreover, we let $\mathfrak{l}_p \subset \mathfrak{g}$ be the Lie algebra $(\theta+\omega)_p^{-1}(\mathfrak{g}_p \cap \mathfrak{h})$ and $pr_p : \mathfrak{g} \cong \mathfrak{g}_p \to V$ the canonical projection.

The following are then standard results on symplectic groups actions. See e.g. [LM, S5].

Proposition 2.4 *Let (M, Ω, ∇) be a symplectic connection on the simply connected manifold M. Let $\pi : F \to M$, θ, ω, $\mathfrak{g}$ and $\mu : M \to \mathfrak{g}^*$ be as above.*

1. *Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then for every $p \in F$, the Lie subgroup $L_p \subset G$ with Lie subalgebra $\mathfrak{l}_p \subset \mathfrak{g}$ is closed.*

2. *For each $p \in F$ the element $\phi_p := (\pi^*\Omega)_p|_{\mathfrak{g}} \in \Lambda^2\mathfrak{g}^*$ is a 2-cocycle, i.e. satisfies $\phi_p([x, y], z) + \phi_p([y, z], x) + \phi_p([z, x], y) = 0$ for all $x, y, z \in \mathfrak{g}$.*

3. *Suppose that for some $p \in F$, the element $\phi_p \in \Lambda^2\mathfrak{g}^*$ is a coboundary, i.e. $\phi_p(x, y) = \eta([x, y])$ for some $\eta \in \mathfrak{g}^*$. Then we may assume that $\mu_0 : M \to \mathfrak{g}^*$ is a G-equivariant submersion onto a coadjoint orbit. Moreover, if ∇ is locally homogeneous, then μ_0 is a local symplectomorphism.*

Thus, in order to decide whether a given locally homogeneous symplectic connection is locally equivalent to a coadjoint orbit, we have to decide if there is an element $\eta \in \mathfrak{g}^*$ such that

$$\eta([x, y]) = \langle \pi_*(x), \pi_*(y)\rangle \tag{2}$$

for all $x, y \in \mathfrak{g}$. The obstruction for the existence of such an element $\eta \in \mathfrak{g}^*$ is represented by the cohomology $H^1(\mathfrak{g}, \mathfrak{g}^*)$ [HS]. As we shall see, this obstruction does not vanish for all homogeneous connections with special symplectic holonomy.

Let us again suppose that there is a G-invariant connection on the homogeneous space $M = \mathrm{G/L}$. Recall that a homogeneous space $M = \mathrm{G/L}$ is called *reductive* if there is a vector space decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ where $\mathfrak{g}$ and $\mathfrak{l}$ are the Lie algebras of G and L, respectively, such that $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$.

Proposition 2.5 *[KN] Let $M = G/L$ be a homogeneous space with G, L connected and with a torsion free G-invariant connection with holonomy H. Let $\pi : F \to M$ be the holonomy bundle, fix $p \in F$ and consider the ismorphism $\mathfrak{g} \to \mathfrak{g}_p \subset \mathfrak{h} \oplus V$ from above.*

Then M is reductive iff there is a L-equivariant map $\tau : V \to \mathfrak{h}$ such that $v + \tau(v) \in \mathfrak{g}_p \subset \mathfrak{h} \oplus V$ for all $v \in V$.

Proof. Since M is homogenous, the projection $pr_p : \mathfrak{g}_p \to V$ is surjective. Thus, linear maps $\tau : V \to \mathfrak{h}$ with $v + \tau(v) \in \mathfrak{g}_p$ for all $v \in V$ exist and are in one-to-one correspondence with vector space decompositions $\mathfrak{g} \cong \mathfrak{g}_p = \{v + \tau(v) \mid v \in V\} \oplus \mathfrak{l}$. If we denote the first summand by $\mathfrak{m}$ then clearly, $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$ iff τ is $\mathfrak{l}$-equivariant. ∎

3 Holonomy and structure equations for symplectic connections

Obviously, a torsion free connection ∇ on a manifold M is symplectic w.r.t. some symplectic form Ω iff its holonomy group H is contained in $\mathrm{Sp}(V)$. Moreover, if H acts irreducibly then Ω is uniquely determined up to constant multiples.

Definition 3.1 *Let $H \subset Sp(V)$ be a proper irreducible Lie subgroup. We call H a* special symplectic holonomy group *if there exists a symplectic connection (M, Ω, ∇) on some symplectic manifold M whose holonomy group is conjugate to H. The corresponding Lie subalgebra $\mathfrak{h} \subset \mathfrak{sp}(V)$ is called a* special symplectic holonomy algebra.

Proposition 3.2 *[S3, S4, CMS1, CMS2] Let $\mathfrak{h} \subset End(V)$ be an irreducible semi-simple Lie subalgebra where V is a finite dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $W := \mathbb{F}^2 \otimes V$ and consider the induced tensor representation of $\mathfrak{h}^+ := \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h} \subset End(W)$. Then the following are equivalent.*

1. *$\mathfrak{h} \subset \mathfrak{sp}(V)$ is the Lie algebra of one of the subgroups $H \subset Sp(V)$ listed in Table 5.*
2. *There is an irreducible symmetric pair $(\mathfrak{g}, \mathfrak{h}^+)$ whose isotropy representation is equivalent to the representation of $\mathfrak{h}^+$ on W.*
3. *There is a symplectic form $\langle\ ,\ \rangle$ on V such that $\mathfrak{h} \subset \mathfrak{sp}(V, \langle\ ,\ \rangle)$, an $\mathfrak{h}$-equivariant map $\circ : \odot^2 V \to \mathfrak{h}$ and an $\mathfrak{h}$-invariant element $(\ ,\ ) \in \odot^2 \mathfrak{h}^*$ which satisfy for all $u, v, w \in V$ and $\mathbf{a} \in \mathfrak{h}$:*

$$\begin{aligned} (\mathbf{a}, u \circ v) &= -2 \langle \mathbf{a}u, v \rangle, \\ (u \circ v)w - (u \circ w)v &= 2 \langle v, w \rangle u + \langle u, w \rangle v - \langle u, v \rangle w. \end{aligned} \tag{3}$$

4. *The following defines a Poisson structure on the vector space* $\mathfrak{h} \oplus V$*:*

$$[x+u, y+v](\mathbf{a}, \mathbf{b}) = (\mathbf{a}, [x,y]) + \langle \mathbf{b}, xv - yu \rangle + \left\langle (2\mathbf{a}^2 + ((\mathbf{a}, \mathbf{a}) + \mathbf{c})Id)u, v \right\rangle, \qquad (4)$$

where $x, y \in \mathfrak{h}$ *and* $u, v \in V$ *are identified with the linear functions* $(x, \ )$, $\langle u, \ \rangle$ *etc. and where* $\mathbf{c} \in \mathbb{F}$ *is an arbitrary constant.*

Proof. The equivalence of the first two is due to the classification of irreducible symmetric spaces [Be2], and one easily calculates that the first Bianchi identity for the $\mathfrak{h}$-invariant curvature tensor on W is equivalent to (3). Finally, the equivalence of (3) and (4) is straightforward. ∎

Table 5: List of special symplectic holonomy groups

Group H	Representation space	Group H	Representation space
$\mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^4 \cong \odot^3\mathbb{R}^2$	E_7^5	$\mathbb{R}^{56}$
$\mathrm{SL}(2,\mathbb{C})$	$\mathbb{C}^4 \cong \odot^3\mathbb{C}^2$	E_7^7	$\mathbb{R}^{56}$
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^{2(p+q)}$, $(p+q) \geq 3$	$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
$\mathrm{SL}(2,\mathbb{C}) \cdot \mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^{2n}$, $n \geq 3$	$\mathrm{Spin}(2,10)$	$\mathbb{R}^{32}$
$\mathrm{Sp}(1) \cdot \mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n \cong \mathbb{R}^{4n}$, $n \geq 2$	$\mathrm{Spin}(6,6)$	$\mathbb{R}^{32}$
$\mathrm{SL}(6,\mathbb{R})$	$\mathbb{R}^{20} \cong \Lambda^3\mathbb{R}^6$	$\mathrm{Spin}(6,\mathbb{H})$	$\mathbb{R}^{32}$
$\mathrm{SU}(1,5)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$	$\mathrm{Spin}(12,\mathbb{C})$	$\mathbb{C}^{32}$
$\mathrm{SU}(3,3)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$	$\mathrm{Sp}(3,\mathbb{R})$	$\mathbb{R}^{14} \subset \Lambda^3\mathbb{C}^6$
$\mathrm{SL}(6,\mathbb{C})$	$\mathbb{C}^{20} \cong \Lambda^3\mathbb{C}^6$	$\mathrm{Sp}(3,\mathbb{C})$	$\mathbb{C}^{14} \subset \Lambda^3\mathbb{C}^6$

The following theorem now summarizes some known results.

Theorem 3.3 *[S3, S4, CMS1, CMS2]*

1. *A proper subgroup* $H \subset Sp(V)$ *is a special symplectic holonomy group iff it is an entry of Table 5.*

2. *Let* M *be a manifold with a torsion free connection whose holonomy is (contained in) the special symplectic holonomy group* $H \subset Sp(V)$*, and let* $\pi : F \to M$ *be the holonomy bundle with connection coframe* $\theta + \omega$*.*

   Then there are maps $\mathbf{a} : F \to \mathfrak{h}$ *and* $\mathbf{b} : F \to V$ *and a constant* $\mathbf{c} \in \mathbb{F}$ *such that the following structure equations are satisfied.*

$$\begin{aligned} d\theta &= -\omega \wedge \theta \\ d\omega &= -\omega \wedge \omega - 2R_{\mathbf{a}}(\theta \wedge \theta) \\ d\mathbf{a} &= -\omega \cdot \mathbf{a} + \mathbf{b} \circ \theta \\ d\mathbf{b} &= -\omega \cdot \mathbf{b} + (2\mathbf{a}^2 + (2(\mathbf{a}, \mathbf{a}) + \mathbf{c})Id_V) \cdot \theta, \end{aligned} \qquad (5)$$

where $R : \mathfrak{h} \to \Lambda^2(V)$ *is determined by*

$$R_{\mathbf{a}}(v, w) = 2\langle v, w\rangle\ \mathbf{a} + v \circ (\mathbf{a}w) - w \circ (\mathbf{a}v) \tag{6}$$

In particular, there is a constant $k_0 \neq 0$ *such that* $\pi^*(scal) = k_0(\mathbf{a}, \mathbf{a})$.

3. *The image of the H-equivariant map* $\rho := \mathbf{a}+\mathbf{b} : F \to \mathfrak{h}\oplus V$ *is contained in a symplectic leaf of the Poisson structure given in (4).*

4. *Every symplectic connection whose holonomy is contained in a special symplectic holonomy group is analytic.*

5. *Let* $\mathfrak{g}$ *be the Lie algebra of infinitesimal symmetries on* F*. Then for any point* $p \in F$ *there is an isomorphism*

$$\begin{array}{rcl} \mathfrak{g} & \longrightarrow & \mathfrak{g}_p := \{(\theta+\omega)(w) \in V \oplus \mathfrak{h} \mid w \in T_pF,\ d\rho_p(w) = 0\} \\ X & \longmapsto & (\theta+\omega)(X_p) \end{array} \tag{7}$$

where $\rho := \mathbf{a} + \mathbf{b} : F \to \mathfrak{h} \oplus V$*. In particular,* $d\rho$ *has constant rank.*

Proof. The first assertion follows from the classification of irreducible holonomy groups [MS1, S3] while the structure equations were determined in [CMS1, Th.3.10]. The third and fourth assertion were demonstrated in [CMS1, Cor.3.12]. The form of the symplectic scalar curvature follows from Definition 2.2 and (6). Finally, the last statement follows from the structure equations (5) together with [S4, Prop.4.8]. ∎

Corollary 3.4 *Let* (M, Ω, ∇) *be a special symplectic manifold, let* $\pi : F \to M$ *be the holonomy bundle and* $\rho : F \to \mathfrak{h} \oplus V$ *be the map from (7). Let* $\Sigma \subset \mathfrak{h} \oplus V$ *be the maximal symplectic leaf containing* $\rho(F)$*. Moreover, let* G *be the (local) group of symmetries of* M*.*

Then there is an open G*-orbit of* M *iff there is an open* H*-orbit of* Σ*. Furthermore, this* G*-orbit is maximal iff* Σ *is* H*-homogeneous.*

Proof. The (local) action of the symmetry group G and the Holonomy group H on the total space of the holonomy bundle $F \to M$ commute. Evidently, the action of G on M is locally homogeneous iff the local action of G × H on F is. The statement then follows from the H-equivariance of ρ and Theorem 3.3.6. ∎

By Theorem 3.3.2, the first and second order derivatives of *scal* vanish at a point $p \in M$ iff they do for the function $(\mathbf{a}, \mathbf{a})$ on the fiber $\pi^{-1}(p)$. From (3) and Theorem 3.3.2 we then get the following.

