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Symplectic connections via integration of Poisson structures

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Dedié à Marguerite Bieliavsky et ses parents

1. INTRODUCTION

An affine connection is one of the basic objects of interest in differential geometry. It provides a simple and invariant way of transferring information from one point of a connected manifold M to another and, not surprisingly, enjoys lots of applications in many branches of mathematics, physics and mechanics. Among the most informative characteristics of an affine connection is its (restricted) holonomy group which is defined, up to conjugacy, as the subgroup of $\mathrm{Aut}(T_pM)$ consisting of all automorphisms of the tangent space T_pM at $p \in M$ induced by parallel translations along p-based loops in M.

One of the basic features of the holonmy group is that any tensor on the tangent bundle which is invariant under the holonomy group establishes a certain 'parallel' geometric structure. For example, if the holonomy equals the (special) orthogonal group, then there is a unique Riemanninan metric which is compatible with the connection.

To make this concept more precise, we shall explain the notion of a *G-structure* on a manifold. In fact, to each connection we may associate a G-structure where G is the holonomy group of the connection. We shall give several examples to explain how canonical geometric structures may be viewed as G-structures for an appropriate G.

There is a natural first order invariant of a connection, namely its *torsion*. We shall see that there is a related invariant of a G-structure, called the *intrinsic torsion*. The vanishing of the intrinsic torsion is an integrability condition for the G-structure, and we shall explain how in examples this corresponds to the 'natural' integrability condition of the underlying geometric structure.

This leads to the question for which subgroups $G \subset \mathrm{Aut}(T_pM)$ there exist G-structures whose intrinsic torsion vanishes, i.e. which satisfy the aforementioned integrability condition. This problem is essentially equivalent to asking which subgroups $G \subset \mathrm{Aut}(T_pM)$ may occur as the holonomy

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group of a torsion free connection. In general, the answer to this question is far from being known, even though a complete classification is known under the hypothesis that G is an *irreducible* subgroup [MS].

A particularly intriguing class of connections is given by those with *special symplectic holonomy*, i.e. torsion free connections whose holonomy is an absolutely irreducible subgroup of the symplectic group. Indeed, for these connections there is a universal construction method which is based on the deformation of a certain linear Poisson structure and the existence of symplectic realizations of this structure [MS], [S]. This deformation can be phrazed entirely with the help of simple Lie algebras, and we shall explain this procedure in detail.

By the same method, we can also construct all symplectic connections part of whose curvature vanishes. These are the so-called *symplectic connections of Ricci type* [BC].

Finally, we shall end our discussion by presenting some open problems concerning the global aspects of these connections.

2. G-STRUCTURES AND INTRINSIC TORSION

As before, let M be a smooth connected (real or complex) manifold of dimension n. Let $\pi : \mathfrak{F} \to M$ be the *coframe bundle* of M, i.e. each $u \in \mathfrak{F}$ is a linear isomorphism $u : T_{\pi(u)}M \xrightarrow{\sim} V$, where V is a fixed n-dimensional (real or complex) vector space. Then $\mathfrak{F}$ is naturally a principal right Aut(V)-bundle over M, where the right action $R_g : \mathfrak{F} \to \mathfrak{F}$ is defined by $R_g(u) = g^{-1} \circ u$. The *tautological* 1*-form* θ on $\mathfrak{F}$ with values in V is defined by $\theta(\xi) = u(d\pi(\xi))$ for $\xi \in T_u\mathfrak{F}$. For θ, we have the Aut(V)-equivariance

$$R_g^*(\theta) = g^{-1}\theta. \tag{1}$$

Let $G \subset$ Aut(V) be a closed Lie subgroup and let $\mathfrak{g} \subset$ End(V) be the Lie algebra of G. A G-structure on M is, by definition, a G-subbundle $F \subset \mathfrak{F}$. For any G-structure, we will denote the restrictions of π and θ to F by the same letters.

It is easy to verify that there is a canonical one-to-one correspondence between G-structures on M and sections of the fiber bundle $\mathfrak{F}/G \to M$ whose fiber is the homogeneous space Aut$(V)/G$. This point of view may sometimes be helpful in order to determine the topological obstructions for the existence of a G-structure on a given manifold M.

For instance, a topological fact states that a fiber bundle with contractible fibers always admits a global section. Applying this to the bundle $\mathfrak{F}/G \to M$, we conclude that for any subgroup $G \subset$ Aut(V) for which Aut$(V)/G$ is contractible, there is a G-structure on any manifold M.

Examples. Fixing a basis of V, we identify Aut(V) with $Gl(n, \mathbb{R})$, the group of invertible $(n \times n)$-matrices.

