Riemannian, Symplectic and Weak Holonomy

Lorenz J. Schwachhöfer*

Dedicated to the memory of Alfred Gray

Abstract

Much of the early work of Alfred Gray was concerned with the investigation of Riemannian manifolds with special holonomy, one of the most vivid fields of Riemannian geometry in the 1960s and the following decades. It is the purpose of the present article to give a brief summary and an appreciation of Gray's contributions in this area on the one hand, and on the other hand to describe some of the more recent developments in the theory of *non-Riemannian* or, more specifically, *symplectic* holonomy groups. Namely, we show that the Merkulov twistor space of a connection on a symplectic manifold M whose holonomy group is irreducible and properly contained in $\mathrm{Sp}(V)$ consists of maximal totally geodesic Lagrangian submanifolds of M.

1 Introduction

A connection on a manifold provides a recipe for parallel translation of tangent vectors along curves. When this is done around a *closed* curve, the final position of the tangent vector typically differs from its initial position, and the *holonomy transformation* expresses this change. The collection of all such transformations constitutes the *holonomy group*.

The notion of the holonomy group was introduced by É. Cartan in 1923 [Car2, Car4]. He used this invariant in order to investigate manifolds of dimensions 2 or 3 with a prescribed holonomy group. Also, in [Car3], he showed that for a *symmetric space*, the holonomy and the isotropy group coincide up to connected components.

It was not until the 1950s when holonomies of torsion free connections received further attention, in particular the classification problem:

Given a finite dimensional vector space V, which are the (closed irreducible) Lie subgroups $H \subset Aut(V)$ that can occur as the holonomy group of a torsion free affine connection?

The condition of *torsion freeness* is an integrability condition which makes this problem non-trivial; namely, by a result of Hano and Ozeki [HO], *any* (closed) Lie subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ can be realized as the holonomy of an affine connection on some manifold M (with torsion, in general). Also, in this article we shall restrict our attention to the case where $\mathrm{H} \subset \mathrm{Aut}(V)$ acts *irreducibly* on V.

*Mathematisches Institut, Universität Leipzig, Augustusplatz 10-11, 04109 Leipzig, Germany; e-mail: schwachh@mathematik.uni-leipzig.de

By the aforementioned work of Cartan, the classification problem contains the classification of (irreducible) symmetric spaces as a "sub-problem". This classification has been completed by Cartan in the Riemannian [Car3] and by Berger in the general irreducible case [Ber2].

It followed tremendous research efforts in the investigation of the classification problem, in particular for the holonomies of Levi-Civita connections of Riemannian manifolds. In section 2, we shall give an account of various aspects of this problem, emphasising Alfred Gray's contributions to the area.

Gray also introduced the notion of the *weak holonomy group* of a Riemannian manifold. He used it to generalise certain structures on Riemannian manifolds that are expressible via the holonomy group. For example, the generalization of a Kähler manifold in this sense is a *nearly Kähler manifold*. We shall give a summary of Gray's results on weak holonomy groups in section 3.

Continuing our survey, we give the developments in the theory of *non-Riemannian* holonomy groups in section 4 and, in particular, discuss the general classification of irreducible holonomy groups.

In section 5, we describe the twistor theory associated to holomorphic torsion free connections introduced by Merkulov. In summary, a manifold M with a torsion free connection ∇ of irreducible holonomy group can be realized as the moduli of Legendre submanifolds of an associated complex contact manifold Y which we call the *Merkulov twistor space* of (M, ∇).

Finally, in section 6 we consider the list of *symplectic* holonomy groups, i.e. the irreducible holonomy groups which admit a parallel symplectic form on M. Historically, these were the last holonomies to be discovered, and there are various ways to describe the list of possible symplectic holonomy groups. A fairly recent description is due to Alekseevskii and Cortés [AC] and allows us to arrive at the following result (cf. Corollary 6.4).

Theorem. *Let (M, ω, ∇) be a symplectic manifold with a torsion free symplectic connection whose holonomy is irreducible and a proper subgroup of $Sp(V)$. Then the Merkulov twistor space of (M, ∇) consists of all maximal conic totally geodesic Lagrangian submanifolds of M.*

This description is remarkable since it states that every point of M can be viewed as a Legendre submanifold of Y, while every point in Y can be associated to a totally geodesic Lagrangian submanifold of M, expressing some unexpected "mirror symmetry" of this twistor correspondence.

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2 Holonomies of Riemannian manifolds

An important class of torsion free connections are the *Levi-Civita connections* of Riemannian manifolds (M, g). If we assume for simplicity that M is simply connected then a connection is the Levi-Civita connection of some Riemannian metric g on M iff the holonomy group H is compact. This follows from the basic work by Borel, Lichnerowicz [BL] and Nijenhuis [N1, N2].

By the *deRham splitting Theorem* [dR], the irreducibility of the Riemannian metric is equivalent to the irreducibility of the holonomy, so this becomes a very natural assumption. Moreover, if H is irreducible then g is uniquely determined up to rescaling.

After Ambrose and Singer gave a description of the Lie algebra of the holonomy group in terms of the curvature of the connection [AS], a tremendous advance was made by Berger in [Ber1]. Namely, he was able to classify all (compact) irreducible Lie groups $H \subset \mathrm{Aut}(V)$ which can possibly occur as the holonomy of a Levi-Civita connection on some Riemannian manifold (M, g). It turns out that the list of non-symmetric Riemannian holonomies is contained (in fact, is almost equal to) the list of transitive group actions on spheres [MoSa1, MoSa2, Bo1, Bo2]. This was later shown directly by Simons [Si].