Corollary 3.5 *Let* (M, Ω, ∇) *be a manifold with special symplectic holonomy. Then the first and second order derivatives of* $scal : M \to \mathbb{F}$ *vanish at a point* $p \in M$ *iff on the fiber* $\pi^{-1}(p)$ *the following equations hold:*

$$\begin{array}{rcl} \mathbf{ab} & = & 0, \\ -3\langle \mathbf{b}, v\rangle\, \mathbf{b} + (\mathbf{b} \circ \mathbf{b})v + \mathbf{a}(2\mathbf{a}^2 + \bar{c}Id_V)v & = & 0 \ \textit{for all } v \in V. \end{array} \tag{8}$$

4 Special symplectic holonomy algebras

There is a one-to-one correspondence between irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{h}^+) = (\mathfrak{g}, \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h})$ and simple Lie algebras $\mathfrak{g}$ which contain a long root space. Namely, let $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be a Cartan decomposition of $\mathfrak{g}_\mathbb{C}$. If $\mathfrak{g} \subset \mathfrak{g}_\mathbb{C}$ contains a long root space $\mathfrak{g}_\alpha$, then it also contains $\mathfrak{g}_{-\alpha}$, whence $H_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}$, and $\mathfrak{s} := span(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}, H_\alpha) \subset \mathfrak{g}$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{F})$. The Eigenspaces of $ad(H_\alpha)$ yield a decomposition

$$\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}_i,$$

where $\mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm\alpha}$, and evidently, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. In particular, we have the symmetric pair $(\mathfrak{g}, \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2)$.

If we let $\mathfrak{h} := \mathfrak{g}_0 \cap H_\alpha^\perp$ then $\mathfrak{h}$ is a Lie subalgebra and $[\mathfrak{h}, \mathfrak{s}] = 0$, i.e. the symmetric pair from above is $(\mathfrak{g}, \mathfrak{s} \oplus \mathfrak{h})$. Moreover, the isotropy representation of this pair is the $\mathfrak{s} \oplus \mathfrak{h}$-module $\mathbb{F}^2 \otimes \mathfrak{g}_1$, and this representation is irreducible if $\mathfrak{g}$ is simple. The skew symmetric form $[\ ,\ ] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_\alpha$ is $\mathfrak{h}$-invariant, whence $\mathfrak{h}$ is a symplectic representation.

Reverting this construction one shows that $\mathfrak{g}$ must be simple for any irreducible symmetric pair $(\mathfrak{g}, \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h})$, and this pair is obtained by this construction above.

A simple Lie algebra $\mathfrak{g}$ contains a long root space iff in the Satake diagram of $\mathfrak{g}$ the nodes corresponding to simple roots α_i which are *not* orthogonal to the maximal root are white [H, OV]. One obtains $\mathfrak{h}$ by deleting these nodes α_i. The representation of $\mathfrak{h}$ on V_1 has one irreducible summand for each deleted node α_i, and it is described by writing on all nodes α_j adjacent to α_i the Cartan number $\langle \alpha_i, \alpha_j \rangle$.

From this and a glance at the Satake diagrams one easily verifies the following:

1. The representation of $\mathfrak{h}$ on V_1 has more than one irreducible summand iff $\mathfrak{g}$ is of type A_n with $n \geq 2$, i.e. $\mathfrak{g}$ is (a real form of) $\mathfrak{sl}(n+1, \mathbb{C})$. In this case, $\mathfrak{h}$ is (a real form of) $\mathfrak{sl}(n, \mathbb{C})$, acting on $V \oplus V^*$ where $V = \mathbb{C}^n$.

2. If $\mathfrak{g}$ is of type C_n then we have $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{F})$ and $\mathfrak{h} = \mathfrak{sp}(n-1, \mathbb{F})$. The representation of $\mathfrak{h}$ on V_1 is equivalent to the standard representation of $\mathfrak{sp}(n-1, \mathbb{F})$ on $\mathbb{F}^{2(n-1)}$.

3. There is a one-to-one correspondence between the entries of Table 5 and the Satake diagrams not of type A_n or C_n for which the node of the adjoint representation is white.

Let us fix the (unique) $ad_\mathfrak{g}$-invariant inner product $(\ ,\ )$ on $\mathfrak{g}$ which satisfies $(\alpha, \alpha) = 2$ ($(\alpha, \alpha) = 3$, respectively) for all long roots α if $\mathfrak{g} \ncong \mathfrak{g}_2$ ($\mathfrak{g} \cong \mathfrak{g}_2$, respectively). Since $\mathfrak{h} \subset \mathfrak{g}$, this restricts to an $ad_\mathfrak{h}$-invariant inner product on $\mathfrak{h}$. With this, we can deduce some statements about the weights of $\mathfrak{h} \subset \mathfrak{sp}(V)$ where we denote by Δ the set of roots of $\mathfrak{h}$ and by Φ the set of weights of V.

Proposition 4.1 *Let $\mathfrak{h} \subset \mathfrak{sp}(V)$ be one of the subalgebras listed in Table 5, and let $(\ ,\ )$ be the $ad_\mathfrak{h}$-invariant inner product on $\mathfrak{h}$ from the preceding paragraph. Then the following hold.*

1. *All weight spaces are one-dimensional.*

2. *For $\lambda \in \Phi$, we have $(\lambda, \lambda) \in \{\frac{3}{2}, \frac{1}{2}\}$ which allows us to refer to* long *and* short *weights. Moreover, if all roots of Δ are long, then so are all weights of Φ.*

3. *For $\lambda \in \Phi$ and $\alpha \in \Delta$ we have $(\lambda, \alpha) \in \{0, \pm 1\}$, provided that λ is a long weight or α is a long root.*

4. *If $\lambda \in \Phi$ is a short weight then $2\lambda \in \Delta$ is a long root.*

5. *Let $\lambda, \mu \in \Phi$ with λ long. Then exactly one of the following holds.*

   (a) $(\lambda, \mu) = \pm\frac{3}{2}$ and $\lambda = \pm\mu$.

   (b) $(\lambda, \mu) = \pm\frac{1}{2}$ and $\lambda = \pm(\mu + \alpha)$ for some $\alpha \in \Delta$ with $(\lambda, \alpha) = 1$.

6. *If $rk(\mathfrak{h}) \geq 3$ then there are long roots $\alpha_i \in \Delta$ with the following properties:*

   (a) $2\lambda_0 = \alpha_1 + \alpha_2 + \alpha_3$, where λ_0 is the maximal weight of Φ,

   (b) $(\alpha_i, \alpha_j) = 2\delta_i^j$,

   (c) α_i is the maximal root in the root system $\{\alpha \in \Delta \mid (\alpha, \alpha_j) = 0, j = 1, \ldots, i-1\}$.

7. *There exists an $\mathfrak{h}$-equivariant map $\circ : \odot^2 V \to \mathfrak{h}$ satisfying (3).*

Proof. Let α_0 be the maximal (long) root of $\mathfrak{g}$. Then $V := V_1$ is spanned by all roots $\alpha \in \Delta_{\mathfrak{g}}$ with $(\alpha, \alpha_0) = 1$. Under the representation of $\mathfrak{h}$, these become weight spaces of weights $\alpha - \frac{1}{2}\alpha_0$. That is,

$$\Phi = \left\{ \alpha - \frac{1}{2}\alpha_0 \;\middle|\; \alpha \text{ a root of } \mathfrak{g} \text{ with } (\alpha, \alpha_0) = 1 \right\}.$$

The proposition follows from this and the standard properties of root systems. We omit further details. ∎

4.1 Complex symplectic holonomy algebras

Throughout this section, all vector spaces and Lie algebras are assumed to be complex. Let $\mathfrak{h} \subset \mathfrak{sp}(V, \langle\ ,\ \rangle)$ be a special symplectic holonomy algebra and let Δ and Φ be the sets of roots of $\mathfrak{h}$ and weights of V, respectively. Moreover, we use the $ad_{\mathfrak{h}}$-invariant inner product $(\ ,\ )$ on $\mathfrak{h}$ for which Proposition 4.1 is valid.

Let $\lambda_0 \in \Phi$ be the dominant weight of a special symplectic holonomy algebra. For $r \in \{\pm 1, \pm 3\}$, we let $V_r := \bigoplus_{\{\lambda \mid (\lambda_0, \lambda) = \frac{r}{2}\}} V_\lambda$. Then, by Proposition 4.1 we have the decomposition

$$V = V_3 \oplus V_1 \oplus V_{-1} \oplus V_{-3} \tag{9}$$

and $V_{\pm 3} = V_{\pm\lambda_0}$. For $r \in \mathbb{Z}$, we define $\Delta_r := \{\alpha \in \Delta \mid (\lambda_0, \alpha) = r\}$ and

$$\mathfrak{n}^{\pm} := \bigoplus_{\alpha\in\Delta_{\pm 1}} \mathfrak{h}_\alpha, \qquad \mathfrak{n}^0 := \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta_0} \mathfrak{h}_\alpha. \tag{10}$$

Since λ_0 is long, Proposition 4.1 implies that $\Delta = \Delta_{-1} \cup \Delta_0 \cup \Delta_1$ and thus,

$$\mathfrak{h} = \mathfrak{n}^- \oplus \mathfrak{n}^0 \oplus \mathfrak{n}^+, \qquad [\mathfrak{n}^i, \mathfrak{n}^j] \subset \mathfrak{n}^{i+j} \quad \text{and} \quad V_r \circ V_s \subset \mathfrak{n}^{\frac{1}{2}(r+s)} \tag{11}$$

with the convention that $\mathfrak{n}^{\pm 2} = \mathfrak{n}^{\pm 3} = 0$. We also define the following subalgebras of $\mathfrak{h}$:

$$\begin{aligned} \mathfrak{p}_s &:= \textstyle\bigoplus_{\alpha\in\Delta_0} \mathfrak{h}_\alpha \oplus \langle\{[\mathfrak{h}_\alpha, \mathfrak{h}_{-\alpha}] \mid \alpha \in \Delta_0\}\rangle \subset \mathfrak{n}^0, \\ \mathfrak{t}^0 &:= \mathfrak{t} \cap \mathfrak{p}_s^\perp, \ \text{ so that } \ \mathfrak{n}^0 = \mathfrak{t}^0 \oplus \mathfrak{p}_s, \\ \mathfrak{p} &:= \{x \in \mathfrak{n}^0 \mid x \cdot V_{\pm 3} = 0\}. \end{aligned} \tag{12}$$

Then $\mathfrak{p}_s \subset \mathfrak{p} \subset \mathfrak{n}^0$ and $\mathfrak{p}_s$ is the maximal semi-simple subalgebra stabilizing λ_0, and $\mathfrak{p}$, $\mathfrak{n}^0$ are central extensions of $\mathfrak{p}_s$. We also have the following:

$$\mathfrak{n}^i \cdot V_r \subset V_{r+2i}, \qquad [\mathfrak{p}, \mathfrak{n}^i] \subset \mathfrak{n}^i, \qquad \mathfrak{p} \cdot V_{\pm 3} = 0. \tag{13}$$

We then define the Lie subgroup $P \subset \mathrm{H}$ by

$$P := \{g \in H \mid g(V_r) \subset V_r, g|_{V_{\pm 3}} = Id_{V_{\pm 3}}\}. \tag{14}$$

Evidently, the Lie algebra of P is $\mathfrak{p}$. Morover, we let N_i and P_s denote the connected Lie subgroups whose Lie algebras are $\mathfrak{n}^i$ and $\mathfrak{p}_s$ which, by definition, is the semi-simple part of $\mathfrak{p}$. From Proposition 4.1, the following is now evident.

Lemma 4.2 *Fix $0 \neq v_\pm \in V_{\pm 3}$. Then the maps*

$$\mathfrak{n}^{\mp} \longrightarrow V_{\pm 1}, \quad x \longmapsto xv_\pm \quad \text{and} \quad V_{\pm 1} \longrightarrow (V_{\mp 1})^*, \quad v \longmapsto \langle v, \_\rangle$$

are P-equivariant isomorphisms.

By (9) and (13), $\mathfrak{n}^\pm$ are nilpotent. Indeed, if $x \in \mathfrak{n}^\pm$ then $x^4 V = 0$ and $x^3 V \subset V_{\pm 3}$.

Definition 4.3 *An element $x \in \mathfrak{n}^\pm$ is said to be* non-degenerate *if $x^3 V \neq 0$. Otherwise, x is called* degenerate.

Proposition 4.4 *If $rk(\mathfrak{h}) \geq 3$ then every P-orbit of an element $x \in \mathfrak{n}^+$ contains an element of the form*

$$\tilde{x} = x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3}, \tag{15}$$

where $x_{\alpha_i} \in \mathfrak{h}_{\alpha_i}$ and the α_i are as in Proposition 4.1,6.

The element $\tilde{x}$ from (15) is called the *normal form* of x. Note that there are only two holonomies $\mathfrak{h}$ with $\mathrm{rk}(\mathfrak{h}) \leq 2$, namely $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{C})$ with $V = \odot^3\mathbb{C}^2$ and $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$ with $V = \mathbb{C}^2 \otimes \mathbb{C}^3$.