(1) $G = Gl(n, \mathbb{R})^+$ which denotes the $(n \times n)$-matrices of positive determinant. Then a G-structure is equivalent to an orientation. Evidently, not any manifold admits an orientation. Indeed, the fiber bundle $\mathfrak{F}/G$ has fiber $Gl(n, \mathbb{R})/G \cong \mathbb{Z}_2$, i.e. either $\mathfrak{F}/G \cong M \dot\cup M$ is disconnected, or $\mathfrak{F}/G$ is connected so that $\mathfrak{F}/G \to M$ is a double cover.

In the first case, M is orientable, and $\mathfrak{F}/G \cong M \dot\cup M$ admits two global sections, corresponding to the two possible orientations on M. In the second case, $\mathfrak{F}/G$ admits no global section, i.e. M is non-orientable; indeed, $\mathfrak{F}/G \to M$ is the oriented double cover of M.

(2) $G = Sl(n, \mathbb{R})$ which denotes the $(n \times n)$-matrices of determinant one. A G-structure is equivalent to a volume form, i.e. a global nowhere vanishing n-form $vol \in \Omega^n(M)$. Clearly, a volume form determines an orientation which is reflected by the fact that $Sl(n, \mathbb{R}) \subset Gl(n, \mathbb{R})^+$. Also, $Gl(n, \mathbb{R})^+/Sl(n, \mathbb{R}) \cong \mathbb{R}^+$ is contractible, which means that any oriented manifold admits a volume form.

(3) $G = Gl(m,\mathbb{C}) \subset Gl(2m,\mathbb{R})$. A G-structure is equivalent to an almost complex structure J on M. Since $Gl(m,\mathbb{C}) \subset Gl(2m,\mathbb{R})^+$, it follows that any almost complex structure induces an orientation. However, $Gl(2m,\mathbb{R})^+/Gl(m,\mathbb{C})$ is the Hermitean symmetric space of all oriented complex structures on $\mathbb{R}^{2m}$, and this space is not contractible. Hence, not every (oriented) manifold admits an almost complex structure.

(4) $G = O(n) \subset Gl(n,\mathbb{R})$ which denotes the orthogonal $(n \times n)$-matrices. In this case, a G-structure corresponds to a Riemannian metric, and since $Gl(n,\mathbb{R})/O(n)$ is the (contractible) space of all positive definite symmetric $(n \times n)$-matrices, it follows that any manifold admits a Riemannian metric.

(5) $G = Sp(m,\mathbb{R}) \subset Gl(2m,\mathbb{R})$ which consists of all automorphisms which preserve the standard symplectic form on $\mathbb{R}^{2m}$. Then a G-structure is equivalent to a pre-symplectic structure on M, i.e. a 2-form $\omega \in \Omega^2(M)$ such that ω^m is a volume form.

(6) $G = U(m) \subset Gl(m,\mathbb{C}) \subset Gl(2m,\mathbb{R})$ which consists of all unitary $(m \times m)$-matrices. A G-structure on M is equivalent to a *Hermitean structure* on M, i.e. a Riemannian metric g and an orthogonal almost complex structure J. Since $Gl(m,\mathbb{C})/U(m)$ is the space of positive self-adjoint $(m\times m)$-matrices and this space is contractible, it follows that any manifold which admits an almost complex structure also admits a Hermitean structure.

Needless to say, this list is by no means complete. It rather serves as an illustration of the usefulness of the concept of G-structures.

Let $\pi : P \to M$ be a principal G-bundle. Recall that a *connection* on P is a distribution $\mathcal{H}$ which is G-invariant and transversal to the fibers of π. Given $A \in \mathfrak{g}$, we define the vector field ξ_A on P by

$$(\xi_A)_u = \frac{d}{dt}\left(R_{exp(tA)}(u)\right)|_{t=0}.$$

It is evident that $d\pi(\xi_A) = 0$; in fact, $\{\xi_A \mid A \in \mathfrak{g}\} = \ker(d\pi)$.