With this, the classification of Riemannian holonomies can be stated in the following way:

Theorem 2.1 (Berger [Ber1], Simons [Si]) *Let (M, g) be a simply connected Riemannian manifold and assume that its holonomy group $H \subset Aut(V)$ acts irreducibly on V. Then either H acts transitively on the unit sphere $S_1 \subset V$, or (M, g) is a locally symmetric space of rank greater than one.*

It is not hard to rule out the subgroup $S^1 \cdot \mathrm{Sp}(n) \subset \mathrm{Aut}(\mathbb{H}^n)$ as a possible holonomy group, even though this group acts transitively on the unit sphere. (Here, S^1 is regarded as a subgroup of $\mathrm{Sp}(1)$ and acts on $\mathbb{H}^n$ by scalar multiplication). Thus, we have as a consequence the following

Corollary 2.2 (Berger [Ber1], Simons [Si]) *Let (M, g) be a simply connected Riemannian manifold of dimension n which is not locally symmetric and whose holonomy group $H \subset Aut(V)$ acts irreducibly on V. Then H is conjugate to one of the groups in Table 1.*

Table 1

No.	H	V	restrictions
1	$\mathrm{SO}(n)$	$\mathbb{R}^n$	$n \geq 2$
2	$\mathrm{U}(m)$	$\mathbb{R}^n \cong \mathbb{C}^m$	$m \geq 2, n = 2m$
3	$\mathrm{SU}(m)$	$\mathbb{R}^n \cong \mathbb{C}^m$	$m \geq 2, n = 2m$
4	$\mathrm{Sp}(1) \cdot \mathrm{Sp}(m)$	$\mathbb{R}^n \cong \mathbb{H}^m$	$m \geq 1, n = 4m$
5	$\mathrm{Sp}(m)$	$\mathbb{R}^n \cong \mathbb{H}^m$	$m \geq 1, n = 4m$
6	G_2	$\mathbb{R}^7 \cong Im\mathbb{O}$	
7	$\mathrm{Spin}(7)$	$\mathbb{R}^8 \cong \mathbb{O}$	
8	$\mathrm{Spin}(9)$	$\mathbb{R}^{16} \cong \mathbb{O}^2$	

We emphasise that Table 1 relates the different possibilities for the holonomy group to one of the four division algebras: while the first entry is related to the reals, the second and third entry are related to the complex numbers, the fourth and fifth to the quaternions and the last three entries are related to the octonions.

An obvious question after the introduction of this list was to determine if all subgroups given in Table 1 actually can occur as the holonomy of a Riemannian manifold (M, g) and if so, to describe the geometric and topological structure of M. This question lead to the introduction of numerous exciting geometric structures of which we shall mention only a few here. For more thorough and complete overviews of the subject we refer to the books by Besse [Bes] and Salamon [Sa].

A "generic" Riemannian metric has holonomy group $\mathrm{H} = \mathrm{SO}(n)$, so the first entry does not encode any geometric structure other than the Riemannian metric itself.

For the *complex* geometries, it turns out that (M, g) has holonomy contained in $\mathrm{U}(m)$ iff (M, g) is a Kähler metric, and the holonomy of a "generic" Kähler metric equals all of $\mathrm{U}(m)$. Moreover, the holonomy of (M, g) is contained in $\mathrm{SU}(m)$ iff (M, g) is a *Ricci-flat* Kähler metric (a so-called *special Kähler metric*).

The first construction of (local) special Kähler metrics in 1960 is due to Calabi [Cal2]. The existence of *compact* special Kähler manifolds follows from the work of Calabi [Cal1] and Yau [Y] from 1978. Indeed, any compact Kähler manifold with vanishing first Chern class admits a unique special Kähler metric.

Next, for the *quaternionic* geometries we view each tangent space as a vector space over $\mathbb{H}$ in a canonical sense. That is, one should expect three almost complex structures representing scalar multiplication by $\mathbb{H}$. This was made precise by Alfred Gray.

Theorem 2.3 (Gray [Gr3], 1965) *Let (M, g) be a $4m$-dimensional Riemannian manifold. Then the following are equivalent.*

1. *The holonomy group of (M, g) is contained in $Sp(1) \cdot Sp(m)$.*

2. *M is covered by open sets on which there exist three orthogonal almost complex structures I_1, I_2, I_3 and 1-forms α, β, γ such that $I_i \cdot I_{i+1} = I_{i+2}$ (with indices mod 3) and such that for all $v \in TM$ the following relations hold:*

$$\begin{array}{rcrcrcr} \nabla_v I_1 & = & & & \gamma(v) \cdot I_2 & - & \beta(v) \cdot I_3 \\ \nabla_v I_2 & = & -\gamma(v) \cdot I_1 & & & + & \alpha(v) \cdot I_3 \\ \nabla_v I_3 & = & \beta(v) \cdot I_1 & - & \alpha(v) \cdot I_2 & & \end{array}$$

If these conditions are satisfied then there is a (globally defined) parallel S^2-subbundle $Q \subset SO(TM)$, consisting of orthogonal almost complex structures. Locally, it is given by $Q = \{\sum c_i I_i \mid \sum c_i^2 = 1\}$.

Riemannian manifolds which satisfy the equivalent conditions from Theorem 2.3 are called *Quaternion-Kähler manifolds*. Examples include the quaternionic symmetric spaces $\mathbb{HP}^n$, but there are also others, for example the homogeneous manifolds found by Alekseevskii [A] from 1968.

Alfred Gray also contributed to the following "negative" result.

Theorem 2.4 (Brown, Gray [BG], 1972) *Let (M, g) be a 16-dimensional Riemannian manifold whose holonomy is $Spin(9) \subset SO(16)$. Then (M, g) is a locally symmetric space and locally isometric either to the Cayley plane $\mathbb{C}\mathrm{a}\mathbb{P}_2$ or to its non-compact dual $\mathbb{C}\mathrm{a}\mathbb{P}_2^*$.*

As it turns out, Spin(9) is the *only* entry of Table 1 which cannot occur as the holonomy of a Riemannian manifold which is not locally symmetric. However, it took much longer to show the existence of Riemannian manifolds whose holonomy is one of the entries 5, 6 and 7 of Table 1. The existence of metrics with holonomy $\mathrm{H} = \mathrm{Sp}(n)$ was shown by Calabi [Cal2] in 1979, while the existence of metrics with holonomy $\mathrm{H} = \mathrm{G}_2$ and $\mathrm{H} = \mathrm{Spin}(7)$ was shown by Bryant [Br1] only in 1987. Compact manifolds with the latter holonomies were constructed by Joyce [J] in 1996.

3 Weak holonomy groups

We now give Alfred Gray's definition of the *weak holonomy group* of a Riemannian manifold (M, g) [Gr1, Gr2].

Definition 3.1 *Let $H \subset Aut(V)$ be a Lie subgroup. A subspace $W \subset V$ is called* special *if*

1. *there exists a proper subspace $U \subset W$ such that $h|_U$ determines $h|_W$ for all $h \in H$.*
2. *W is maximal in the sense that if $W \subset W'$ is such that $h|_U$ determines $h|_{W'}$ for all $h \in H$, then $W = W'$.*

Examples:

1. $\mathrm{H} = \mathrm{SO}(n)$: in this case, $W = V$ is the only special subspace.