Proof. Let $\alpha \in \Delta$ be a root. Then $(\alpha, \beta) \geq -2$ for all $\beta \in \Delta$ and thus, $(ad(\mathfrak{h}_\alpha))^3 = 0$. Therefore,

$$Ad_{exp(x_\alpha)}x = x + [x_\alpha, x] + \frac{1}{2}[x_\alpha, [x_\alpha, x]] \quad \text{for all } x_\alpha \in \mathfrak{h}_\alpha,\ \alpha \in \Delta \text{ and } x \in \mathfrak{h}. \tag{16}$$

Let $\alpha_1 \in \Delta$ be the maximal long root, and let $x \in \mathfrak{n}^+$. We decompose x as

$$x = \sum_{\alpha \in \Delta_1} x_\alpha,$$

where $x_\alpha \in \mathfrak{h}_\alpha$. W.l.o.g. we assume that $x_{\alpha_1} \neq 0$. Moreover, we let $\Delta_1' := \{\alpha \in \Delta_1 \mid (\alpha, \alpha_1) = 0\}$.

Since $[\mathfrak{n}^+, \mathfrak{n}^+] = 0$ by (11), we have $(\beta, \gamma) \geq 0$ for all $\beta, \gamma \in \Delta_1$. Let $\beta \in \Delta_1$ be a root with $(\beta, \alpha_1) = 1$, i.e. $\beta - \alpha_1 \in \Delta_0$. We first assert that $[\mathfrak{h}_{\beta-\alpha_1}, \mathfrak{n}^+] \subset \mathfrak{h}_\beta \oplus \bigoplus_{\alpha \in \Delta_1'} \mathfrak{h}_\alpha$. To see this, let $\gamma \in \Delta_1, \gamma \neq \alpha_1$ be a root for which $[\mathfrak{h}_{\beta-\alpha_1}, \mathfrak{h}_\gamma] = \mathfrak{h}_{\beta-\alpha_1+\gamma} \neq 0$, i.e. $\beta - \alpha_1 + \gamma \in \Delta_1$. Then $0 \leq (\beta - \alpha_1 + \gamma, \alpha_1) = 1 - 2 + (\gamma, \alpha_1)$, and since $\gamma \neq \alpha_1$ and α_1 is long, this implies that $(\gamma, \alpha_1) = 1$, i.e. $\beta - \alpha_1 + \gamma \in {\Delta_1}'$ as claimed.

Thus, if $\beta \in \Delta_1$ is a root such that $(\beta, \alpha_1) = 1$ and $x_\beta \neq 0$, then replacing x by $x' := Ad_{exp(x_{\beta-\alpha_1})}(x)$ for a suitable element $x_{\beta-\alpha_1} \in \mathfrak{h}_{\beta-\alpha_1}$ then, using (16) and the above, we have $x'_\beta = 0$ so that the number of roots $\beta \in \Delta_1$ with $(\beta, \alpha_1) = 1$ and $x_\beta \neq 0$ can be reduced by one. Repeating this process, we conclude that the P-orbit of x contains an element of the form

$$x' = x_{\alpha_1} + \sum_{\alpha \in {\Delta_1}'} x_\alpha.$$

Next, let α_2 be the maximal root in the root system $\{\alpha \in \Delta \mid (\alpha, \alpha_1) = 0\}$. By Proposition 4.1, $(\lambda_0, \alpha_2) = 1$. Then a similar discussion as above implies that the P-orbit of x contains an element of the form

$$x'' = x_{\alpha_1} + x_{\alpha_2} + \sum_{\alpha \in {\Delta_1}''} x_\alpha,$$

where $\Delta_1'' = \{\alpha \in \Delta_1 \mid (\alpha, \alpha_i) = 0,\ i = 1, 2\}$. Let $\beta \in {\Delta_1}''$. Then by Proposition 4.1, $(\beta, \alpha_3) = (\beta, 2\lambda_0) = 2$, hence $\beta = \alpha_3$ since α_3 is long, i.e. ${\Delta_1}'' = \{\alpha_3\}$. ∎

From here, one immediately deduces the following.

Corollary 4.5 *1. Suppose $rk(\mathfrak{h}) \geq 3$. Then an element $x \in \mathfrak{n}^+$ is non-degenerate iff in its normal form (15) we have $x_{\alpha_i} \neq 0$ for $i = 1, 2, 3$.*

2. *Suppose* $rk(\mathfrak{h}) \geq 3$. *Then every* $v \in V_{\pm 1}$ *lies in the* P*-orbit of an element of the form*

$$v = v_{\pm(\lambda_0-\alpha_1)} + v_{\pm(\lambda_0-\alpha_2)} + v_{\pm(\lambda_0-\alpha_3)} \tag{17}$$

with $v_{\pm(\lambda_0-\alpha_i)} \in V_{\pm(\lambda_0-\alpha_i)}$ *where* $\alpha_i \in \Delta$ *are as in Proposition 4.1,6.*

3. *Let* $v \in V_1 \oplus V_3$ *Then either* $v \in V_3$ *or the* N^+*-orbit of* v *contains an element in* V_1.

4. *Let* $v \in V_1 \oplus V_3$. *Then either* $v \circ v \in \mathfrak{n}^+$ *is non-degenerate or* $(v \circ v)^2 = 0$*; in the latter case,* $v \circ v$ *lies in the* P*-orbit of* $\mathfrak{h}_\alpha$ *where* $\alpha \in \Delta$ *is a long root.*

   Thus, if $v \circ v$ *is degenerate then either* $v \circ v = 0$ *or* $rk(v \circ v) = \#\{\lambda \in \Phi \mid (\lambda, \alpha) = 1\}$.

5. *Fix* $0 \neq v_- \in V_{-3}$. *For any constant* $c \neq 0$*, the set*

$$S_c := \{x \in \mathfrak{n}^+ \mid \langle x^3 v_-, v_- \rangle = c\} \tag{18}$$

is a single P*-orbit of codimension* 1. *Moreover, if* $x \in \mathfrak{n}^+$ *is non-degenerate then every element* $y \in \mathfrak{n}^+$ *satisfies* $y = rx + [p, x]$ *for some* $r \in \mathbb{C}$ *and* $p \in \mathfrak{p}$.

6. *If* $x \in \mathfrak{n}^+$ *is non-degenerate then* $x : V_{-1} \to V_1$ *and* $ad(x)^2 : \mathfrak{n}^- \to \mathfrak{n}^+$ *are isomorphisms.*

Proof. The first three assertions are straightforward. For the fourth part, we may assume by (11) that $v \in V_1$ is in its normal form (17) $v = \sum_{i=1}^3 v_{\lambda_0-\alpha_i}$. Since $\lambda_0 - \alpha_i$ is a long weight, it follows that $2(\lambda_0 - \alpha_i)$ is not a root. Thus,

$$v \circ v = 2 \sum_{i<j} v_{\lambda_0-\alpha_i} \circ v_{\lambda_0-\alpha_j}.$$

But now, $v_{\lambda_0-\alpha_i} \circ v_{\lambda_0-\alpha_j} \in \mathfrak{h}_{2\lambda_0-\alpha_i-\alpha_j} = \mathfrak{h}_{\alpha_k}$ where $\{i,j,k\} = \{1,2,3\}$. Thus, by Corollary 4.5, $v \circ v$ is non-degenerate iff $v_{\lambda_0-\alpha_i} \circ v_{\lambda_0-\alpha_j} \neq 0$ for all $i \neq j$ which happens iff $v_{\lambda_0-\alpha_i} \neq 0$ for all i. On the other hand, if, say, $v_{\lambda_0-\alpha_1} = 0$ then $v \circ v = 2 v_{\lambda_0-\alpha_2} \circ v_{\lambda_0-\alpha_3} \in \mathfrak{h}_{\alpha_1}$, and the claim follows.

The fifth statement is easily verified for the two holonomies with $\mathrm{rk}(\mathfrak{h}) \leq 2$, and for higher rank it follows since any two elements of S_c in their normal form lie in the same P-orbit. Therefore, $[\mathfrak{p}, x]$ is a hyperplane in $\mathfrak{n}^+$ and $x \notin [\mathfrak{p}, x]$.

The last statement is clear for $\mathrm{rk}(\mathfrak{h}) \leq 2$. Assume that $\mathrm{rk}(\mathfrak{h}) \geq 3$ and write $x \in \mathfrak{n}^+$ in its normal form (15). Then clearly, there is an element $y \in \mathfrak{n}^-$ such that $[x, y] = v_+ \circ v_-$. From there the assertions follow. ∎

Let us fix a non-degenerate element $\mathbf{a}_+ \in \mathfrak{n}^+$ and let $P_0 \subset P$ and $\mathfrak{p}_0 \subset \mathfrak{p}$ be the stabilizers of $\mathbf{a}_+$. We define the symmetric bilinear form $\sigma = \sigma_{\mathbf{a}_+}$ on V_{-1} by the equation

$$\sigma(v, w) = \sigma_{\mathbf{a}_+}(v, w) := \langle \mathbf{a}_+ v, w \rangle \quad \text{for all } v, w \in V_{-1}. \tag{19}$$

Thus,

$$P_0 = P \cap \mathrm{O}(V_{-1}, \sigma) \quad \text{and} \quad \mathfrak{p}_0 := \mathfrak{p} \cap \mathfrak{so}(V_{-1}, \sigma). \tag{20}$$

Table 6: $\mathfrak{p}$ and $\mathfrak{p}_0$ for special symplectic holonomies

$\mathfrak{h}$	V	$\mathfrak{p}$	V_1	$\mathfrak{p}_0$	$W_1 \subset V_1$
$\mathfrak{sl}(2,\mathbb{C})$	$\odot^3\mathbb{C}^2$	0	$\mathbb{C}$	0	0
$\mathfrak{sp}(3,\mathbb{C})$	$(\Lambda^3\mathbb{C}^6)_0$	$\mathfrak{sl}(3,\mathbb{C})$	$\odot^2\mathbb{C}^3$	$\mathfrak{so}(3,\mathbb{C})$	$(\odot^2\mathbb{C}^3)_0$
$\mathfrak{sl}(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6$	$\mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C})$	$\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{C})$
$\mathfrak{spin}(12,\mathbb{C})$	$\Delta_{12}^{\mathbb{C}}$	$\mathfrak{sl}(6,\mathbb{C})$	$\Lambda^2\mathbb{C}^6$	$\mathfrak{sp}(3,\mathbb{C})$	$(\Lambda^2\mathbb{C}^6)_0$
$\mathfrak{e}_7$	$\mathbb{C}^{56}$	$\mathfrak{e}_6$	$\mathbb{C}^{27}$	$\mathfrak{f}_4$	$\mathbb{C}^{26}$
$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathbb{C} \oplus \mathfrak{so}(n-2,\mathbb{C})$	$\mathbb{C} \oplus \mathbb{C}^{n-2}$	$\mathfrak{so}(n-3,\mathbb{C})$	$\mathbb{C} \oplus \mathbb{C}^{n-3}$

Of course, P_0 fixes $\mathbf{a}_+ v_- \in V_{-1}$ and hence its orthogonal complement

$$W_{-1} := V_{-1} \cap \mathbf{a}_+^\perp = \{y \in V_{-1} \mid \sigma(\mathbf{a}_+ v_-, y) = \langle \mathbf{a}_+^2 v_-, y \rangle = 0\} \tag{21}$$

and the space

$$W_1 := \mathbf{a}_+ W_{-1} \subset V_1.$$

Evidently, $W_1 \cong W_{-1}$ as a P_0-module, and one can show that

$$\begin{aligned} W_{-1} &= \ker(\mathbf{a}_+^2 : V_{-1} \to V_3) = \mathfrak{p}(\mathbf{a}_+ v_-) = [\mathfrak{p}, \mathbf{a}_+] v_-, \\ \text{and} \qquad & \\ W_1 &= \ker(\mathbf{a}_+ : V_1 \to V_3) = \mathfrak{p}(\mathbf{a}_+^2 v_-) = [\mathfrak{p}, \mathbf{a}_+^2] v_-. \end{aligned} \tag{22}$$

As it turns out, $\mathfrak{p}_0$ is again semi-simple for almost all special symplectic holonomy algebras $\mathfrak{h} \subset \mathfrak{sp}(n,\mathbb{C})$. (Some exceptions occur if $\mathrm{rk}(\mathfrak{h}) \leq 3$.) Also, $\mathfrak{p}_s = \mathfrak{p}$ for all entries except $\mathfrak{h} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$ in which case $\mathfrak{p} = \mathfrak{p}_s \oplus \mathbb{C}$. We list the Lie algebras $\mathfrak{p}$ and their representation on V_1 as well as the representations of $\mathfrak{p}_0$ on W_1 in Table 6.

An important observation we make is that $(\mathfrak{p}_s, \mathfrak{p}_0)$ is an *irreducible symmetric pair* in all cases. Next, we let

$$Sym(W_{-1}, \sigma) := \{\phi \in \mathrm{End}(W_{-1}) \mid \sigma(\phi(v), w) = \sigma(\phi(w), v) \text{ for all } v, w \in W_{-1}\}$$

and also, let

$$\mathfrak{n}^\pm_{\mathbf{a}_+} := \{y \in \mathfrak{n}^\pm \mid y V_{\mp 3} \subset W_{\mp 1}\} = \{y \in \mathfrak{n}^\pm \mid \mathbf{a}_+^2 y V_{\mp 3} = 0\}, \tag{23}$$

where the equality of these sets follows from (22). By Lemma 4.2, $\mathfrak{n}^\pm_{\mathbf{a}_+} \cong W_{\pm 1}$ as a P_0-module.