Let us now consider the special situation where $P = F \subset \mathfrak{F}$ is a G-structure. Then from the preceding, we have $\theta(\xi_A) = 0$ for all $A \in \mathfrak{g}$. Also, for given $x \in V$ and $u \in F$, there is a unique $\xi_x \in \mathcal{H}_u \subset T_uF$ with $\theta(\xi_x) = x$. Evidently, the vector fields $\{\xi_x \mid x \in V\}$ span $\mathcal{H}$, so that we get a frame of F via the identification

$$\begin{array}{ccc} \mathfrak{g} \oplus V & \longleftrightarrow & T_uP \\ \\ A + x & \longmapsto & \xi_A + \xi_x. \end{array}$$

The vector fields ξ_A and ξ_x are called the *fundamental vertical vector fields on F* and the *fundamental horizontal vector fields on F*, respectively. For their bracket relations, we obtain the following equations for all $A, B \in \mathfrak{g}$ and $x, y \in V$:

$$[\xi_A, \xi_B] = \xi_{[A,B]} \quad [\xi_A, \xi_x] = \xi_{Ax} \quad [\xi_x, \xi_y] = \xi_{\Theta(x,y)} + \xi_{\Omega(x,y)}, \tag{2}$$

where $\Theta(x,y) \in V$ and $\Omega(x,y) \in \mathfrak{g}$. Indeed, the first of these bracket relation is equivalent to saying that the vector fields ξ_A are induced by a right Lie group action, whereas the second relation is equivalent to the G-invariance of $\mathcal{H}$ and the G-equivariance of the tautological form θ (1). The decomposition of the brackets $[\xi_x, \xi_y]$ into their horizontal and vertical part yields the forms

$$\Theta \in \Omega^2(F) \otimes V \quad \text{and} \quad \Omega \in \Omega^2(F) \otimes \mathfrak{g},$$

called the *torsion* and the *curvature* of the connection, respectively.

Suppose now that $\mathcal{H}'$ is another connection on F. Then the fundamental horizontal vector fields w.r.t. $\mathcal{H}'$ have the form $\xi'_x = \xi_x + \xi_{\alpha(x)}$ for some $\alpha \in \Omega^1(F) \otimes \mathfrak{g}$. Thus, the bracket relations (2) yield that α is G-equivariant, and

$$[\xi'_x, \xi'_y] = [\xi_x, \xi_y] + \xi_{\alpha(x)y - \alpha(y)x} + \xi_{\xi'_x(\alpha(y)) - \xi'_y(\alpha(x))} + \xi_{[\alpha(x),\alpha(y)]}.$$

Note that the last two terms are vertical, hence the torsion changes to

(3) $$\Theta'(x,y) = \Theta(x,y) + \alpha(x)y - \alpha(y)x.$$

This motivates the following definition. Given a Lie subalgebra $\mathfrak{g} \subset \mathrm{End}(V)$, we define the map

$$\delta : V^* \otimes \mathfrak{g} \longrightarrow \Lambda^2 V^* \otimes V, \quad (\delta(\alpha))(x,y) := \alpha(x)y - \alpha(y)x$$

for all $\alpha \in V^* \otimes \mathfrak{g}$ and $x, y \in V$. This is actually one of the boundary maps of the *Spencer complex* of $\mathfrak{g} \subset \mathrm{End}(V)$ which we will not explain in detail here but rather refer the reader to e.g. [G], [O]. At least, this explains the notation for the kernel and cokernel of δ, given in the following exact sequence:

$$0 \longrightarrow \mathfrak{g}^{(1)} \longrightarrow V^* \otimes \mathfrak{g} \stackrel{\delta}{\longrightarrow} \Lambda^2 V^* \otimes V \longrightarrow H^{0,1}(\mathfrak{g}) \longrightarrow 0.$$

Here, $\mathfrak{g}^{(1)} = \{\alpha \in V^* \otimes \mathfrak{g} \mid \alpha(x)y = \alpha(y)x \text{ for all } x, y \in V\}$ is called the *first prolongation of* $\mathfrak{g}$. Note that with this notation, we can rewrite (3) as

$$\Theta' = \Theta + \delta(\alpha).$$

Therefore, as an immediate consequence, we obtain the following

Theorem 2.1. *Let $F \subset \mathfrak{F}$ be a G-structure on M, and the $\mathcal{H}$ be any connection on F with torsion $\Theta \in \Omega^2(F) \otimes V$. Then*

(1) $[\Theta_u] \in H^{0,1}(\mathfrak{g})$ *is well-defined, independent of the choice of connection, i.e. it depends only on the G-structure F.* $[\Theta_u] \in H^{0,1}(\mathfrak{g})$ *is called the* intrinsic torsion *of F at $u \in F$.*
(2) *F admits a torsion free connection iff its intrinsic torsion vanishes everywhere.*
(3) *If $\mathfrak{g}^{(1)} = 0$ then there is at most one torsion free connection on F.*