2. $\mathrm{H} = \mathrm{U}(m), SU(m)$: $0 \neq W \subset V$ is special iff W is J-invariant, i.e. W is a *complex* subspace.

3. $\mathrm{H} = \mathrm{Sp}(1) \cdot \mathrm{Sp}(m), \mathrm{Sp}(m)$: $0 \neq W \subset V$ is special iff W is I, J, K-invariant, i.e. W is a *quaternionic* subspace.

4. $\mathrm{H} = \mathrm{G}_2$, $V = Im\mathbb{O}$: $W \subset V$ is special iff and $span(1, W) \subset \mathbb{O}$ is a subalgebra and $\dim W > 1$.

5. $\mathrm{H} = \mathrm{Spin}(7)$, $V = \mathbb{O}$: $W \subset V$ is special iff there is an $x \in \mathbb{O}$ such that $x \cdot W \subset \mathbb{O}$ is a subalgebra and $\dim W > 1$.

Definition 3.2 *A Riemannian manifold (M, g) is said to have* weak holonomy group H *iff for each p-based loop γ and each special subspace $W \subset T_pM$ of minimal dimension with $\gamma'(0) \in W$ we have that $P_\gamma|_W = h|_W$ for some $h \in H$. Here, P_γ denotes parallel translation along γ.*

Evidently, if (M, g) has holonomy H then it has weak holonomy H. Note, however, that the weak holonomy group of a Riemannian manifold is not uniquely determined: indeed, if $\mathrm{H} \subset \mathrm{H}'$ and H is a weak holonomy group of (M, g) then so is H'.

From the example above it is evident that every oriented Riemannian manifold has weak holonomy $\mathrm{SO}(n)$. An investigation of the remaining entries of Table 1 yields the following result.

Theorem 3.3 (Gray, [Gr1, Gr2]) *Let (M, g) be a Riemannian manifold, and suppose that M has weak holonomy group H.*

1. *If G is one of the groups $Sp(1) \cdot Sp(m)$, $Sp(m)$ or $Spin(7)$ then $H \subset G$ iff the holonomy of M is contained in G.*

2. *$H \subset U(m)$ iff M is nearly Kähler.*

3. *$H \subset SU(m)$ iff M is nearly Kähler and the structure group of the tangent bundle of M can be reduced to $SU(m)$.*

Recall that a nearly Kähler manifold is an almost Hermitian manifold with almost complex structure J such that $(\nabla_v J)v = 0$ for all $v \in TM$.

The only entry of Table 1 not mentioned in Theorem 3.3 is $\mathrm{G} = \mathrm{G}_2$. For this case, Gray shows the following.

Theorem 3.4 (Gray, [Gr2]) *Let (M, g) be a 7-dimensional Riemannian manifold with weak holonomy group G_2. Then M is an Einstein manifold.*

It is well known that a manifold with *holonomy* G_2 is Ricci-flat. Thus, it follows from this result that the weak holonomy G_2 is indeed a more general notion.

4 The classification of irreducible holonomy groups

After first important advances on the classification of Riemannian holonomy groups and the understanding of the underlying geometric and topological structures had been made, the interest in the *non-Riemannian* irreducible holonomy groups grew. The starting point is the idea of Berger [Ber1] to give a necessary condition for the Lie algebra of the holonomy group of a torsion free connection. In a more modern language, we can describe this as follows.

Definition 4.1 *Let $H \subset Aut(V)$ be a Lie subgroup and $\mathfrak{h} \subset End(V)$ be the corresponding Lie subalgebra.*

1. *A* formal curvature map of $\mathfrak{h}$ *is a linear map $R : \Lambda^2 V \to \mathfrak{h}$ satisfying $R(x,y)z + R(y,z)x + R(z,x)y = 0$ for all $x, y, z \in V$.*

2. *A* curvature value of $\mathfrak{h}$ *is an element $R(x,y) \in \mathfrak{h}$ where R is a formal curvature map and $x, y \in V$.*

3. $\mathfrak{h}$ *is called a* Berger algebra *if it is generated as a Lie algebra by its curvature values. In this case, H is called a* Berger group.

4. *A Berger algebra $\mathfrak{h}$ is called* symmetric *if – up to scalar multiples – there is only one formal curvature map of $\mathfrak{h}$. In this case, H is also called symmetric.*

Then the following is an immediate consequence of the Ambrose-Singer Holonomy Theorem [AS] and a calculation involving the second Bianchi identity [KoNo].

Theorem 4.2 (Berger [Ber1]) *The holonomy group of a torsion free connection is a Berger group. Moreover, if H is a symmetric Berger group, then any connection whose holonomy is contained in H is locally symmetric.*

The classification of irreducible symmetric Berger groups follows from that of irreducible symmetric spaces. This has been achieved in [Car3, Ber2]. Berger then gave a partial classification of irreducible non-symmetric *Riemannian* Berger groups, i.e. those Berger groups which are contained in $\mathrm{SO}(V)$. In fact, Table 1 contains all non-symmetric Riemannian Berger groups (and Spin(9) which actually *is* symmetric, cf. Theorem 2.4).

In Berger's classification of *pseudo-Riemannian holonomies*, i.e. those holonomy groups H which are contained in $\mathrm{SO}(p,q)$ for some $p, q \geq 1$, there were some omissions which were later fixed by Bryant [Br3]. Also, Berger's list of *non-Riemannian* irreducible holonomies was incomplete. Further Berger groups were discovered by Bryant, Chi, Merkulov and this author [Br2, Br3, CS, CMS1, CMS2, MeSc1, MeSc2]. Finally, in [MeSc1, MeSc2] the classification of irreducible holonomy groups (and thus of irreducible Berger groups) was completed.

To avoid the use of tables, we state here only the classification theorem for *complex* irreducible Berger groups. For the detailed list of all Berger groups we refer the reader to [MeSc1, Sc].