Let $\pi : V_{-1} \to W_{-1}$ be the σ-orthogonal projection. It is then straightforward to verify that the maps

$$\begin{array}{rccc} \imath : & \mathfrak{n}^- & \longrightarrow & Sym(W_{-1}, \sigma) \\ & y & \longmapsto & (\pi \circ y \circ \mathbf{a}_+)|_{W_{-1}} \end{array} \quad \text{and} \quad \begin{array}{rccc} \jmath : & \odot^2 \mathfrak{p}_0 & \longrightarrow & Sym(W_{-1}, \sigma) \\ & (x, y) & \longmapsto & \frac{1}{2}(xy + yx)|_{W_{-1}} \end{array} \tag{24}$$

are well defined and P_0-equivariant.

Proposition 4.6 *Let $\mathfrak{h}$ be a complex special symplectic holonomy group and fix a non-degenerate element $\mathbf{a}_+ \in \mathfrak{n}^+$. Then the map $\imath : \mathfrak{n}^- \to Sym(W_{-1}, \sigma)$ from above is injective. Moreover, the solutions of the equation*

$$\imath(y) = \jmath(p^2) \quad \text{for } p \in \mathfrak{p}_0 \text{ and } y \in \mathfrak{n}^-_{\mathbf{a}_+} \tag{25}$$

can be classified as follows.

1. *If $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{C})$ or $\mathfrak{h} = \mathfrak{sl}(6, \mathbb{C})$ then $y = 0$ and $p = 0$.*
2. *If $\mathfrak{h} = \mathfrak{sp}(3, \mathbb{C})$ then $\imath$ is an isomorphism. Either*
   - *(a) $y = 0$ and $p = 0$, or*
   - *(b) p lies in the orbit of maximal root of $\mathfrak{p}_0 \cong \mathfrak{sl}(2, \mathbb{C})$ and y lies in the orbit of the maximal weight vector of $\mathfrak{n}^-_x \cong M_4$ where M_k denotes the (unique) $(k+1)$-dimensional $\mathfrak{sl}(2, \mathbb{C})$-module.*
3. *If $\mathfrak{h} = \mathfrak{spin}(12, \mathbb{C})$ or $\mathfrak{h} = \mathfrak{e}_7$ then $y = 0$ and either*
   - *(a) $p = 0$ or*
   - *(b) p lies in the orbit of the maximal root of $\mathfrak{p}_0$.*
4. *If $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$ then $y = 0$ and $p \in \mathfrak{so}(n-3, \mathbb{C})$ is such that $p^2\mathbb{C}^{n-3} = 0$.*

Proof. The cases with $\mathrm{rk}(\mathfrak{h}) \leq 2$ are easily proven, thus we assume the contrary.

Let $y \in \ker(\imath) \subset \mathfrak{n}^-$, i.e. $yW_1 \subset \mathbf{a}_+ V_{-3}$. This is equivalent to saying that $y^+ W_{-1} \subset \mathbf{a}_+^2 V_{-3}$ for $y^+ := [\mathbf{a}_+, [\mathbf{a}_+, y]]$ by (22).

Let $z^+ \in \mathfrak{n}^+_{\mathbf{a}_+}$. Then $y^+ z^+ W_{-1} = z^+ y^+ W_{-1} \subset z^+ \mathbf{a}_+^2 V_{-3} = \mathbf{a}_+^2 z^+ V_{-3} \subset \mathbf{a}_+^2 W_{-1} = 0$ by (22) so that $y^+ (\mathfrak{n}^+_{\mathbf{a}_+})^2 V_{-3} = 0$. But now, one shows that $(\mathfrak{n}^+_{\mathbf{a}_+})^2 V_{-3} = V_1$ using the normal form (15) of elements of $\mathfrak{n}^+_{\mathbf{a}_+}$. This implies that $y^+ V_1 = 0$ and thus, $y^+ = 0$ by Lemma 4.2. Therefore, $y = 0$ by Corollary 4.5 and hence, $\imath$ is injective.

Now let us investigate equation (25) for each holonomy group separately. Note that $Sym(W_{-1}, \sigma) \cong \odot^2 W_{-1} \cong \odot^2 W_1$ as a $\mathfrak{p}_0$-module.

1. $\mathfrak{h} \cong \mathfrak{sp}(3, \mathbb{C})$:

   In this case, $\dim(\odot^2 \mathfrak{p}_0) = \dim(\odot^2 W_{\pm 1}) = 6$, so that $\imath$ is an isomorphism. Moreover, $\odot^2 \mathfrak{p}_0 \cong \odot^2 M_2 \cong M_4 \oplus M_0$ by the Clebsch-Gordon formula and thus, $\mathfrak{n}^-_{\mathbf{a}_+} \cong W_{-1} \cong M_4$ as a $\mathfrak{sl}(2, \mathbb{C})$-module. The M_0-summand on $\mathfrak{p}_0$ is represented by the Killing form B of $\mathfrak{p}_0$ so that $\jmath(p^2) \in M_4$ iff $B(p, p) = 0$ iff $p = 0$ or p lies in the orbit of a root space.

2. $\mathfrak{h} \cong \mathfrak{sl}(6, \mathbb{C})$:

   We decompose $\odot^2 \mathfrak{p}_0 \cong M_{max} \oplus \imath(\mathfrak{n}^-_{\mathbf{a}_+}) \oplus M'$ where M_{max} is the irreducible summand whose maximal weight is given by 2α with $\alpha \in \mathfrak{sl}(3, \mathbb{C})$ the maximal root. Then $\jmath(M_{max}) \neq 0$ since $(ad_{\mathfrak{p}_\alpha})^2 \neq 0$.

Let $\pi : \odot^2\mathfrak{p}_0 \to M_{max}$ be the projection. Then the set $\{p \in \mathfrak{p}_0 \mid \pi(\jmath(p^2)) = 0\}$ is closed, $\mathfrak{p}_0$-invariant and does not intersect the orbit of the highest weight vector. It is well-known that for an irreducible representation, the only set with these properties is $\{0\}$, i.e. $\pi(\jmath(p^2)) = 0$ iff $p = 0$. Since $\pi(\jmath(p^2)) = \pi(\imath(y)) = 0$, the claim follows.

3. $\mathfrak{h} \cong \mathfrak{spin}(12, \mathbb{C})$:

   There is only one summand of $\odot^2\mathfrak{p}_0$ isomorphic to $\mathfrak{n}^-_{\mathbf{a}_+} \cong (\Lambda^2\mathbb{C}^6)_0$, namely the one given by the image of the map $\kappa : (\Lambda^2\mathbb{C}^6)_0 \to Sym(\Lambda^2\mathbb{C}^6)$ characterized by the equation $\kappa(\alpha)(\beta) \wedge \omega^2 = \alpha \wedge \beta \wedge \omega$ where ω is the symplectic form preserved by $\mathfrak{sp}(3, \mathbb{C})$. Thus, we must have $\jmath(p^2) = \kappa(\alpha)$ for some $\alpha \in (\Lambda^2\mathbb{C}^6)_0$.

   The set $\{\alpha \in (\Lambda^2\mathbb{C}^6)_0 \mid \kappa(\alpha) = \jmath(p^2)$ for some $p \in \mathfrak{p}_0\} \subset (\Lambda^2\mathbb{C}^6)_0$ is closed and $\mathfrak{p}_0$-invariant and hence is either $\{0\}$ or contains the orbit of the maximal weight vector.

   Let us suppose that there is a $p \in \mathfrak{p}_0$ such that $\jmath(p^2) = \kappa(\alpha)$ where $\alpha \in (\Lambda^2\mathbb{C}^6)_0$ is a maximal weight vector. Fixing a basis $\{e_{\pm i} \mid i = 1, 2, 3\}$ of $\mathbb{C}^6$ such that $\omega = \sum_i e_i \wedge e_{-i}$, we may assume that $\alpha = e_1 \wedge e_2$. Thus, $\kappa(\alpha)(e_i \wedge \mathbb{C}^6) = 0$ for $i = 1, 2, \pm 3$. Now, a straightforward investigation yields that $p^2(e_i \wedge \mathbb{C}^6) = 0$ for $i = 1, 2, \pm 3$ and $p \in \mathfrak{sp}(3, \mathbb{C})$ implies that $p(v) = \omega(v, u)u$ for some fixed $u \in \mathbb{C}^6$. But this means that $\jmath(p^2) = 0$ which is a contradiction.

   Thus, if $\jmath(p^2) \in \imath(\mathfrak{n}^-_{\mathbf{a}_+}) \subset \kappa(\Lambda^2\mathbb{C}^6)_0$ then we must have $\jmath(p^2) = 0$ which implies that either $p = 0$ or p lies in the orbit of the maximal root. Also, $y = 0$ by the first part.

4. $\mathfrak{h} \cong \mathfrak{f}_4$:

   The decomposition of $\odot^2(\mathfrak{p}_0)$ into its irreducible components yields that there is no summand isomorphic to $\mathfrak{n}^-_{\mathbf{a}_+} \cong W_{-1}$. Thus, $\imath(\mathfrak{n}^-_{\mathbf{a}_+}) \cap \jmath(\odot^2\mathfrak{p}_0) = 0$ which means that (25) implies that $\imath(y) = \jmath(p^2) = 0$, thus $y = 0$ by the first part.

   One can then show that each $\mathfrak{p} \in \mathfrak{f}_4$ satisfies $\jmath(p^2) = 0$ iff $p = 0$ or p lies in the orbit of a maximal root. We omit the details.

5. $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$:

   In this case, $\odot^2 W_1 \cong \odot^2\mathbb{C} \oplus (\mathbb{C} \otimes \mathbb{C}^{n-3}) \oplus (\odot^2\mathbb{C}^{n-3})_0 \oplus \mathbb{C}Id_{\mathbb{C}^{n-3}}$. Since $\mathfrak{p}_0$ acts trivially on the $\mathbb{C}$-factor, it follows that $\jmath(\odot^2\mathfrak{p}_0) \subset (\odot^2\mathbb{C}^{n-3})_0 \oplus \mathbb{C}Id_{\mathbb{C}^{n-3}}$. On the other hand, $\mathfrak{n}^-_{\mathbf{a}_+}$ does not contain $(\odot^2\mathbb{C}^{n-3})_0$ as a summand. Comparing these decompositions yields $\imath(\mathfrak{n}^-_{\mathbf{a}_+}) \cap \jmath(\odot^2\mathfrak{p}) \subset \mathbb{C}Id_{\mathbb{C}^{n-3}}$.

   However, a glance at the normal form (15) reveals that $\imath(\mathfrak{n}^-) \cap \mathbb{C}Id_{\mathbb{C}^{n-3}} = 0$ so that (25) implies that $\imath(y) = 0$ and therefore, $y = 0$ by the first part. ∎

4.2 Real symplectic holonomy algebras

The facts about real forms and the notation used in this section are taken from [OV, ch.3,4]. Throughout this section, all vector spaces and Lie groups are assumed to be real unless they are indexed by $\mathbb{C}$.

Let $\mathrm{H} \subset \mathrm{Aut}(V)$ be a real special symplectic holonomy group where V is a real vector space, let $\mathfrak{h} \subset \mathrm{End}(V)$ be its Lie algebra and let $\mathrm{H}_{\mathbb{C}} \subset \mathrm{Aut}(V_{\mathbb{C}})$ and $\mathfrak{h}_{\mathbb{C}} \subset \mathrm{End}(V_{\mathbb{C}})$ be the complexifications. Thus, $\mathfrak{h} \oplus V \subset \mathfrak{h}_{\mathbb{C}} \oplus V_{\mathbb{C}}$ as a real subspace. Let $\mathcal{C}_{\mathbb{C}} \subset \mathbb{P}(V_{\mathbb{C}})$ be the $\mathrm{H}_{\mathbb{C}}$-orbit of the highest weight vector and let $\mathcal{C} := \mathcal{C}_{\mathbb{C}} \cap \mathbb{RP}(V)$.

We choose a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{h}_{\mathbb{C}}$ and let $\mathfrak{t} := \mathfrak{t}_{\mathbb{C}} \cap \mathfrak{h}$. Since $\mathfrak{h}_{\mathbb{C}} \subset \mathrm{End}(V_{\mathbb{C}})$ is also a special symplectic holonomy algebra, there is the decomposition $\mathfrak{h}_{\mathbb{C}} = \mathfrak{n}^-_{\mathbb{C}} \oplus \mathfrak{n}^0_{\mathbb{C}} \oplus \mathfrak{n}^+_{\mathbb{C}}$ and the subalgebra $\mathfrak{p}_{\mathbb{C}} \subset \mathfrak{n}^0_{\mathbb{C}}$ from (10) and (12).

Let $\mathfrak{a} \subset \mathfrak{t} \subset \mathfrak{h}$ be the maximal $\mathbb{R}$-diagonalizable subalgebra. Then there is a decomposition

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{h}_\lambda \quad \text{where } \mathfrak{h}_0 = \mathfrak{a} \oplus \mathfrak{m} \tag{26}$$

for some subset $\Sigma \subset \mathfrak{a}^*$ and some compact subalgebra $\mathfrak{m}$. The following results are standard; we omit the proofs.