Examples

(1) $G = Sl(n, \mathbb{R})$. In this case, δ is surjective so that $H^{0,1} = 0$. On the other hand, by a dimension count, δ cannot be injective, i.e. $\mathfrak{g}^{(1)} \neq 0$. This reflects the (easy) fact that for a given volume form, there are many *unimodular connections*, i.e. torsion free connections for which the volume form is parallel.
(2) $G = Gl(m, \mathbb{C}) \subset Gl(2m, \mathbb{R})$. Here, one calculates that $H^{0,1}(\mathfrak{g}) \cong \{\phi \in \Lambda^2 V^* \otimes V \mid \phi(Jx, Jy) + J\phi(x,y) = 0\}$, where $J : V \to V$ denotes the complex structure on V. Indeed, the intrinsic torsion is represented by the Nijenhuis tensor, hence an almost complex structure is torsion free iff its Nijenhuis tensor vanishes iff it is integrable. On the other hand, $\mathfrak{g}^{(1)} \neq 0$, which implies that on a complex manifold, there are many torsion free connections for which the complex structure is parallel.
(3) $G = O(n) \subset Gl(n, \mathbb{R})$. For this example, let us calculate the first prolongation explicitly. I.e., let $\alpha : V \to \mathfrak{so}(n)$ be linear such that $\alpha(x)y = \alpha(y)x$. Then, since $\alpha(x)$ is skew symmetric, we have for $x, y, z \in V$

$$\begin{aligned}(\alpha(x)y) \cdot z &= -y \cdot (\alpha(x)z) = -y \cdot (\alpha(z)x) = (\alpha(z)y) \cdot x \\ &= (\alpha(y)z) \cdot x = -z \cdot (\alpha(y)x) = -z \cdot (\alpha(x)y).\end{aligned}$$

Thus, $(\alpha(x)y) \cdot z = 0$ for all $x, y, z \in V$, so that $\alpha = 0$, i.e. $\mathfrak{g}^{(1)} = 0$. On the other hand, $\dim \mathfrak{g} = \dim \Lambda^2 V$ so that δ is an isomorphism and hence, $H^{0,1}(\mathfrak{g}) = 0$ as well.

These two observations together reflect the fact that for a Riemannian metric, there is always a unique torsion free connection compatible with the metric, namely the Levi-Civita-connection.

(4) $G = Sp(m, \mathbb{R}) \subset Gl(2m, \mathbb{R})$. In this case, one calculates that $H^{0,1}(\mathfrak{g}) = \Lambda^3 V^*$, and the intrinsic torsion is represented by $d\omega$, where $\omega \in \Omega^2(M)$ is the almost symplectic structure. Thus, the G-structure is torsion free iff $d\omega = 0$, i.e. iff ω is a *symplectic form* on M. Moreover, a dimension count yields that $\mathfrak{g}^{(1)} \neq 0$, hence on a symplectic manifold, there are many torsion free connections for which the symplectic form is parallel.

(5) $G = U(m) \subset Gl(m, \mathbb{C}) \subset Gl(2m, \mathbb{R})$. A G-structure corresponds to a Hermitean structure (M, g, J). Thus, a connection on this G-structure is a connection for which g and J are parallel. It follows that any torsion free connection compatible with G must be the Levi-Civita-connection of g. Thus, the G-structure is torsion free iff the orthogonal almost complex structure J is parallel w.r.t. the Levi-Civita connection of g. This implies, in particular, that the almost complex structure is integrable. Moreover, the *Kähler form* $\omega(x, y) := g(x, Jy)$ must also be parallel and hence closed, that is (M, g, J) is a Kähler manifold.

Thus, a $U(m)$-structure, i.e. a Hermitean structure, is torsion free iff the Hermitean structure yields a Kähler metric.

These examples should convince the reader that the integrability condition which is equivalent to the vanishing of the intrinsic torsion is indeed the most natural integrability condition which can be imposed on the underlying geometric structure.

3. Constructing local torsion free connections

We have seen that a connection on a G-structure $F \subset \mathfrak{F}$ induces a framing of F by vector fields $\{\xi_x \mid x \in V\}$ and $\{\xi_A \mid A \in \mathfrak{g}\}$, satisfying the structure equations (2). Conversely, suppose that we are given vector fields ξ_x, ξ_A on a manifold F which induce pointwise an isomorphism $T_u F \leftrightarrow V \oplus \mathfrak{g}$ and satisfy the bracket relations (2).

The first of the equations in (2) implies that the flow along the vector fields ξ_A induces a free local action of G on F. Indeed, after shrinking F if necessary, we may assume that the orbit space $M := F/G$ is a manifold. Let $\pi : F \to M$ be the canonical projection.