Theorem 4.3 *[MeSc1, MeSc2, Sc] Let V be a finite dimensional complex vector space, let $H_{\mathbb{C}} \subset Aut(V)$ be an irreducible semi-simple complex connected Lie subgroup with Lie algebra $\mathfrak{h}_{\mathbb{C}} \subset End(V)$. Then the following holds.*

1. *If the first prolongation of $\mathbb{C}Id_V \oplus \mathfrak{h}_{\mathbb{C}}$ does not vanish, then both $H_{\mathbb{C}}$ and $(\mathbb{C}^* Id_V) \cdot H_{\mathbb{C}}$ are non-symmetric Berger groups.*

2. *If there is an irreducible symmetric space of the form $M = G/(SL(2, \mathbb{C}) \cdot H_{\mathbb{C}})$, then $H_{\mathbb{C}}$ is a non-symmetric Berger group. If $\dim V = 4$ then $(\mathbb{C}^* Id_V) \cdot H_{\mathbb{C}}$ is also a non-symmetric Berger group.*

3. *If $H_{\mathbb{C}}$ is the complexification of one of the entries in Table 1 then $H_{\mathbb{C}}$ is a non-symmetric Berger group.*

4. *1. – 3. yield all irreducible complex connected non-symmetric Berger groups.*

Here, we use the standard notation $\mathrm{G} \cdot \mathrm{H} = (\mathrm{G} \times \mathrm{H})/\Gamma$ for some finite group Γ. Also, recall that the *first prolongation of a Lie subalgebra* $\mathfrak{g} \subset \mathrm{End}(V)$ is the space $\{\phi \in V^* \otimes \mathfrak{g} \mid \phi(x)y = \phi(y)x \ \forall x, y \in V\}$. The irreducible Lie subalgebras whose first prolongation does not vanish have been classified by Cartan [Car1] and Kobayashi and Nagano [KoNa].

The groups specified in the three parts of Theorem 4.3 are not disjoint: namely, the standard representations of $\mathrm{SO}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C})$ belong to both 1 and 3, while the standard representation of $\mathrm{Sp}(n, \mathbb{C})$ even belongs to all three parts.

The last Berger groups to be discovered were the ones corresponding to the groups in part 2 of Theorem 4.3. These are all contained in $\mathrm{Sp}(n, \mathbb{C})$ and hence every manifold with one of these holonomies carries a parallel symplectic form. These connections share some remarkable features and we shall discuss them in some more detail in section 6.

5 The Merkulov twistor space of torsion free connections

In this section, we shall give a brief exposition of a twistor theory which can be associated to a holomorphic torsion free connection on a complex manifold M. This twistor theory has been developed by Merkulov in [Me1, Me2, Me3, Me4]. Throughout this section, we shall work in the complex category. That is, all manifolds, functions, vector fields, forms etc. are understood to be *holomorphic*. Also, TM and T^*M stand for the *holomorphic* (co-)tangent bundle of the manifold M.

Definition 5.1 *Let Y be a manifold, let $\mathcal{D}$ be a codimension-1 distribution on Y, and define the line bundle L by the exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow TY \longrightarrow L \longrightarrow 0. \tag{1}$$

If the L-valued 2-form θ on $\mathcal{D}$ given by $\theta(x, y) := [x, y]$ mod $\mathcal{D}$ is non-degenerate, then $\mathcal{D}$ is called a contact structure on Y*, and L is called the* contact line bundle *of Y.*

A submanifold $X \subset Y$ is called a contact submanifold *if $TX \subset \mathcal{D}$. If X is a contact submanifold with* $\dim X = (\dim Y - 1)/2$ *then X is called a* Legendre submanifold.

Note that from the maximal non-integrability of $\mathcal{D}$ it follows that Legendre submanifolds are contact submanifolds of maximal dimension.

Given a contact manifold Y and a compact Legendre submanifold $X_0 \subset Y$, a natural question is when the moduli space of "close-by" Legendre submanifolds carries the structure of a manifold. To make this more precise, we need the following definition.

Definition 5.2 *Let Y be a contact manifold. An* analytic family of compact Legendre manifolds *is a submanifold $S \hookrightarrow M \times Y$ with some manifold M such that the restriction $\pi_1 : S \to M$ is a submersion, and $X_p := \pi_2(\pi_1^{-1}(p)) \subset Y$ is a compact Legendre submanifold for all $p \in M$. Here, π_i is the projection of $M \times Y$ onto the i-th factor. In this case, we call M a* moduli space of Legendre submanifolds*, and say that the submanifolds X_p, $p \in M$, are contained in the analytic family.*

S is called maximal (locally maximal, respectively) *if for every analytic family $S' \subset M' \times Y$ with $M \subset M'$ and $S \subset S'$, it follows that $S = S'$ and $M = M'$ (S open in S' and M open in M', respectively).*

From the deformation theory of Merkulov [Me1], one arrives at the following result.

Theorem 5.3 *[Me1] Let Y be a contact manifold with contact line bundle $L \to Y$, and let $X \subset Y$ be a compact Legendre submanifold such that the restriction L_X is a very ample line bundle. Then there exists a locally maximal analytic family $S \hookrightarrow Y \times M$ containing X, and the line bundles $L_{X_p} \to X_p$ with $p \in M$ are pairwise equivalent. Moreover, there is a canonical isomorphism $T_pM \cong H^0(X_p, L_{X_p})$, and hence,* $\dim M = h^0(X, L_X)$.

The most astonishing feature of this theory is that every manifold M with a torsion free connection ∇ with irreducible holonomy H can be realized as such a moduli space. Namely, to such a connection we can associate a contact manifold Y (cf. Theorem 5.6 below), and we shall now give a brief outline of the construction.

To begin with, let $F \subset \mathfrak{F}_V$ be the holonomy reduction of ∇, i.e. F is an H-reduction of $\mathfrak{F}_V$. Next, let $\pi : T^*M \to M$ be the holomorphic cotangent bundle. We let λ denote the *Liouville form* on T^*M which is given by the equation

$$\lambda(v_\theta) := \theta(\pi_*(v_\theta))$$

for all $v_\theta \in T_\theta(T^*M)$. The 2-form

$$\omega := d\lambda$$

is non-degenerate and is called the *canonical symplectic form* on T^*M. Then

$$m_t^*\lambda = t\lambda \quad \text{and} \quad m_t^*\omega = t\omega,$$

where $m_t : T^*M \to T^*M$ denotes the scalar multiplication by $t \in \mathbb{C}^*$.

The following relates contact structures to the symplectic form.