Lemma 4.7 *Let $H \subset End(V)$ be a real special symplectic holonomy group and choose a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a} \subset \mathfrak{t} \subset \mathfrak{h}$ and $\mathfrak{m}$ as above. There are real long weights in Φ iff $\mathcal{C} \neq \emptyset$. In this case, we have the decompositions (9) and (10).*

In particular, these conditions are satisfied unless $H \subset End(V)$ is one of the following real holonomy groups:

1. $H = SL(2,\mathbb{R}) \cdot SO(n) \subset End(\mathbb{R}^2 \otimes \mathbb{R}^n)$,
2. $H = Sp(1) \cdot SO(n,\mathbb{H}) \subset End(\mathbb{H}^n)$,
3. $H = SU(1,5) \subset End(\mathbb{R}^{20})$,
4. $H = Spin(2,10) \subset End(\Delta_{2,10})$.

This follows since in terms of the Satake diagram of $\mathfrak{h}$, the dominant weight is real iff in the description of the representation of $\mathfrak{h}$ on V via Dynkin diagrams, there are no non-zero coefficient over a black node, as these correspond to the roots of $\mathfrak{m}$.

Lemma 4.8 *Let $H \subset End(V)$ be a real special symplectic holonomy group. Then either $\mathcal{C} = \emptyset$ or H acts transitively on $\mathcal{C}$.*

Suppose now that we have a fixed decomposition of $\mathfrak{h}$ and V from (9) and (11) and elements $v_\pm \in V_{\pm 3}$ with $\langle v_+, v_- \rangle = 1$ and define for a constant $c \neq 0$ the set $S_c \subset \mathfrak{n}^+$ as in (18). We denote its complexification by $(S_c)_{\mathbb{C}} \subset (\mathfrak{n}^+)_{\mathbb{C}}$. Recall that by Corollary 4.5, $(S_c)_{\mathbb{C}}$ is a single $P_{\mathbb{C}}$-orbit.

Proposition 4.9 *Let $H \subset End(V)$ be a real special symplectic holonomy group for which $\mathcal{C} \neq \emptyset$, fix a Cartan decomposition, $v_\pm \in V_{\pm 3}$ and $c \neq 0$ as above.*

Then there is a one-to-one correspondence between P-orbits of S_c and isomorphism classes of real subalgebras $\mathfrak{p}_0 \subset \mathfrak{p}$ such that $\mathfrak{p}_0 \otimes \mathbb{C} = (\mathfrak{p}_0)_{\mathbb{C}}$. This correspondence is given by associating to each element $x \in S_c$ its infinitesimal stabilizer.

Table 7: $\mathfrak{p}_s$ and $\mathfrak{p}_0$ for simple real special symplectic holonomies with $\mathcal{C} \neq \emptyset$

$\mathfrak{h}$	$\mathfrak{p}_s$	V_1	$\mathfrak{p}_0$	# P_0-orbits of solutions of (25)
$\mathfrak{sl}(2,\mathbb{R})$	0	$\mathbb{R}$		1
$\mathfrak{sp}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{R})$	$\odot^2\mathbb{R}^3$	$\mathfrak{so}(3)$ $\mathfrak{so}(2,1)$	2 2
$\mathfrak{sl}(6,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$	$\mathbb{R}^3 \otimes (\mathbb{R}^3)^*$	$\mathfrak{sl}(3,\mathbb{R})$	1
$\mathfrak{su}(3,3)$	$\mathfrak{sl}(3,\mathbb{C})$	$\left\{ \begin{array}{c} A \in M_{\mathbb{C}}(3) \\ A = A^* \end{array} \right\}$	$\mathfrak{su}(3)$ $\mathfrak{su}(2,1)$	1 1
$\mathfrak{spin}(6,6)$	$\mathfrak{sl}(6,\mathbb{R})$	$\Lambda^2\mathbb{R}^6$	$\mathfrak{sp}(3,\mathbb{R})$	3
$\mathfrak{spin}(6,\mathbb{H})$	$\mathfrak{sl}(3,\mathbb{H})$	$\Lambda^2\mathbb{H}^3$	$\mathfrak{sp}(3)$ $\mathfrak{sp}(2,1)$	1 1
$\mathfrak{e}_7^{(5)}$	$\mathfrak{e}_6^{(1)}$	$\mathbb{R}^{27}$	$\mathfrak{f}_4^{(1)}$	2
$\mathfrak{e}_7^{(7)}$	$\mathfrak{e}_6^{(4)}$	$\mathbb{R}^{27}$	$\mathfrak{f}_4$ or $\mathfrak{f}_4^{(2)}$	1
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(p,q)$ $p \geq q \geq 1,\ p \geq 2$	$\mathfrak{so}(p-1,q-1)$	$\mathbb{R}^{p-1,q-1}$	$\mathfrak{so}(p-2,q-1)$ $\mathfrak{so}(p-1,q-2)$	$[\frac{min(p,q+1)}{2}]$ $[\frac{q}{2}]$

Proof. Evidently, the infinitesimal stabilizer $\mathfrak{p}_0$ of $x \in S_c$ must be a real form of $(\mathfrak{p}_0)_{\mathbb{C}}$ since $S_c = (S_c)_{\mathbb{C}} \cap \mathfrak{n}^+$. Also, if two elements lie in the same P-orbit then their stabilizers are conjugate and hence isomorphic.

Let $\mathfrak{p}_0 \subset \mathfrak{p}$ be a real form of $(\mathfrak{p}_0)_{\mathbb{C}}$. Then the representation of $\mathfrak{p}_0$ on $\mathfrak{n}^+ \cong V_1$ is a real form of the representation of $(\mathfrak{p}_0)_{\mathbb{C}}$ on $(V_1)_{\mathbb{C}}$. By Table 6, this means that there is a $\mathfrak{p}_0$-invariant subspace which intersects S_c for all $c \neq 0$. Thus, $\mathfrak{p}_0$ is the infinitesimal stabilizer of some element $x \in S_c$.

Finally, suppose that $x, x' \in S_c$ have isomorphic infinitesimal stabilizers $\mathfrak{p}_0$ and $\mathfrak{p}_0'$, respectively. A glance at Table 6 shows that $((\mathfrak{p}_s)_{\mathbb{C}}, (\mathfrak{p}_0)_{\mathbb{C}})$ is an irreducible symmetric pair in each case, hence so are $(\mathfrak{p}_s, \mathfrak{p}_0)$ and $(\mathfrak{p}_s, \mathfrak{p}_0')$. Since $\mathfrak{p}_0 \cong \mathfrak{p}_0'$, this implies that there is an automorphism $\imath : \mathfrak{p}_s \to \mathfrak{p}_s$ with $\imath(\mathfrak{p}_0') = \mathfrak{p}_0$. Moreover, one verifies that each class of the outer automorphisms of $\mathfrak{p}_s$ contains an element which leaves $\mathfrak{p}_0$ invariant, thus $\imath$ can be chosen to be inner, hence $\mathfrak{p}_0$ and $\mathfrak{p}_0'$ are conjugate. Therefore, after replacing x' by an element in its P-orbit, we may assume that $\mathfrak{p}_0 = \mathfrak{p}_0'$.

If H is simple then the representation of $\mathfrak{p}_0$ on $\mathfrak{n}^+$ has a one-dimensional invariant subspace, hence x, x' are linearly dependent. Since $x, x' \in S_c$, we conclude that $x = x'$.

If H is not simple then there is a two-dimensional $\mathfrak{p}_0$-invariant subspace W_0 of $\mathfrak{n}^+$, and one verifies directly that P acts transitively on $W_0 \cap S_c$. ∎

Proposition 4.10 *Let $H \subset End(V)$ be a real special symplectic holonomy group with $\mathcal{C} \neq \emptyset$, suppose decompositions (9), (11) and $\mathbf{a}_+ \in S_c \subset \mathfrak{n}^+$ for some $c \neq 0$ has been fixed, and $P_0 \subset P$ and $\mathfrak{p}_0 \subset \mathfrak{p}$ are defined as in (20). Then the number of P_0-orbits of solutions of (25) is as specified in Table 7.*

Proof. Since $\mathfrak{h} \oplus V \subset \mathfrak{h}_{\mathbb{C}} \oplus V_{\mathbb{C}}$, each solution $(y, p) \in \mathfrak{n}^-_{\mathbf{a}_+} \oplus \mathfrak{p}_0$ of (25) must also be a solution in the complexification $(\mathfrak{n}^-_{\mathbf{a}_+})_{\mathbb{C}} \oplus (\mathfrak{p}_0)_{\mathbb{C}}$. Thus, by Proposition 4.6, we have either $(y, p) = (0, 0)$, or $\mathfrak{h}$ is simple and p lies in the orbit of the maximal root of $(\mathfrak{p}_0)_{\mathbb{C}}$, or $\mathfrak{h}$ is not simple and $y = 0$.

If $\mathfrak{h}$ is simple and $(y, p) \neq (0, 0)$ then by Proposition 4.6 $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{spin}(12, \mathbb{C})$, $\mathfrak{e}_7^{\mathbb{C}}$ or $\mathfrak{sp}(3, \mathbb{C})$, and $p \in \mathcal{R}_{\mathbb{C}} \cap \mathfrak{p}_0$ where $\mathcal{R}_{\mathbb{C}}$ is the orbit of the maximal root of $(\mathfrak{p}_0)_{\mathbb{C}}$. But $\mathcal{R}_{\mathbb{C}} \cap \mathfrak{p}_0 \neq \emptyset$ iff in the Satake diagram of $\mathfrak{p}_0$ the nodes corresponding to the simple roots of $(\mathfrak{p}_0)_{\mathbb{C}}$ which are not perpendicular to the maximal root are white. Verifying this condition it follows that this is the case iff $\mathfrak{p}_0 \subset (\mathfrak{p}_0)_{\mathbb{C}}$, $\mathfrak{p} \subset \mathfrak{p}_{\mathbb{C}}$ and $\mathfrak{h} \subset \mathfrak{h}_{\mathbb{C}}$ are the split forms. Thus each P_0-orbit of $\mathcal{R}_{\mathbb{C}} \cap \mathfrak{p}_0$ intersects the maximal root space. It remains to decide if p and $-p$ lie in the same P_0-orbit.

A more careful analysis (cf. [S5] for details) reveals that p and $-p$ lie in the same P_0-orbit if $\mathfrak{h} \cong \mathfrak{sp}(3, \mathbb{R})$, $\mathfrak{e}_7^{(5)}$ but in different P_0-orbits if $\mathfrak{h} \cong \mathfrak{spin}(6, 6)$.

Finally, if $\mathfrak{h}$ is not simple then $y = 0$ and $p \in \mathfrak{so}(p-1, q-2)$ ($\mathfrak{so}(p-2, q-1)$, respectively) is an endomorphism with $p^2 = 0$. It is easy to show that two such p's are P_0-equivalent iff they have equal rank, and the rank can be any even integer $\leq min(p-1, q-2)$ ($\leq min(p-2, q-1)$, respectively). From this, the asserted numbers follow. ∎

5 Classification of degenerate pairs

Let $\mathrm{H} \subset \mathrm{Sp}(V, \langle\ ,\ \rangle)$ be a special symplectic holonomy group with Lie algebra $\mathfrak{h} \subset \mathfrak{sp}(V, \langle\ ,\ \rangle)$ for which we fix a Cartan decomposition $\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{h}_\alpha$. Let (M, Ω, ∇) be a symplectic manifold with a symplectic connections with holonomy H. By Corollary 3.5, the degenerate critical points of the scalar curvature are in one-to-one correspondence with pairs $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ satisfying (8). As it turns out, it is convenient to distinguish the following types of these solutions.

Definition 5.1 *A pair $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ with $\mathbf{b} \neq 0$ which satisfies (8) is called* degenerate. *A degenerate pair is said to be of*

type 1 if $\overline{c} \neq 0$.

type 2 if $\overline{c} = 0$ and $\mathbf{b} \circ \mathbf{b} \neq 0$, or $rk(\mathfrak{h}) = 1$.

type 3 if $\overline{c} = 0$, $\mathbf{b} \circ \mathbf{b} = 0$ and $rk(\mathfrak{h}) \geq 2$.

Lemma 5.2 *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ be a degenerate pair. Then we may replace $(\mathbf{a}, \mathbf{b})$ by an element in its H-orbit such that the following hold:*

1. $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_+$ *with* $\mathbf{a}_i \in \mathfrak{n}^i$,

2. $\mathbf{b} \in V_1$ *and* $0 \neq \mathbf{b} \circ \mathbf{b} \in \mathfrak{n}^+$, *or* $\mathbf{b} \in V_3$ *and* $\mathbf{b} \circ \mathbf{b} = 0$,

3. $\mathbf{a}_0\mathbf{b} = \mathbf{a}_+\mathbf{b} = 0$,

4. $$\mathbf{a}_0(2\mathbf{a}_0^2 + \bar{c}Id_V) = 0, \tag{27}$$

5. *if* $\bar{c} \neq 0$ *then* $[\mathbf{a}_0, \mathbf{a}_+] = 0$ *and* $\mathbf{a}_0 \in \mathfrak{t}$.

Proof. Let $\mathbf{a} = \mathbf{a}_s + \mathbf{a}_n$ be the Jordan decomposition of $\mathbf{a}$ into its semi-simple and nilpotent part such that $[\mathbf{a}_s, \mathbf{a}_n] = 0$. After replacing $\mathbf{a}$ by an element in its H-orbit, we may assume that $\mathbf{a}_s \in \mathfrak{t}$ and $\mathbf{a}_n \in \bigoplus_{\alpha\in\Delta^+} \mathfrak{h}_\alpha$. This implies the first assertion.