If $\mathcal{H}$ is the distribution on F spanned by the ξ_x's, then evidently, $\mathcal{H}$ is a distribution transversal to the fibers of π, and the second equation of (2) implies that this distribution is G-invariant. For $u \in F$, the restriction $d\pi : \mathcal{H} \to T_{\pi(u)} M$ is an isomorphism, and since $\mathcal{H}_u \cong V$ via the framing, we can associate to $u \in F$ an isomorphism $T_{\pi(u)} M \to V$, i.e. a point in $\mathfrak{F}$.

This yields a G-equivariant immersion $\imath : F \to \mathfrak{F}$, and – again after shrinking F if necessary – we may assume that $\imath$ is an embedding. Thus, $\hat{F} := G \cdot \imath(F) \subset \mathfrak{F}$ is a G-structure, and $d\imath(\mathcal{H})$ extends in a canonical way to a connection on $\hat{F}$.

Now the third equation of (2) implies that this connection has Θ and Ω as its torsion and curvature, respectively, so that all together we have the

Proposition 3.1. *Let $\mathfrak{g} \subset Aut(V)$ be the Lie algebra of the closed subgroup $G \subset Aut(V)$, let F be a manifold with $\dim F = \dim V + \dim \mathfrak{g}$, and let $\{\xi_x \mid x \in V\}$ and $\{\xi_A \mid A \in \mathfrak{g}\}$ be vector fields on F which induce a pointwise isomorphism $T_u F \leftrightarrow V \oplus \mathfrak{g}$. If these vector fields satisfy the bracket relations (2), then – at least locally – we can embedd F into a G-structure $\hat{F} \subset \mathfrak{F}$ of a manifold M of $\dim M = \dim V$ which carries a connection whose torsion and curvature are given by Θ and Ω, respectively.*

Thus, in order to construct local examples of torsion free connections with holonomy group G, we must find vector fields ξ_x and ξ_A, satisfying (2) with $\Theta = 0$.

4. POISSON TYPE CONNECTIONS

Definition 4.1. *Let $(P, \{\ ,\ \})$ be a Poisson manifold. A* symplectic realization of P *is a symplectic manifold (S, ω) and a submersion*

$$\pi : S \longrightarrow P$$

which is compatible with the Poisson structures, i.e.

$$\{\pi^*(f), \pi^*(g)\}_S = \pi^*(\{f, g\}) \quad \textit{for all } f, g \in C^\infty(P, \mathbb{R}), \tag{4}$$

where the Poisson bracket $\{\ ,\ \}_S$ on S is induced by the symplectic structure.

The following result ensures the existence of such realizations, at least locally.

Proposition 4.2. [W] *Let $(P, \{\ ,\ \})$ be a Poisson manifold. Then for every point $p_0 \in P$, there is an open neighborhood U of p_0 and a symplectic realization $\pi : S \longrightarrow U$.*

If $\pi : S \to P$ is a symplectic realization then, in order to avoid confusion, we denote the Hamiltonian vector fields on S by ξ_h where $h \in C^\infty(S)$, while we denote the Hamiltonian vector fields on P by η_f for $f \in C^\infty(P)$. With this, we have for all $f, g \in C^\infty(P)$

$$d\pi(\xi_{\pi^*(f)}) = \eta_f \quad \text{and} \quad \begin{array}{rcl} [\xi_{\pi^*(f)}, \xi_{\pi^*(g)}] & = & \xi_{\{\pi^*(f), \pi^*(g)\}_S} \\ & = & \xi_{\pi^*(\{f,g\})} \end{array} \quad \text{by (4).} \tag{5}$$

This implies that the distribution Ξ on S given by

$$\Xi_s = \{(\xi_{\pi^*(f)})_s \mid f \in C^\infty(P)\} \text{ for all } s \in S$$

is integrable. Evidently, $(\xi_{\pi^*(f)})_s$ only depends on $df_{\pi(s)}$, and since π is a submersion, the map $\pi^* : T^*_{\pi(s)}P \to T^*_s S$ is injective. Thus, the canonical map

$$\begin{array}{rccc} \Theta : & T^*_{\pi(s)}P & \longrightarrow & \Xi_s \\ & df_{\pi(s)} & \longmapsto & (\xi_{\pi^*(f)})_s \end{array}$$

is a linear isomorphism and hence, Ξ has constant rank equal to the dimension of P. Moreover, if $F \subset S$ is an integral leaf of Ξ then by (5), there is a symplectic leaf $\Sigma \subset P$ such that $\pi : F \to \Sigma$ is a submersion.