Lemma 5.4 *Let Y be a manifold, let $\mathcal{D}$ be a codimension-1 distribution on Y, and let L be the line bundle from (1). Consider the dual embedding $\imath : L^* \hookrightarrow T^*Y$. Then $\mathcal{D}$ is a contact structure iff $\imath^*\omega$ is non-degenerate where ω denotes the canonical symplectic form on T^*Y.*

Let V be a vector space with $\dim V = \dim M =: n$, and identify the holonomy group with the subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$. We let $\tilde{\mathcal{C}} \subset V^*\backslash\{0\}$ be the G-orbit of a highest weight vector of the dual representation, and let $\mathcal{C} \subset \mathbb{P}(V^*)$ be its projectivization. $\mathcal{C}$ is called the *sky* of H.

Clearly, the cotangent bundle of M and its projectivization can be expressed as $T^*M = F \times_H V^*$ and $\mathbb{P}T^*M = F \times_H \mathbb{P}(V^*)$. Let

$$\tilde{S} := F \times_H \tilde{\mathcal{C}} \subset T^*M\backslash\{0\},$$

and

$$S := F \times_H \mathcal{C} \subset \mathbb{P}T^*M.$$

Obviously, S is the quotient of $\tilde{S}$ by the natural $\mathbb{C}^*$-action. The restriction $\omega_{\tilde{S}}$ of ω is no longer non-degenerate, and we let $\mathcal{N} \subset T\tilde{S}$ be its annihilator, i.e.

$$\mathcal{N} := \{v \in T\tilde{S} \mid v \lrcorner \omega_{\tilde{S}} = 0\}. \tag{2}$$

If we denote the canonical projection by $\pi : \tilde{S} \to M$, then it is easy to see that for all $p \in M$,

$$\mathcal{N} \cap T\pi^{-1}(p) = 0. \tag{3}$$

Now a calculation in local coordinates yields the following important result.

Proposition 5.5 *[Me1] Let M be a manifold with torsion free connection ∇, and suppose that the holonomy group H of ∇ is irreducible. Let $F \subset \mathfrak{F}$ be the holonomy reduction, and let $\tilde{S} \subset T^*M$ be as before.*

Then the distribution $\mathcal{N}$ from (2) is contained in the horizontal distribution, and rk$(\mathcal{N})$ = codim$(\tilde{\mathcal{C}} \subset V)$ = codim$(\mathcal{C} \subset \mathbb{P}(V))$ is constant. Moreover, $\mathcal{N}$ is integrable.

Thus, restricting to a sufficiently small open subset of M, we may assume that the set of integral leafs of $\mathcal{N}$ is a *manifold* $\tilde{Y}$, i.e. we have a submersion

$$\tilde{\mu} : \tilde{S} \longrightarrow \tilde{Y}$$

such that $\mathcal{N}$ is precisely the tangent space of the fibers of $\tilde{\mu}$.

Let v be a vector field on $\tilde{S}$ with $v_s \subset \mathcal{N}$ for all $s \in \tilde{S}$. Then $\mathfrak{L}_v\omega_{\tilde{S}} = v \lrcorner d\omega_{\tilde{S}} + d(v \lrcorner \omega_{\tilde{S}}) = 0$, and therefore $\omega_{\tilde{S}}$ can be pushed down to $\tilde{Y}$ via $\tilde{\mu}$; in other words, there is a 2-form $\tilde{\omega}$ on $\tilde{Y}$ with

$$\omega_{\tilde{S}} = \tilde{\mu}^*(\tilde{\omega}).$$

It is obvious that $\tilde{\omega}$ is nondegenerate. Moreover, $0 = d\omega_{\tilde{S}} = \tilde{\mu}^*(d\tilde{\omega})$, and since $\tilde{\mu}$ is a submersion, it follows that $d\tilde{\omega} = 0$, i.e. $(\tilde{Y}, \tilde{\omega})$ is a symplectic manifold.

Since the distribution $\mathcal{N}$ is invariant under the natural $\mathbb{C}^*$-action on $\tilde{S}$, there is an induced $\mathbb{C}^*$-action on $\tilde{Y}$ for which

$$m_t^*\tilde{\omega} = t\tilde{\omega} \text{ for all } t \in \mathbb{C}^*. \tag{4}$$

Also, $\mathcal{N}$ factors through to an integrable distribution on $S = \tilde{S}/\mathbb{C}^*$, and if we denote the leaf space of this distribution by Y then we get a submersion $\mu : S \to Y$, and Y is the quotient of $\tilde{Y}$ by the $\mathbb{C}^*$-action. We denote the canonical projection by $p : \tilde{Y} \to Y$.

Let ∂_t denote the vector field on $\tilde{Y}$ whose flow induces this $\mathbb{C}^*$-action. Then by (4), $\mathfrak{L}_{\partial_t}\tilde{\omega} = \tilde{\omega}$, and since $\tilde{\omega}$ is closed, this implies that

$$\tilde{\omega} = d\tilde{\lambda}, \quad \text{where} \quad \tilde{\lambda} = \partial_t \lrcorner \tilde{\omega}.$$

Evidently, $\tilde{\lambda}(\partial_t) = 0$, and $\tilde{\lambda}$ is nowhere vanishing. Thus, for each $\tilde{y} \in \tilde{Y}$, there is a unique 1-form $0 \neq \underline{\lambda}_{\tilde{y}} \in T_y^*Y$ where $y = p(\tilde{y})$, such that $p^*(\underline{\lambda}_{\tilde{y}}) = \tilde{\lambda}_{\tilde{y}}$. Hence, the map $\imath : \tilde{Y} \hookrightarrow T^*Y\backslash\{0\}$ with $\imath(\tilde{y}) := \underline{\lambda}_{\tilde{y}}$ is well-defined and, by (4), a $\mathbb{C}^*$-equivariant embedding whose image is a $\mathbb{C}^*$-subbundle. It is now evident that $\tilde{\lambda} = \imath^*\lambda_Y$ where λ_Y denotes the Liouville 1-form on T^*Y, and thus $\tilde{\omega} = \imath^*\omega_Y$ where ω_Y is the canonical symplectic form on T^*Y. But since $\tilde{\omega}$ is non-degenerate on $\tilde{Y}$, Lemma 5.4 implies that the distribution $\mathcal{D}$ on Y which is annihilated by $\imath(\tilde{Y})$ defines a *contact structure on* Y, and $\imath(\tilde{Y}) \subset T^*Y\backslash\{0\}$ is precisely the dual of the contact line bundle $L \to Y$. Thus, identifying $\tilde{Y}$ with its image under this inclusion, we get the following commutative diagram:

$$\begin{array}{ccccc} & & \tilde{S} & \xrightarrow{\tilde{\mu}} & L^*\backslash\{0\} \\ & \swarrow & \downarrow \mathbb{C}^* & & \downarrow \mathbb{C}^* \\ M & \leftarrow & S & \xrightarrow{\mu} & Y \end{array} \tag{5}$$