Let $\mathbf{b} = \sum_r \mathbf{b}_r$ with $\mathbf{b}_r \in V_r$ be the decomposition of $\mathbf{b} \in V$.

Substitute $v = v_+ \in V_3$ into (8). Since $\mathbf{a}V_3 \subset V_3$ and $(\mathbf{b} \circ \mathbf{b})V_3 \subset V_1 \oplus V_3$, the V_{-3}-component of this equation reads $-3\langle \mathbf{b}_{-3}, v_+\rangle \mathbf{b}_{-3} = 0$, whence $\mathbf{b}_{-3} = 0$ and thus, $\langle \mathbf{b}, v_+\rangle = 0$. Thus, the vanishing of the V_1-component implies that $(\mathbf{b} \circ \mathbf{b})v_+ \subset V_3$, i.e. $\mathbf{b} \circ \mathbf{b} \in \mathfrak{n}^0 \oplus \mathfrak{n}^+$ by Lemma 4.2.

Next, substitute $v \in V_1$ into (8). By the above, both $\mathbf{a}$ and $\mathbf{b} \circ \mathbf{b}$ preserve $V_1 \oplus V_3$, whence the vanishing of the V_{-1}-component of (8) implies $-3\langle \mathbf{b}_{-1}, v\rangle \mathbf{b}_{-1} = 0$ for all $v \in V_1$, whence $\mathbf{b}_{-1} = 0$ so that $\mathbf{b} \in V_1 \oplus V_3$. The second assertion follows from Corollary 4.5 since N^+ preserves the decomposition of $\mathbf{a}$, and the third is the decomposition of $\mathbf{ab} = 0$.

For $v \in V_r$, $r = 1, 3$, the vanishing of the V_r-component of (8) implies that $\mathbf{a}_0(2\mathbf{a}_0^2 + \bar{c}Id_V)v = 0$ for all $v \in V_1 \oplus V_3$, and since $\mathbf{a}_0(2\mathbf{a}_0^2 + \bar{c}Id_V)V_i \subset V_i$ for all i, this implies (27).

Finally, note that $0 = [\mathbf{a}_s, \mathbf{a}] = [\mathbf{a}_s, \mathbf{a}_0] + [\mathbf{a}_s, \mathbf{a}_+]$ and since $\mathbf{a}_s \in \mathfrak{t}$ and $[\mathfrak{t}, \mathfrak{n}^i] \subset \mathfrak{n}^i$, this implies that $[\mathbf{a}_0, \mathbf{a}_s] = 0$. If $\bar{c} \neq 0$ then $\mathbf{a}_0$ is semisimple by (27), hence so is $\mathbf{a}_0 - \mathbf{a}_s$. On the other hand, $\mathbf{a}_0 - \mathbf{a}_s = \mathbf{a}_n - \mathbf{a}_+ \in \bigoplus_{\alpha\in\Delta^+} \mathfrak{h}_\alpha$ is nilpotent. Therefore, $\mathbf{a}_0 = \mathbf{a}_s$ and $\mathbf{a}_n = \mathbf{a}_+$ which implies the last statement. ∎

5.1 Degenerate pairs of type 1

Proposition 5.3 *Let* $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ *be a degenerate pair of type 1. Then we can choose the Cartan decomposition of* $\mathfrak{h}$ *such that* $\mathbf{b} \in V_\lambda$ *where* $\lambda \in \Phi$ *is a short weight, and* $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_+$ *with* $\mathbf{a}_0 \in \mathfrak{t}$ *and*

$$\lambda(\mathbf{a}_0) = 0, \quad \mathbf{b} \circ \mathbf{b} = 2\bar{c}\mathbf{a}_+, \quad \textit{and} \quad \langle \mathbf{b}, v\rangle \mathbf{b} = (2\mathbf{a}_0^2 + \bar{c}Id_V)\mathbf{a}_+ v \quad \textit{for all } v \in V. \tag{28}$$

In particular, degenerate pairs of type 1 exist only for those holonomies $\mathfrak{h}$ *which have short roots, i.e. for* $\mathfrak{h} \cong \mathfrak{sp}(3, \mathbb{F})$, $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2p+1, 2q)$ *and* $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(2n+1, \mathbb{C})$.

Proof. By Lemma 5.2, we may assume that $\mathbf{b} \in V_1 \oplus V_3$ and $\mathbf{b} \circ \mathbf{b} \in \mathfrak{n}^+$, $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_+$ with $\mathbf{a}_0 \in \mathfrak{t}, \mathbf{a}_+ \in \mathfrak{n}^+$ and $[\mathbf{a}_0, \mathbf{a}_+] = 0$. Using (27) we get

$$\mathbf{a}(2\mathbf{a}^2 + \bar{c}Id_V) = \mathbf{a}_+(6\mathbf{a}_0^2 + \bar{c}Id_V) + 6\mathbf{a}_0\mathbf{a}_+^2 + 2\mathbf{a}_+^3. \tag{29}$$

Let $r := \lambda_0(\mathbf{a}_0)$. Then by (27), $2r^2 \in \{0, -\bar{c}\}$. Applying (8) with $v_- \in V_{-3}$, the V_{-1}-component reads $(\mathbf{b} \circ \mathbf{b})v_- + (6r^2 + \bar{c})\mathbf{a}_+ v_- = 0$ and since $6r^2 + \bar{c} \in \{\bar{c}, -2\bar{c}\}$, Lemma 4.2 implies that

$$\mathbf{b} \circ \mathbf{b} = k\bar{c}\mathbf{a}_+, \quad \text{where } k \in \{2, -1\}. \tag{30}$$

Suppose that $\mathbf{b} \circ \mathbf{b} = 0$. Then by (30), $\mathbf{a}_+ = 0$, i.e. $\mathbf{a} = \mathbf{a}_0$ and hence by (27), $\mathbf{a}(2\mathbf{a}^2 + \bar{c}Id_V) = 0$. But then (8) reads $-3\langle \mathbf{b}, v\rangle \mathbf{b} = 0$ for all $v \in V$, implying that $\mathbf{b} = 0$ which was excluded.

Thus, $\mathbf{b} \circ \mathbf{b} \neq 0$, hence by Lemma 5.2 we may assume that $\mathbf{b} \in V_1$. Then the V_3-component of (8) with $v = v_- \in V_{-3}$ reads $2\mathbf{a}_+^3 v_- = 0$, thus $\mathbf{a}_+$ and hence $\mathbf{b} \circ \mathbf{b}$ are degenerate. In particular, Corollary 4.5 implies that $(\mathbf{b} \circ \mathbf{b})^2 = 0$ and hence, $\mathbf{a}_+^2 = 0$. Thus, (8), (29) and (30) imply

$$-3\langle \mathbf{b}, v\rangle \mathbf{b} + (6\mathbf{a}_0^2 + (k+1)\bar{c}Id_V)\mathbf{a}_+ v = 0 \quad \text{for all } v \in V, \text{ where } k \in \{2, -1\}. \tag{31}$$

Next, multiplying (31) by $\mathbf{a}_0$ and using (27) and $\mathbf{a}_0\mathbf{b} = 0$ by Lemma 5.2 we get

$$(k-2)\bar{c}\mathbf{a}_0\mathbf{a}_+ = 0.$$

If $k = -1$ this would imply $\mathbf{a}_0\mathbf{a}_+ = 0$, and hence by (31), $-3\langle \mathbf{b}, v\rangle \mathbf{b} = 0$ for all $v \in V$, thus, $\mathbf{b} = 0$ which is impossible. Thus, we have $k = 2$, and (30) and (31) yield the last two equations of (28).

Let $V = W_0 \oplus W_r \oplus W_{-r}$ be the decomposition of V into the $\mathbf{a}_0$-Eigenspaces. Since $\mathbf{a}_0$ preserves $\langle\ ,\ \rangle$, it follows that $\langle W_0, W_{\pm r}\rangle = \langle W_r, W_r\rangle = \langle W_{-r}, W_{-r}\rangle = 0$; since $[\mathbf{a}_0, \mathbf{a}_+] = 0$ it follows that $\mathbf{a}_+$ preserves this decomposition, and $\mathrm{rk}(\mathbf{a}_+|_{W_r}) = \mathrm{rk}(\mathbf{a}_+|_{W_{-r}})$. Moreover, from $\mathbf{b} \in W_0$ and (28) we obtain $\mathbf{a}_+(W_0) = \mathbf{b}$.

Thus, $\mathrm{rk}(\mathbf{a}_+) = 2\mathrm{rk}(\mathbf{a}_+|_{W_r}) + 1$ is odd, and hence, by (30), so is the rank of $\mathbf{b} \circ \mathbf{b}$. But then, since $\mathbf{b} \circ \mathbf{b}$ is degenerate, Corollary 4.5 implies that $\mathbf{b} \circ \mathbf{b}$ lies in the orbit of $\mathfrak{h}_\alpha$ where $\alpha \in \Delta$ is a long root such that the set

$$S := \{\lambda \in \Phi \mid (\lambda, \alpha) = 1\}$$

contains an odd number of elements. Note, however, that there is an involution $\sigma : S \to S$, $\sigma(\lambda) := \alpha - \lambda$ which has at most one fixed point $\lambda = \frac{1}{2}\alpha$. That is to say, S contains an odd number of elements iff $\frac{1}{2}\alpha \in \Phi$. In particular, Φ contains short weights.

Replacing $\mathbf{b}$ by an element in its P-orbit, we may assume that $\mathbf{b}$ is in its normal form (17). Since $(\mathbf{b} \circ \mathbf{b})^2 = 0$ and by Proposition 4.1,6. this means that

$$2\lambda_0 = \alpha_1 + \alpha_2 + \alpha_3 = \alpha_1 + \alpha_2 + 2\lambda.$$

and

$$\mathbf{b} = \mathbf{b}_{\lambda_0 - \alpha_1} + \mathbf{b}_{\lambda_0 - \alpha_2} = \mathbf{b}_{\lambda + \beta} + \mathbf{b}_{\lambda - \beta},$$

where $\beta = \frac{1}{2}(\alpha_1 - \alpha_2)$. Since $V_\lambda \subset V_1$ by Proposition 4.1,2, $\lambda_0 - \lambda \in \Delta_1$, and $(\lambda_0 - \lambda, \alpha_2) = 1$, whence $\beta = \lambda_0 - \lambda - \alpha_2 \in \Delta$.

Let $\mathrm{H}_\beta := \exp(\langle \mathfrak{h}_\beta, \mathfrak{h}_{-\beta} \rangle)$. Then $\mathrm{H}_\beta \cong \mathrm{SL}(2, \mathbb{C})$ and since $(\beta, \lambda_0) = 0$ it follows that $\mathrm{H}_\beta \subset P$. Moreover, since $\mathbf{a}_0 \mathbf{b} = 0$, we have $\beta(\mathbf{a}_0) = \lambda(\mathbf{a}_0) = 0$ so that H_β leaves $\mathbf{a}_0$ invariant, while it acts on $V_{\lambda+\beta} \oplus V_\lambda \oplus V_{\lambda-\beta}$ via the irreducible 3-dimensional representation. Thus, the H_β-orbit of $\mathbf{b}$ either contains an element in $V_{\lambda+\beta}$ or an element in V_λ. Since $\mathbf{b} \circ \mathbf{b} \neq 0$, the latter is the case, and this completes the proof. ∎

From this explicit description of the degenerate pairs we also get the following result by a straightforward calculation which we omit.

Corollary 5.4 *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ be a degenerate pair of type 1. Then the image of $R_{\mathbf{a}} : \Lambda^2 V \to \mathfrak{h}$ generates all of $\mathfrak{h}$.*

The following proposition is proven by a not very eluminating calculation which we omit. We refer the interested reader to [S5] for details.

Proposition 5.5 *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ be a degenerate pair of type 1, and let $\mathbf{a} = \mathbf{a}_+ + \mathbf{a}_0$ be the Jordan decomposition of $\mathbf{a}$ where $\mathbf{a}_0$ is semi-simple and $\mathbf{a}_+$ is nilpotent. Let $\mathbf{b}_- \in V$ be such that $\mathbf{a}_0 \mathbf{b}_- = 0$ and $\langle \mathbf{b}, \mathbf{b}_- \rangle = 1$. Then the linear map $\tau : V \to \mathfrak{h}$ given by*

$$\tau(v) := \frac{2}{\bar{c}} \mathbf{b} \circ (\mathbf{a}_0 v) + \mathbf{b}_- \circ \left(\frac{\bar{c}}{2} \langle \mathbf{b}, v \rangle \mathbf{b}_- - (2\mathbf{a}_0^2 + \bar{c} Id_V) v \right)$$

satisfies the identity

$$\rho_v(\mathbf{a} + \mathbf{b}) := \xi_{v+\tau(v)}(\mathbf{a} + \mathbf{b}) = 0 \quad \text{for all } v \in V. \tag{32}$$

Moreover, if we let $\mathfrak{l} := stab(\mathbf{a}, \mathbf{b}) \subset \mathfrak{h}$ then there is no $\mathfrak{l}$-equivariant map $\tau : V \to \mathfrak{h}$ which satisfies (32).