In order to use this approach to solve the structure equations (2), let us assume that $P = (\mathfrak{g} \oplus V)^*$ where $\mathfrak{g} \subset \mathrm{End}(V)$ is a Lie subalgebra. Then, since P is a vector space, we have $T^*_{\pi(s)}P \cong P^* = \mathfrak{g} \oplus V$ canonically. Thus, if we regard $x \in V$ and $A \in \mathfrak{g}$ as (linear) functions on P, then it follows that the Hamiltonian vector fields $\{\xi_x := \xi_{\pi^*(x)} \mid x \in V\}$ and $\{\xi_A := \xi_{\pi^*(A)} \mid A \in \mathfrak{g}\}$ yield a framing on F.

Now it is clear from (5) that these vector fields satisfy the first two equations of (2) if the Poisson bracket on P is given as

$$\{A + x, B + y\}(\lambda) = \lambda([A, B] + Ay - Bx) + \Phi(\lambda)(x, y), \tag{6}$$

where we regard $A, B \in \mathfrak{g}$ and $x, y \in V$ as (linear) functions on P, where $\lambda \in P = (\mathfrak{g} \oplus V)^*$, and where $\Phi : P \to \Lambda^2 V^*$ is a smooth map. Moreover, we calculate

$$d\{x, y\}(\lambda) = (d\Phi_\lambda)^*(x, y) \in P^* = \mathfrak{g} \oplus V. \tag{7}$$

Thus, again by (5), the condition $\Theta = 0$ is equivalent to saying that $(d\Phi_\lambda)^*(x, y) \in \mathfrak{g}$ for all $\lambda \in P$ and $x, y \in V$, or, equivalently, $d\Phi_\lambda(V^*) = 0$ for all $\lambda \in P$, i.e. Φ depends only on $\mathfrak{g}^*$.

Thus, we can construct local solutions to the structure equations (2) and hence by Proposition 3.1 torsion free connections on a G-structure by this method if the Poisson structure on P is given by (6) with a map $\Phi : \mathfrak{g}^* \to \Lambda^2 V^*$. Note that the curvature of the corresponding connection is then represented by $(d\Phi_\lambda)^* : \Lambda^2 V \to \mathfrak{g}$ by (7).

Now a straightforward calculation yields the following

Lemma 4.3. *The equation (6) with $\Phi : \mathfrak{g}^* \to \Lambda^2 V^*$ defines a Poisson structure on P iff Φ satisfies the following conditions:*

(i) *Φ is H-equivariant,*

(ii) *for every $\lambda \in \mathfrak{g}^*$, the dual of the differential $(d\Phi_\lambda)^* : \Lambda^2 V \to \mathfrak{g}$ satisfies the first Bianchi identity $(d\Phi_\lambda)^*(x,y)z + (d\Phi_\lambda)^*(y,z)x + (d\Phi_\lambda)^*(z,x)y = 0$ for all $x, y, z \in V$.*

Note that the second condition is obviously necessary since by (7), $(d\Phi_\lambda)^*$ represents the curvature of a torsion free connection and hence must satisfy the first Bianchi identity.

Definition 4.4. *Let $\mathfrak{g} \subset End(V)$, and let $\Phi : \mathfrak{g}^* \to \Lambda^2 V^*$ be a smooth map which satisfies the hypotheses of Lemma 4.3. Then any (local) torsion free connection which comes from the symplectic realization of the corresponding Poisson structure on $P = (\mathfrak{g} \oplus V)^*$ is called a* Poisson type connection.

For instance, the map $\Phi = 0$ would satisfy these hypotheses, but the corresponding Poisson type connections are flat. A priori, it is not clear if there exist maps Φ satisfying the conditions of Lemma 4.3 and such that the curvature $(d\phi_\lambda)^*$ is 'interesting'. The following section will be devoted to the construction of such maps for a special class of subalgebras $\mathfrak{g} \subset \mathrm{End}(V)$.

5. SPECIAL SYMPLECTIC CONNECTIONS

Theorem 5.1. *Let $\mathfrak{g} \subset End(V)$ be a Lie subalgebra where V is a finite dimensional vector space over $\mathbb{R}$. Let $W := \mathbb{R}^2 \otimes V$ and consider the induced tensor representation of $\mathfrak{g}^+ := \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{g} \subset End(W)$. Then the following are equivalent.*

(1) *There is an irreducible symmetric pair $(\hat{\mathfrak{g}}, \mathfrak{g}^+)$ whose isotropy representation is equivalent to the representation of $\mathfrak{g}^+$ on W.*