For $p \in M$, we let $S_p := \pi^{-1}(p) \subset S$. Since $\mathcal{N} \cap TS_p = 0$ by (3), it follows that the map $\pi \times \mu : S \to M \times Y$ is an embedding. Moreover, it follows easily from the construction that $X_p := \mu(S_p) \subset Y$ is a *contact submanifold.* But from Proposition 5.5, $\dim Y = \dim S - \dim \mathcal{N} = 2\dim X_p + 1$ so that $X_p \subset Y$ is a *Legendre submanifold* and hence, S determines an analytic family of compact Legendre submanifolds. Moreover, from (5) it follows that $L^*|_{X_p} \cong \mathcal{O}(-1)|_{\mathcal{C}}$ and hence, $L_{X_p} \cong \mathcal{O}(1)|_{\mathcal{C}}$ is very ample and $H^0(\mathcal{C}, L_{X_p}) \cong V$. Thus, Theorem 5.3 implies

Theorem 5.6 *Let M be a manifold with a torsion free connection ∇ with irreducible holonomy group H, and let Y be the contact manifold constructed above. Then the twistor correspondence from (5) describes M as a locally maximal analytic family of Legendre submanifolds of Y.*

Definition 5.7 *Let M be a manifold with a torsion free connection ∇ with irreducible holonomy group H. Then the contact manifold Y from Theorem 5.6 is called the* Merkulov twistor space of *(M, ∇).*

Now we wish to interpret the points of Y as certain submanifolds of M. For this, we need some preliminary notation.

Definition 5.8 *Let $F \subset \mathfrak{F}$ be an H-structure on a manifold M with $H \subset Aut(V)$ irreducible, and consider the submanifold $\tilde{S} \subset T^*M$ from above.*

1. *A subspace $W_p \subset T_pM$ is called* conic *if there is a $\tilde{s} \in \tilde{S}_p \subset T_p^*M$ such that $Ann(W_p) = T_{\tilde{s}}(\tilde{S}_p)$. (Note that $T_{\tilde{s}}(\tilde{S}_p) \subset T_{\tilde{s}}(T_p^*M)$, and the latter can be canonically identified with the vector space T_p^*M.)*

2. *An (immersed) submanifold $\Sigma \subset M$ is called* conic *if $T_p\Sigma$ is conic for all $p \in \Sigma$.*

Note that from (2) it follows that $W_p \subset T_pM$ is a conic subspace iff there is a $\tilde{s} \in \tilde{S}_p$ such that $W_p = \pi_*(\mathcal{N}_{\tilde{s}})$. Moreover, $\tilde{s}$ is uniquely determined up to scalar multiples. From this, we can now show the following.

Proposition 5.9 *Let $M \xleftarrow{\pi} S \xrightarrow{\mu} Y$ be the lower line of the twistor correspondence in (5) associated to a torsion free H-structure $F \subset \mathfrak{F}$. Then $\pi(\mu^{-1}(y)) \subset M$ is totally geodesic and conic for all $y \in Y$.*

Conversely, every totally geodesic conic submanifold $\Sigma \subset M$ is contained in $\pi(\mu^{-1}(y))$ for some (unique) $y \in Y$.

Proof. Let $y \in Y$. From (5) it follows that $\pi(\mu^{-1}(y)) = \tilde{\pi}(\tilde{\mu}^{-1}(l))$ where $l \in L^*\backslash\{0\}$ lies in the fiber over y. By definition of $\tilde{Y} = L^*\backslash\{0\}$, the tangent space of $\tilde{\mu}^{-1}(l)$ is precisely $\mathcal{N}$. By (3), it follows that $\pi_*|_{\mathcal{N}_{\tilde{s}}}$ is injective so that $\pi|_{\tilde{\mu}^{-1}(l)}$ is an immersion, and its image is a conic submanifold. Moreover, it is totally geodesic since $\mathcal{N}_{\tilde{s}}$ is contained in the horizontal distribution by Proposition 5.5.

To see the converse, note that every totally geodesic submanifold $\Sigma \subset M$ is locally determined by its tangent space at a single point. Since $T_p\Sigma$ is conic, i.e. $T_p\Sigma = \pi_*(\mathcal{N}_{\tilde{s}})$ for some $\tilde{s} \in \tilde{S}_p$, it follows that $\Sigma \subset \pi(\tilde{\mu}^{-1}(l))$ where $l = \tilde{\mu}(\tilde{s})$. Finally, note that $\tilde{s}$ is determined up to the $\mathbb{C}^*$-action, so that $l \in \tilde{Y}$ is determined up to the $\mathbb{C}^*$-action, and this means that $y \in Y$ with $\Sigma \subset \pi(\mu^{-1}(y))$ is unique. ∎

This allows us to interpret the twistor spaces in the diagram $M \xleftarrow{\pi} S \xrightarrow{\mu} Y$ as follows: While M may be regarded as the moduli of Legendre deformations of a compact submanifold $X \subset Y$, we may regard Y as the space of maximal totally geodesic conic submanifolds of M.

6 Holonomies of symplectic manifolds

We call a pair $(V, \langle\ ,\ \rangle)$ a *symplectic vector space* if $\langle\ ,\ \rangle$ is a non-degenerate skew-symmetric bilinear form on the vector space V. The Lie group (Lie algebra, respectively) of symplectic automorphisms (endomorphisms, respectively) of V is given by

$$\mathrm{Sp}(V, \langle\ ,\ \rangle) = \{g \in \mathrm{Aut}(V) \mid \langle gx, gy\rangle = \langle x, y\rangle \text{ for all } x, y \in V\},$$

$$\mathfrak{sp}(V, \langle\ ,\ \rangle) = \{A \in \mathrm{End}(V) \mid \langle Ax, y\rangle + \langle x, Ay\rangle = 0 \text{ for all } x, y \in V\}.$$

We shall frequently omit the explicit reference to $\langle\ ,\ \rangle$. It is known that $\mathfrak{sp}(V)$ is the Lie algebra of $\mathrm{Sp}(V)$ and that both are simple. Moreover, there is a canonical $\mathrm{Sp}(V)$-equivariant isomorphism $\mathfrak{sp}(V) \cong \odot^2 V$, given by

$$(xy) \cdot z := \langle x, z\rangle\, y + \langle y, z\rangle\, x. \tag{6}$$

Also, the formula

$$(xy, zw) := \langle x, z\rangle\, \langle y, w\rangle + \langle x, w\rangle\, \langle y, z\rangle \tag{7}$$

for $x, y, z, w \in V$ yields a $\mathrm{Sp}(V)$-invariant symmetric bilinear form on $\mathfrak{sp}(V) \cong \odot^2 V$ and thus, must be a multiple of the Killing form.