Proposition 5.6 *Let $(\mathbf{a}, \mathbf{b}) \subset \mathfrak{h} \oplus V$ be a degenerate pair of type 1 and let a_0 be the semisimple part of $\mathbf{a}$. Then the maximal symplectic leaf $\Sigma \subset \mathfrak{h} \oplus V$ containing $(\mathbf{a}, \mathbf{b})$ is diffeomorphic to the homogeneous vector bundle*

$$\Sigma \cong H \times_{L_0} U, \tag{33}$$

where $L_0 = stab(\mathbf{a}_0)$ and $U := \ker(\mathbf{a}_0) \subset V$. Moreover, the complement of the zero section is the H-orbit of $(\mathbf{a}, \mathbf{b})$.

Proof. By Proposition 5.5, the H-orbit of $(\mathbf{a}, \mathbf{b})$ is open in Σ, whence (28) and the analyticity of Σ imply that the image of the H-equivariant map $s : \Sigma \to \mathfrak{h}$, $s(\mathbf{a}, \mathbf{b}) := \mathbf{a} - 1/2\bar{c}\mathbf{b} \circ \mathbf{b}$ consists of semisimple elements satisfying (27) and $\mathbf{b} \in \ker s(\mathbf{a}, \mathbf{b})$. Since H-orbits of semisimple elements are closed in $\mathfrak{h}$, we obtain an $\mathfrak{h}$-equivariant embedding $\phi : \Sigma \to H_0$, where H_0 is the homogeneous bundle on the right of (33).

To show that ϕ is a diffeomorphism, note that L_0 acts transitively on $U\backslash 0$ so that H_0 consists of only two H-orbits, namely the 0-section and its complement.

Consider the curve $\gamma(t) = (\mathbf{a}_s = 2\bar{c}t^2\mathbf{b} \circ \mathbf{b}, 2\bar{c}t\mathbf{b}) =: (\mathbf{a}(t), \mathbf{b}(t))$. Then by (5), we have $(\xi_{2\mathbf{b}})_\gamma(t) = (2\mathbf{b}(t) \circ \mathbf{b}, 2(2\mathbf{a}(t)^2 + \bar{c}Id)\mathbf{b}) = \gamma'(t)$ as $\mathbf{a}(t)\mathbf{b} = 0$ for all t. Thus, $\gamma(t)$ is contained in a symplectic leaf, and since $\gamma(1/2\bar{c}) = (\mathbf{a}, \mathbf{b})$ it follows that $\gamma(0) = (\mathbf{a}_s, 0) \in \Sigma$, whence ϕ is a diffeomorphism. ∎

Theorem 5.7 *Let $(\mathbf{a}, \mathbf{b}) \subset \mathfrak{h} \oplus V$ be a degenerate pair of type 1. Then there is a G-invariant symplectic connection on the total space of the homogeneous vector bundle $E \to G/L_0$ from Table 1 corresponding to holonomy group H whose curvature at any point $p \in E\backslash 0$ is represented by $(\mathbf{a}, \mathbf{b})$. Moreover, the homogeneous space $E\backslash 0 = G/L$ is not reductive, and the momentum map $\mu : G/L \to \mathfrak{g}^*$ is a double covering of a coadjoint orbit.*

Proof. Since the Lie algebra $\mathfrak{g}$ of infinitesimal symmetries is isomorphic for all points of a symplectic leaf, we may calculate this at a point $(\mathbf{a}_s, 0) \in \Sigma$ by Proposition 5.6. Using (7) from Theorem 3.3, we obtain

$$\mathfrak{g} = \mathfrak{l}_0 \oplus \{\xi_v \mid (2\mathbf{a}_s^2 + \bar{c})v = 0\}, \tag{34}$$

where $\mathfrak{l}_0$ is the Lie algebra of L_0. Evidently, $(\mathfrak{g}, \mathfrak{l}_0)$ is a symmetric pair, and from here one easily calculates the symmetry algebras for the holonomy groups with short roots.

Conversely, from our calculations so far it follows that the principal G-bundle $F := (\mathrm{G} \times H) \times_{\Delta L_0} W \to \Sigma$ carries solutions to the structure equations (7) from Theorem 3.3 and therefore yields a torsion free special symplectic connection on $F/\mathrm{H} = \mathrm{G} \times_{L_0} W$ whose curvature is represented by the map $F \to \Sigma \subset \mathfrak{h} \oplus V$. By Corollary 5.4 and the *Ambrose-Singer Holonomy theorem* [AS] it follows that the holonomy of this connection equals *all of* H, and Propositions 2.5 and 5.5 imply that this space is not reductive.

That $\mu : E\backslash 0 \to \mathfrak{g}^*$ is a double covering of a coadjoint orbit follows from another explicit calculation which we omit. ∎

Of course, Proposition 2.4 already implies that $\mu(\mathrm{G/L}) \subset \mathfrak{g}^*$ must be a coadjoint orbit since $H^1(\mathfrak{g}, \mathfrak{g}^*) = 0$ for simple $\mathfrak{g}$ by the *Whitehead Lemma* [HS].

5.2 Degenerate pairs of type 2 or 3

Proposition 5.8 *Let $\mathfrak{h} \subset End(V)$ be a complex symplectic holonomy algebra and fix a decomposition of V and $\mathfrak{h}$ as in (9) and (11). Let $v_\pm \in V_{\pm 3}$ be such that $\langle v_+, v_- \rangle = 1$ and define $S_c \subset \mathfrak{n}^+$ as in (18). Fix an element $\mathbf{a}_+ \in S_{\frac{3}{2}} \subset \mathfrak{n}^+$, i.e.*

$$\mathbf{a}_+^3 v_- = \frac{3}{2} v_+ \tag{35}$$

and let $P_0 \subset P$, $\mathfrak{p}_0 \subset \mathfrak{p}$, $\mathfrak{n}^-_{\mathbf{a}_+} \subset \mathfrak{n}^-$ and $W_{-1} \subset V_{-1}$ as defined in (20), (21) and (23).

Then $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ *is a degenerate pair of type 2 or 3 iff the* H*-orbit of* $(\mathbf{a}, \mathbf{b})$ *contains a pair of the form* $(\mathbf{a}_+ + \mathbf{a}_0 + \mathbf{a}_-, v_+ - 2\mathbf{a}_-\mathbf{a}_+^2 v_-)$ *where* $\mathbf{a}_0 \in \mathfrak{p}_0$ *and* $\mathbf{a}_- \in \mathfrak{n}^-_{\mathbf{a}_+}$ *are such that*

$$\jmath(\mathbf{a}_0^2) = -3\imath(\mathbf{a}_-) \tag{36}$$

with $\imath, \jmath$ *from (24).*

The proof consists of a lengthy calculation which has been written in detail in [S5]. We omit it here. Thus, Propositions 4.6 and 5.8 immediately yield the following

Corollary 5.9 *Let* $H \subset Aut(V)$ *be a complex symplectic holonomy group with holonomy algebra* $\mathfrak{h}$. *Choose* $v_\pm \in V_{\pm 3}$ *with* $\langle v_+, v_- \rangle = 1$ *and a non-degenerate* $\mathbf{a}_+ \in \mathfrak{n}^+$ *with* $\mathbf{a}_+^3 v_- = \frac{3}{2} v_+$ *and define* $\mathfrak{p}_0 \subset \mathfrak{p}$ *as in (20). Then* $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ *is a degenerate pair of type 3 iff one of the following holds.*

1. $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$ *and the* H*-orbit of* $(\mathbf{a}, \mathbf{b})$ *contains a pair* $(\mathbf{a}_+ + \mathbf{a}_0, v_+)$ *where* $\mathbf{a}_0 \in \mathfrak{p}_0$ *is such that* $\mathbf{a}_0^2 = 0$. *The* $P_0 \cong SO(n-3, \mathbb{C})$*-orbit of* $\mathbf{a}_0 \in \mathfrak{p}_0$ *with* $\mathbf{a}_0^2 = 0$ *is determined by the rank of* $\mathbf{a}_0$ *which can be any even integer* $\leq \frac{n-3}{2}$. *Thus, there are exactly* $[\frac{n+1}{4}]$ *such orbits.*
2. $\mathfrak{h} \cong \mathfrak{sp}(3, \mathbb{C})$ *or* $\mathfrak{h} \cong \mathfrak{sl}(6, \mathbb{C})$ *and* $(\mathbf{a}, \mathbf{b})$ *lies in the* H*-orbit of the pair* $(\mathbf{a}_+, v_+)$.
3. $\mathfrak{h} \cong \mathfrak{spin}(12, \mathbb{C})$ *or* $\mathfrak{h} \cong \mathfrak{e}_7^{\mathbb{C}}$ *and* $(\mathbf{a}, \mathbf{b})$ *lies in the* H*-orbit of the pair* $(\mathbf{a}_+ + \mathbf{a}_0, v_+)$, *where either* $\mathbf{a}_0 = 0$ *or* $\mathbf{a}_0$ *is a long root vector of* $\mathfrak{p}_0$.

Moreover, $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ *is a degenerate pair of type 2 iff* $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C})$ *and* $(\mathbf{a}, \mathbf{b})$ *lies in the* H*-orbit of the pair* $(\mathbf{a}_+, v_+)$, *or* $\mathfrak{h} \cong \mathfrak{sp}(3, \mathbb{C})$ *and* $(\mathbf{a}, \mathbf{b})$ *lies in the* H*-orbit of the pair* $(\mathbf{a}_+ + \mathbf{a}_0 + \mathbf{a}_-, v_+ - 2\mathbf{a}_-\mathbf{a}_+^2 v_-)$ *where* $\mathbf{a}_0 \in \mathfrak{p}_0 \cong \mathfrak{sl}(2, \mathbb{C})$ *is a root vector and* $\mathbf{a}_-$ *is uniquely determined by (36).*

Corollary 5.10 *Let* $H \subset Aut(V)$ *be a real symplectic holonomy algebra.*

1. *There are no degenerate pairs of type 2 or type 3 if* $\mathcal{C} = \emptyset$, *i.e. for those holonomy groups listed in Lemma 4.7.*
2. *If* H *satisfies* $\mathcal{C} \neq \emptyset$ *then there is a one-to-one correspondence between* H*-orbits of degenerate pairs of type 2 and type 3, and* P_0*-orbits of solutions of (25) for a fixed* $v_+ \in \mathcal{C}$ *and* $\mathbf{a}_+ \in S_{\frac{3}{2}}$. *In particular, the number of such orbits is the one specified in Table 7 on page 23. Moreover, for both* $H = SL(2, \mathbb{R})$ *and* $H = Sp(3, \mathbb{R})$, *there is exactly one* H*-orbit of degenerate pairs of type 2.*

Proof. In the proof, we use the notational conventions of section 4.2.

If $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V \subset \mathfrak{h}_{\mathbb{C}} \oplus V_{\mathbb{C}}$ is a degenerate pair of type 3 then $[\mathbf{b}] \in \mathcal{C}$, hence the first part follows from Lemma 4.7. Moreover, by Lemma 4.8, we may assume that $\mathbf{b} = v_+ \in V_3$ for some fixed decomposition of V and $\mathfrak{h}$ as in (9) and (11), and $\mathbf{a} = \mathbf{a}_+ + \mathbf{a}_0$ with $\mathbf{a}_+ \in S_{\frac{3}{2}}$

and $\jmath(\mathbf{a}_0^2) = 0$. Any two such pairs $(\mathbf{a}, v_+)$ lie in the same H-orbit iff they lie in the same P-orbit, and the statement then follows from Proposition 4.10.

If $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V \subset \mathfrak{h}_{\mathbb{C}} \oplus V_{\mathbb{C}}$ is a degenerate pair of type 2 then, by Corollary 5.9 we must have $\mathrm{H} = \mathrm{Sp}(3, \mathbb{R})$, hence $\mathcal{C} \neq \emptyset$. Again Lemma 4.8 and Proposition 4.10 apply. ∎

Another calculation then yields the following result:

Proposition 5.11 *[S5] Let $H \subset Aut(V)$ be a (real or complex) special symplectic holonomy group. If $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ is a symmetric pair of type 2 or type 3 then the image of $R_{\mathbf{a}} : \Lambda^2 V \to \mathfrak{h}$ generates all of $\mathfrak{h}$.*

5.2.1 Degenerate pairs of type 2

If H is a special symplectic holonomy group then by Corollary 5.9, degenerate pairs of type 2 may exist only if (the complexification of) H equals $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{Sp}(3, \mathbb{C})$, i.e. $\mathrm{H} = \mathrm{SL}(2, \mathbb{F})$ or $\mathrm{H} = \mathrm{Sp}(3, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $V = \odot^3 \mathbb{F}^2$ or $V = (\Lambda^3 \mathbb{F}^6)_0$, respectively.

The case where $\mathrm{H} = \mathrm{SL}(2, \mathbb{F})$ has been treated in [Br1, S1]. It follows that there is exactly one degenerate pair of type 2 which represents the curvature of a homogeneous symplectic connection on the coadjoint orbit specified in Table 2. This homogeneous space is known to be reductive [S1].