(2) *There is a symplectic form $\omega \in \Lambda^2 V^*$ such that $\mathfrak{g} \subset \mathfrak{sp}(V, \omega)$, an $ad_{\mathfrak{g}}$-invariant non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{g}$ and a surjective $\mathfrak{g}$-equivariant map $\circ : S^2 V \to \mathfrak{g}$ which satisfy for all $x, y, z \in V$ and $A \in \mathfrak{g}$*

$$(8)\qquad \begin{aligned} (x \circ y)z - (x \circ z)y &= 2\,\omega(y,z)x + \omega(x,z)y - \omega(x,y)z, \\ (A, x \circ y) &= \omega(Ax, y). \end{aligned}$$

If these conditions are satisfied, then – identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via $(\ ,\ )$ – the map $\Phi^c : \mathfrak{g} \to \Lambda^2 V^*$ defined by

$$(9)\qquad \Phi^c(A)(x,y) := \omega((A^2 + (c - (A,A))Id)x, y)$$

satisfies the hypotheses of Lemma 4.3 for any constant $c \in \mathbb{R}$. Indeed, the map

$$(10)\qquad (d\Phi^c_A)^* =: R_A, \quad \textit{is given by} \quad R_A(x,y) = 2\,\omega(x,y)\,A + x \circ (Ay) - y \circ (Ax).$$

Proof. Suppose that $(\hat{\mathfrak{g}}, \mathfrak{g}^+)$ is the requested symmetric pair, i.e. as a vector space, $\hat{\mathfrak{g}} = \mathfrak{g}^+ \oplus W = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{g} \oplus \mathbb{R}^2 \otimes V$. Let $R := [\ ,\ ] : \Lambda^2 W \to \mathfrak{g}^+ = \mathfrak{sl}(2,\mathbb{F}) \oplus \mathfrak{g}$ be the Lie bracket. Since $\Lambda^2 W = S^2 V \oplus \mathfrak{sl}(2,\mathbb{F}) \otimes \Lambda^2 V$ as a $\mathfrak{g}^+$-module, and since R is $\mathfrak{g}^+$-equivariant, it follows that there is a $\mathfrak{g}$-invariant $\omega \in \Lambda^2 V^*$ and a $\mathfrak{g}$-equivariant map $\circ : S^2 V \to \mathfrak{g}$ such that

$$(11)\qquad R(e \otimes x, f \otimes y) = \omega(x,y)ef + \langle e, f\rangle\, x \circ y$$

for all $e, f \in \mathbb{R}^2$ and $x, y \in V$, where $\langle \_, \_ \rangle$ is the determinant on $\mathbb{R}^2$, and where we identify $\mathfrak{sl}(2,\mathbb{R}) \cong S^2(\mathbb{R}^2)$. The irreducibility of $(\hat{\mathfrak{g}}, \mathfrak{g}^+)$ implies that ω is non-degenerate and $\circ$ is surjective. Moreover, one calculates that (8) is equivalent to the Jacobi identity on W.

Conversely, given the map $\circ$ satisfying (8), we can define a Lie algebra structure on $\hat{\mathfrak{g}} := \mathfrak{g}^+ \oplus W$ by the action of $\mathfrak{g}^+$ on W and (11), so that $(\hat{\mathfrak{g}}, \mathfrak{g}^+)$ becomes an irreducible symmetric pair.

That the second equation implies the rest is shown by a straightforward calculation. ∎

Definition 5.2. *Let $\mathfrak{g} \subset \mathfrak{sp}(V, \omega)$ be a subalgebra for which the two equivalent conditions of Theorem 5.1 are satisfied, and let $G \subset Sp(V, \omega)$ be the corresponding connected Lie subgroup. Then $\mathfrak{g} \subset \mathfrak{sp}(V, \omega)$ is called a* special symplectic subalgebra, *$G \subset Sp(V, \omega)$ is called a* special symplectic subgroup, *and any torsion free connection on a G-structure whose curvature is everywhere of the form R_A from (10) is called a* special symplectic connection.

Thus, for any $c \in \mathbb{R}$ we can now define the Poisson structure (6) on $P = (\mathfrak{g} \oplus V)^*$ for any special symplectic subalgebra $\mathfrak{g} \subset \mathfrak{sp}(V, \omega)$, using the map $\Phi^c : \mathfrak{g}^* \cong \mathfrak{g} \to \Lambda^2 V^*$ from (9), and hence construct Poisson type connections as in the preceding section whose curvature is of the form (10). Thus, we obtain the

Corollary 5.3. *Let $\mathfrak{g} \subset \mathfrak{sp}(V, \omega)$ be a special symplectic Lie subalgebra. Then for every $A_0 \in \mathfrak{g}$, there exists a special symplectic connection whose curvature at some point equals R_{A_0}, and these connections are of Poisson type.*

In order to apply this result, we now need to take a closer look at the irreducible symmetric pairs $(\hat{\mathfrak{g}}, \mathfrak{g}^+)$ of the form considered above.