Let $\mathfrak{h} \subset \mathfrak{sp}(V)$ be a Lie subalgebra. The orthogonal projection w.r.t. $(\ ,\ )$ from (7) yields an $\mathfrak{h}$-equivariant map $\circ : \odot^2 V \cong \mathfrak{sp}(V) \to \mathfrak{h}$, given by

$$(x \circ y, A) = \langle Ax, y\rangle \quad \text{for all } x, y \in V \text{ and } A \in \mathfrak{h}. \tag{8}$$

Let (M, ω) be a symplectic manifold. A torsion free connection ∇ on M is called *symplectic* if $\nabla\omega \equiv 0$. Thus, ∇ is a symplectic connection iff the holonomy group H of ∇ is contained in $\mathrm{Sp}(T_pM, \omega_p) \cong \mathrm{Sp}(V)$ where V is a vector space of the same dimension as M. Symplectic torsion free connections always exists. However, they are not uniquely determined by ω, and this establishes an important difference between symplectic and Riemannian geometry.

We call a proper irreducible subgroup $\mathrm{H} \subset \mathrm{Sp}(V)$ which can occur as the holonomy of a torsion free (symplectic) connection on M a *proper symplectic holonomy group.*

The first proper symplectic holonomy group was discovered by Bryant in 1991 [Br2]. Since then, various examples of proper symplectic holonomy groups have been discovered [CS, CMS1, CMS2, MeSc1, MeSc2]. The construction of connections with these holonomies is based on the realization of a certain Poisson structures, and we shall give a brief outline.

Theorem 6.1 *([CMS1, CMS2]) Let $(V, \langle\ ,\ \rangle)$ be a symplectic vector space and let $H \subset Sp(V)$ be an irreducible Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{sp}(V)$ such the product $\circ$ from (8) satisfies for all $x, y, z \in V$*

$$(x \circ y)z - (x \circ z)y = 2\,\langle y, z\rangle\, x - \langle x, y\rangle\, z + \langle x, z\rangle\, y. \tag{9}$$

Then there are torsion free connections on some manifold M whose holonomy equals H.

Sketch of proof. Let $W := \mathfrak{h} \oplus V$. Using the Killing form on $\mathfrak{h}$ and the symplectic form on V, we may canonically identify W and W^*. In particular, for $A \in \mathfrak{h}$, and $x \in V$ we define the linear functions $\tilde{A} := (A, \_)$ and $\tilde{x} := \langle x, \_\rangle$. Then, for an arbitrary constant c_0 we define the following bracket of functions on W.

$$\{\tilde{A}+\tilde{x}, \tilde{B}+\tilde{y}\}(C+z) := (C, [A, B]) + \langle z, A\cdot y - B\cdot x\rangle + 2\left\langle (C^2 + ((C, C) + c_0)Id_V)x, y\right\rangle \tag{10}$$

Here, $A, B, C \in \mathfrak{h}$ and $x, y, z \in \mathfrak{h}$. The first important observation is to verify that (9) implies that (10) yields a *Poisson bracket* on W for any c_0.

Next, it follows from general principles [W] that W admits *local symplectic realizations*, i.e. each point $p \in W$ is contained in an open set $U \subset W$ such that there is a symplectic manifold S and a submersion $\pi : S \to U$ which is compatible with the Poisson structures, that is

$$\{\pi^* f, \pi^* g\}_S = \pi^*(\{f, g\})$$

for all $f, g \in C^\infty(U)$ and where $\{\ ,\ \}_S$ denotes the Poisson structure on S induced by the symplectic structure.

One then defines the distribution $\Xi := \{X_{\pi^* f} \mid\ f \in C^\infty(U)\}$ on S where $X_{\pi^* f}$ denotes the Hamiltonian vector field of the function $\pi^* f$. Since $[X_{\pi^* f}, X_{\pi^* g}] = X_{\{\pi^* f, \pi^* g\}_S} = X_{\pi^*\{f,g\}}$ for all $f, g \in C^\infty(U)$, it follows that Ξ is integrable, and since π is a submersion, the correspondence $X_{\pi^* f} \leftrightarrow df$ yields a canonical isomorphism $\Xi_s \cong T^*_p U \cong W^* \cong W$ with $p = \pi(s)$.

Now let $F \subset S$ be an integral leaf of Ξ. Then the tangent space $T_s F = \Xi_s \cong W$ so that we obtain a $W = \mathfrak{h} \oplus V$-valued coframing on F. From the construction, it follows that this coframing satisfies the *structure equations* for a torsion free connection with holonomy contained in H, so that F gives rise to such a connection. The curvature of this connection depends on the choice of $U \subset W$, but for a generic U the holonomy of this connection equals *all* of H. ∎

Now it turns out that the Lie algebras of the proper symplectic holonomy groups all satisfy (9) [MeSc1, MeSc2]. Moreover, from Theorem 3.10 in [CMS1] and the calculations of the curvature maps of the proper symplectic holonomies in [MeSc1] it even follows that the construction of torsion free connections from Theorem 6.1 is *universal* for connections with proper symplectic holonomies. More precisely, we have

Theorem 6.2 *[MeSc1] Let $H \subset Sp(V)$ be a connected irreducible proper Lie subgroup. Then H is a proper symplectic holonomy group iff (H, V) is an entry of Table 2. This occurs iff the Lie algebra $\mathfrak{h}$ satisfies (9).*

Moreover, if $H \subset Sp(V)$ is a proper symplectic holonomy group then every torsion free connection with holonomy group H is locally equivalent to a connection which comes from the construction in the proof of Theorem 6.1. In particular, the moduli space of torsion free connections with holonomy group H is finite dimensional.