Thus, we shall concentrate on the case where $\mathrm{H} = \mathrm{Sp}(3, \mathbb{F})$. We fix a basis $e_{\pm i}$, $i = 1, 2, 3$ of $\mathbb{F}^6$ such that the symplectic form $\omega = \sum_i e_i \wedge e_{-i}$ is H-invariant, and the symplectic form $\langle\ ,\ \rangle$ on V is determined by the equation

$$-6\ \alpha \wedge \beta = \langle \alpha, \beta \rangle\, \omega^3 \quad \text{for all } \alpha, \beta \in V.$$

To convenience our notation, we let $\alpha_{ijk} := e_i \wedge e_j \wedge e_k$ and $\beta_i := e_i \wedge (e_j \wedge e_{-j} - e_k \wedge e_{-k})$ where $(|i|, j, k)$ is an even permutation of $(1, 2, 3)$. Evidently, $\beta_i \in V$ spans the weight space of weight θ_i while α_{ijk} spans the weight space of weight $\theta_i + \theta_j + \theta_k$ if $\{|i|, |j|, |k|\} = \{1, 2, 3\}$.

One then shows that after replacing $(\mathbf{a}, \mathbf{b})$ by an element in its H-orbit we may assume that

$$(\mathbf{a}, \mathbf{b}) = (-2e_1^2 + e_{-1}e_3,\ \ 2\beta_3). \tag{37}$$

The infinitesimal stabilizer of this pair is the Lie subalgebra

$$\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1,$$

where

$$\begin{aligned} \mathfrak{l}_0 &:= span\{e_{\pm 2}^2, e_2 e_{-2}\} \cong \mathfrak{sl}(2, \mathbb{F}), \\ \mathfrak{l}_1 &:= span\{e_{\pm 2} e_3, e_3^2, \mathbf{a}\}. \end{aligned}$$

Evidently, $[\mathfrak{l}_i, \mathfrak{l}_j] \subset \mathfrak{l}_{i+j}$. Also, one calculates that the linear map $\tau : V \to \mathfrak{h}$ given by

$$\begin{aligned} \tau(\alpha_{\pm 1, \pm 2, 3}) &:= 0 \\ \tau(\alpha_{-1, \pm 2, -3}) &:= -4 e_{\pm 2} e_{-3} \\ \tau(\alpha_{1, \pm 2, -3}) &:= e_{-1} e_{\pm 2} \end{aligned} \qquad \begin{aligned} \tau(\beta_1) &:= -2e_1^2, & \tau(\beta_{-1}) &:= 4e_3 e_{-3} \\ \tau(\beta_3) &:= 4e_1 e_3, & \tau(\beta_{-3}) &:= 4e_1 e_{-3} + \tfrac{1}{2} e_{-1}^2 \end{aligned}$$

$$\tau(\beta_{\pm 2}) := -4 e_1 e_{\pm 2}$$

satisfies the identity

$$\rho_v(\mathbf{a}+\mathbf{b}) := \xi_{v+\tau(v)}(\mathbf{a}+\mathbf{b}) = 0 \quad \text{for all } v \in V. \tag{38}$$

If there was a $\mathfrak{l}$-equivariant map $\sigma : V \to \mathfrak{h}$ satisfying (38) then $\sigma = \tau + \delta$ with $\delta : V \to \mathfrak{l}$. But then, since $\mathbf{a}\beta_3 = 0$, we would have $0 = [\mathbf{a}, \sigma(\beta_3)] = [\mathbf{a}, 4e_1e_3 + \delta(\beta_3)] = -4e_3^2 + [\mathbf{a}, \delta(\beta_3)]$, and since $[\mathbf{a}, \delta(\beta_3)] \in [\mathbf{a}, \mathfrak{l}] = 0$ this yields a contradiction. Thus, there cannot be a $\mathfrak{l}$-equivariant map $\tau : V \to \mathfrak{h}$ satisfying (38), i.e. the corresponding homogeneous space is *not* reductive by Proposition 2.5.

Next, one calculates that

$$\mathfrak{g} \cong \begin{cases} \mathfrak{g}_2^{4,3} \rtimes \mathbb{R}^7 & \text{if } \mathbb{F} = \mathbb{R} \\ \mathfrak{g}_2^{\mathbb{C}} \rtimes \mathbb{C}^7 & \text{if } \mathbb{F} = \mathbb{C}, \end{cases}$$

where $\mathfrak{g}_2^{4,3}$ stands for the (unique) non-compact real form of the exceptional Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$.

Another calculation then yields that the element $\eta \in \mathfrak{g}^*$ determined by

$$\eta(\mathbf{a}) = -\eta(\rho_{\beta_1}) = \frac{1}{8}, \quad \eta(\rho_\lambda) = 0, \text{ all weights } \lambda \neq \beta_1, \quad \eta([\mathfrak{l}, \mathfrak{l}]) = 0$$

satisfies the identity (2). Therefore, the homogeneous space G/L is the coadjoint G-orbit of $\mathrm{G} \cdot \eta \subset \mathfrak{g}^*$ with its canonical symplectic form by Proposition 2.4, and its holonomy is all of H by Corollary 5.11. In conclusion, we have the following

Theorem 5.12 *Let $(\mathbf{a}, \mathbf{b}) \subset \mathfrak{h} \oplus V$ be a degenerate pair of type 2. Then there exists a Lie group G and a coadjoint orbit $\Sigma := G \cdot \eta \subset \mathfrak{g}^*$ for some $\eta \in \mathfrak{g}^*$ with its canonical symplectic structure and a G-invariant symplectic connection on Σ whose holonomy is conjugate to H and whose curvature is represented by $(\mathbf{a}, \mathbf{b})$. This homogeneous space is reductive if rk(H) $= 1$ and not reductive otherwise. The possible choices for $\eta \in \mathfrak{g}^*$ are listed in Table 2.*

5.2.2 Degenerate pairs of type 3

Proposition 5.13 *Let $\mathfrak{h} \subset \mathfrak{sp}(V)$ be a special holonomy algebra and let $\Sigma \subset \mathfrak{h} \oplus V$ be a maximal symplectic leaf of the Poisson structure (4). If Σ contains a degenerate element of type 3 then it also contains an element of the form $(\mathbf{a}, 0)$ with $\mathbf{a}^2 \neq 0$ and $\mathbf{a}^3 = 0$.*

Proof. Let $\mathbf{a} \in \mathfrak{h}$ be such that $\mathbf{a}^2 \neq 0$ and $\mathbf{a}^3 = 0$, and choose $v \in \mathcal{C}$ such that $\mathbf{a}^2 v \neq 0$. A calculation using (3) shows that the curve

$$\gamma(t) = (\mathbf{a} + t^2(\mathbf{a}^2 v) \circ v,\ 2t\mathbf{a}^2 v) =: (\mathbf{a}(t), \mathbf{b}(t)) \tag{39}$$

is a trajectory of ξ_v, i.e. $\gamma'(t) = (v \circ \mathbf{b}(t), \mathbf{a}(t)v)$.

Next, let $(\mathbf{a}_+ + \mathbf{a}_0, v_+) \in \Sigma$ be a degenerate of type 3. Then one can show that there is a decomposition $\mathbf{a}_+ = \mathbf{a}_1 + \mathbf{a}_2$ with $\mathbf{a}_i \in \mathfrak{n}^+$, $[\mathbf{a}_0, \mathbf{a}_i] = 0$, $\mathbf{a}_1^2 = 0$ and $\mathbf{a}_2^3 = 0$. For the complex holonomies and the split forms, this follows from the normal form (15), while in the remaining cases one has to perform a direct investigation.

Let $v := \mathbf{a}_1 v_- \in V_1$. Then $v \circ v = [\mathbf{a}_1, [\mathbf{a}_1, v_- \circ v_-]] = 0$, whence $v \in \mathcal{C}$. Moreover, since $[\mathbf{a}_1, \mathbf{a}_2] \in [\mathfrak{n}^+, \mathfrak{n}^+] = 0$, we have $v_+ = \frac{3}{2}\mathbf{a}_+^3 v_- = 3\mathbf{a}_1\mathbf{a}_2^2 v_- = 3\mathbf{a}_2^2 v$.

Thus, the trajectory γ of ξ_v passing through $(\mathbf{a}_2 + \mathbf{a}_0, 0)$ satisfies $\gamma(1) = (\mathbf{a}_+ + \mathbf{a}_0, v_+)$ by (39), whence $(\mathbf{a}_2 + \mathbf{a}_0, 0) \in \Sigma$. ∎

Therefore, since the Lie algebra of infinitesimal symmetry groups is isomorphic at each point of a symplectic leaf, we obtain

$$\mathfrak{g} \cong \mathfrak{l}_0 \oplus \{\xi_v \mid \mathbf{a}_s^2 v = 0\},$$

where $(\mathbf{a}_s, 0) \in \Sigma$ is the element given above and where $\mathfrak{l}_0$ is the centralizer of $\mathbf{a}_s$. From this, $\mathfrak{g}$ can be explicitly computed for all special symplectic holonomy groups. The resulting calculations are long and not very eluminating. We therefore omit the details, again referring the interested reader to [S5] where the calculations are spelled out in detail. The result can be described as follows.

Theorem 5.14 *Let $(\mathbf{a}, \mathbf{b}) \in \mathfrak{h} \oplus V$ be the degenerate pair of type 3 from Corollary 5.9, let $\mathfrak{g}$ be the symmetry algebra from (34), G be a Lie group with Lie algebra $\mathfrak{g}$ and $L \subset G$ be the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{l} := stab(\mathbf{a}, \mathbf{b}) \subset \mathfrak{h}$.*

Then there is a G-invariant connection with special symplectic holonomy H on the homogeneous space G/L whose curvature at any point is represented by $(\mathbf{a}, \mathbf{b})$.

Moreover, the pair (G, L) corresponds to one of the entries of Table 3. None of these homogeneous spaces are maximal, none of them are reductive and none of them are equivalent to the coadjoint orbit of $\mathfrak{g}^*$.

6 Proofs of the main results

As before, we let (M, Ω, ∇) be a symplectic manifold with a connection of special symplectic holonomy and let $\pi : F \to M$ be its holonomy bundle and $\rho := \mathbf{a} + \mathbf{b} : F \to \mathfrak{h} \oplus V$ be the map from Theorem 3.3. We define $F_0 \subset F$ and $M_0 \subset M$ by

$$F_0 := \{p \in F \mid \mathbf{b}(p) = 0\}, \qquad M_0 := \pi(F_0).$$

Thus, M_0 is the set of *symmetric points*, i.e. of those $p \in M$ with $(\nabla R)_p = 0$. Note that there are no locally symmetric connections with special symplectic holonomy [S3, S4], whence $M_0 \subset M$ and $F_0 \subset F$ are *proper* subsets.

Proof of Theorem 1.1. The constancy of the symplectic scalar curvature and Definition 5.1 implies that $\rho(F \backslash F_0)$ consists of degenerate pairs. Note that for degenerate pairs $(\mathbf{a}, \mathbf{b})$ of type 2 or 3, $\mathbf{a}$ is nilpotent and hence, $(\mathbf{a}, \mathbf{a}) = 0$. Thus, by the non-vanishing of the symplectic scalar curvature, $\rho(F \backslash F_0)$ must consist of degenerate pairs of type 1. It follows that the maximal symplectic leaf $\Sigma \subset \mathfrak{h} \oplus V$ containing $\rho(F)$ is the one given in (33), whence M must be locally equivalent to one of the entries of Table 1 by Theorem 5.7 ∎

Proof of Theorem 1.2. (1) $\Longrightarrow$ (2): If M contained one symmetric point then – by homogeneity – M would be locally symmetric which was already excluded. Thus, M contains no symmetric points and since the symplectic scalar curvature is preserved by the symmetry group, it must be constant.

(2) $\Longrightarrow$ (3): Trivial

(3) $\Longrightarrow$ (1): Let $p \in F \backslash F_0$ such that $scal - scal(\pi(p))$ vanishes of order at least 3. Thus, the first and second order derivatives of $(\mathbf{a}, \mathbf{a})$ vanish at p whence $\rho(p)$ is degenerate by Definition 5.1. Thus, the local homogeneity of ∇ follows from Theorem 3.3,4. together with Theorems 5.7, 5.12 and 5.14. ∎

Proof of Theorem 1.3. Let (M, Ω, ∇) be a manifold with a homogeneous symplectic connection with vanishing symplectic scalar curvature and let $\pi : F \to M$ and $\rho : F \to \Sigma \subset \mathfrak{h} \oplus V$ as before. Thus, $\rho(F)$ consists of degenerate pairs only. Since the scalar curvature vanishes, these pairs can be of type 2 or 3 only.

Suppose Σ contains a pair of type 2. Comparing the Lie algebras of infinitesimal symmetries, it follows that Σ cannot contain any pair of type 3. If Σ contained a symmetric point $(\mathbf{a}, 0)$ then by (8), we must have $\mathbf{a}^3 = 0$, and the trajectory (39) would contain pairs which are not of type 2, unless $\mathbf{a}^2 v = 0$ for all $v \in \mathcal{C}$.

However, the latter would imply that $\mathbf{a}^2 = 0$, which by (5) would imply that $\mathbf{b} \equiv 0$ which is a contradiction.

Thus, either Σ contains only degenerate pairs of type 2 and whence is H-homogeneous, or it contains only degenerate pairs of type 3 and symmetric points. The result then follows from Corollary 3.4 and Theorem 5.14. ∎

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