Table 2

H	V	H	V
$\mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^4 \simeq \odot^3\mathbb{R}^2$	E_7^5	$\mathbb{R}^{56}$
$\mathrm{SL}(2,\mathbb{C})$	$\mathbb{C}^4 \simeq \odot^3\mathbb{C}^2$	E_7^7	$\mathbb{R}^{56}$
$\mathrm{SL}(2,\mathbb{R})\cdot\mathrm{SO}(p,q)$	$\mathbb{R}^{2(p+q)}$, $(p+q)\geq 3$	$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
$\mathrm{SL}(2,\mathbb{C})\cdot\mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^{2n}$, $n\geq 3$	$\mathrm{Spin}(2,10)$	$\mathbb{R}^{32}$
$\mathrm{Sp}(1)\mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n \simeq \mathbb{R}^{4n}$, $n\geq 2$	$\mathrm{Spin}(6,6)$	$\mathbb{R}^{32}$
$\mathrm{SL}(6,\mathbb{R})$	$\mathbb{R}^{20}\simeq\Lambda^3\mathbb{R}^6$	$\mathrm{Spin}(12,\mathbb{C})$	$\mathbb{C}^{32}$
$\mathrm{SU}(1,5)$	$\mathbb{R}^{20}\subset\Lambda^3\mathbb{C}^6$	$\mathrm{Sp}(3,\mathbb{R})$	$\mathbb{R}^{14}\subset\Lambda^3\mathbb{R}^6$
$\mathrm{SU}(3,3)$	$\mathbb{R}^{20}\subset\Lambda^3\mathbb{C}^6$	$\mathrm{Sp}(3,\mathbb{C})$	$\mathbb{C}^{14}\subset\Lambda^3\mathbb{C}^6$
$\mathrm{SL}(6,\mathbb{C})$	$\mathbb{C}^{20}\simeq\Lambda^3\mathbb{C}^6$		

There is a natural interpretation of the crucial identity (9) (cf. [Sc]). Namely, let us consider an irreducible symmetric space of the form $G/(\mathrm{SL}(2,\mathbb{F})\cdot\mathrm{H})$ with $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Then on the Lie algebra level we may write

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{F})\oplus\mathfrak{h}\oplus\mathbb{F}^2\otimes V.$$

Note that we may identify $\mathfrak{sl}(2,\mathbb{F})\cong\odot^2\mathbb{F}^2$ by (6). Then the curvature is a linear $\mathrm{SL}(2,\mathbb{F})\cdot\mathrm{H}$-equivariant map $R:\Lambda(\mathbb{F}^2\otimes V)\to\mathfrak{sl}(2,\mathbb{F})\oplus\mathfrak{h}$. Decomposing into components and using the H-equivariance, we observe that there are non-vanishing constants $c_1, c_2\in\mathbb{F}$ such that for all $e,f\in\mathbb{F}^2$ and $x,y\in V$,

$$R(e\otimes x, f\otimes y) = c_1\,\langle x,y\rangle\,ef + c_2\,\langle e,f\rangle\,x\circ y.$$

Here, we use $\langle\ ,\ \rangle$ to denote both the symplectic form on $\mathbb{F}^2$ and on V. Of course, we can replace $\langle\ ,\ \rangle$ by a suitable multiple to achieve that $c_1=c_2$, and then it is a straightforward calculation to verify that (9) is equivalent to the first Bianchi identity for R.

There is yet another characterization of the entries of Table 2. Namely, consider the irreducible $\mathrm{Sp}(V)$-module $W:=(\Lambda^2V)_0$. Then the entries of Table 2 are precisely those Lie subgroups $\mathrm{H}\subset\mathrm{Sp}(V)$ for which W is H-irreducible. This has been observed in [C], where also a direct proof of (9) from the H-irreducibility of W is given.

Finally, if V and H are both complex and $\mathrm{H}\subset\mathrm{Sp}(V)$ is a proper subgroup, then H is a proper symplectic holonomy group iff the sky $\mathcal{C}\subset\mathbb{P}(V)$ of H is a Legendre submanifold and $\mathrm{H}=\mathrm{Aut}(\mathcal{C})$. This description follows from the work of Alekseevskii and Cortés [AC].

We collect these various descriptions in the following

Theorem 6.3 *Let $H\subset Sp(V)$ be an irreducible connected closed subgroup and let $\mathfrak{h}\subset\mathfrak{sp}(V)$ be its Lie algebra. Then the following are equivalent.*

1. *H is the holonomy group of a symplectic connection.*
2. *$H = Sp(V)$ or H is an entry of Table 2.*

3. *After replacing $\langle\ ,\ \rangle$ by a suitable multiple, the product $\circ$ from (8) satisfies (9).*

4. *There is an irreducible symmetric space of the form $G/(SL(2,\mathbb{F})\cdot H)$ with $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$.*

5. *The H-module $(\Lambda^2 V)_0$ is irreducible.*

Furthermore, if V and H are both complex and $H \neq Sp(V)$, then the above conditions hold iff the sky $\mathcal{C} \subset \mathbb{P}(V)$ of H is a Legendre submanifold and $H = Aut(\mathcal{C})$.

It is particularly the latter description which is important in the context of twistor theory, since the sky $\mathcal{C} \subset \mathbb{P}(V)$ is the Legendre submanifold which is being deformed in the Merkulov twistor space. Namely, from this it follows that every conic subspace $W_p \subset T_pM$ is *Lagrangian* and thus, we obtain the following result from Proposition 5.9.

Corollary 6.4 *Let (M,ω,∇) be a symplectic manifold with proper exotic holonomy H. Then every conic submanifold of M is Lagrangian. In fact, the Merkulov twistor space is the moduli space of all maximal conic totally geodesic Lagrangian submanifolds $\Sigma \subset M$.*

In fact, one can show from the structure equations for connections with proper symplectic holonomies [CMS1, Sc] that if the curvature tensor at some point is sufficiently "generic" then a Lagrangian submanifold $\Sigma \subset M$ is totally geodesic iff it is conic. Thus, under some generic curvature assumptions we may regard the Merkulov twistor space as the space of all totally geodesic Lagrangian submanifolds of M.

Also, if ∇ is a generic connection with $\nabla\omega \equiv 0$ then one can show that there are no totally geodesic Lagrangians at all. Thus, the mere existence of such Lagrangians is by itself a special feature of connections with proper symplectic holonomy group.

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