Let $\hat{\mathfrak{g}}$ be a real simple Lie algebra which contains a maximal root space $\mathfrak{g}_\alpha$. Then $\hat{\mathfrak{g}}$ also contains $\mathfrak{g}_{-\alpha}$ and the semisimple element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}$ with $\alpha(H_\alpha) = 2$. Then $\hat{\mathfrak{g}}$ decomposes into the $ad(H_\alpha)$-eigenspaces as a graded Lie algebra:

$$\hat{\mathfrak{g}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where $\mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}_0 = \mathbb{R}H_\alpha \oplus \mathfrak{g}$ for some reductive Lie algebra $\mathfrak{g}$. Thus, if we let

$$\mathfrak{g}_{ev} := \mathfrak{g}_2 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2 \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{g}, \qquad \mathfrak{g}_{odd} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \cong \mathbb{R}^2 \otimes \mathfrak{g}_1 \text{ as a } \mathfrak{g}_{ev}\text{-module},$$

then $(\hat{\mathfrak{g}}, \mathfrak{g}_{ev})$ is an irreducible symmetric pair of the form considered in Theorem 5.1. Conversely, from the well known classification of irreducible symmetric spaces [B] it follows that any such symmetric pair comes from the decomposition of a simple real Lie algebra which contains a maximal root. Thus, we have the following

Corollary 5.4. *Table 1 yields the complete list of (real or complex) special symplectic subalgebras $\mathfrak{g} \subset \mathfrak{sp}(V, \omega)$.*

In order to understand how general the class of special symplectic connections is, we cite the following result.

Proposition 5.5. [MS] *Let $\mathfrak{g} \subset \mathfrak{sp}(V, \omega)$ be a special symplectic subalgebra corresponding to one of the entries (iv) – $(xviii)$ in Table 1. Then any linear map $R : \Lambda^2 V \to \mathfrak{g}$ which satisfies the first Bianchi identity is of the form R_A as in (10) for some $A \in \mathfrak{g}$. In particular, any torsion free connection on a G-structure with one of these special symplectic subgroups G is special symplectic.*

In fact, by the classification of all irreducible holonomy groups [MS], it follows that the subgroups $G \subset Sp(V, \omega)$ from (iv) – $(xviii)$ in Table 1 exhaust all absolutely irreducible proper subgroups of the symplectic group which can occur as the holonomy of a torsion free connection.

Note that for the entries in (iii), we have $G = Sp(V, \omega)$. In this case, the special symplectic connections, i.e. those connections whose curvature is of the form (10), have been investigated in [BC] where they were called **connections of Ricci type**. Thus, connections of Ricci type are merely a particular class of special symplectic connections.

One might wonder if there are special symplectic connections which do not arise as Poisson type connections. Remarkably, this is not the case. Namely, we have the following result.

Table 1: Special symplectic subalgebras (Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

	Dynkin type of $\hat{\mathfrak{g}}$	$\hat{\mathfrak{g}}$	$\mathfrak{g}$	V
(i)	A_k, $k \geq 2$	$\mathfrak{sl}(n+2, \mathbb{F})$, $n \geq 1$	$\mathfrak{gl}(n, \mathbb{F})$	$U \oplus U^*$ with $U \cong \mathbb{F}^n$
(ii)		$\mathfrak{su}(p+1, q+1)$, $p+q \geq 1$	$\mathfrak{u}(p, q)$	$\mathbb{C}^{p+q}$
(iii)	C_k, $k \geq 2$	$\mathfrak{sp}(n+1, \mathbb{F})$	$\mathfrak{sp}(n, \mathbb{F})$	$\mathbb{F}^{2n}$
(iv)	B_k, D_{k+1}, $k \geq 3$	$\mathfrak{so}(n+4, \mathbb{C})$	$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$, $n \geq 3$
(v)		$\mathfrak{so}(p+2, q+2)$, $p+q \geq 3$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(p, q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q}$
(vi)		$\mathfrak{so}(n+1, \mathbb{H})$, $n \geq 2$	$\mathfrak{sp}(1) \oplus \mathfrak{so}(n, \mathbb{H})$	$\mathbb{H}^n$
(vii)	G_2		$\mathfrak{sl}(2, \mathbb{F})$	$S^3(\mathbb{F}^2)$
(viii)	F_4		$\mathfrak{sp}(3, \mathbb{F})$	$\mathbb{F}^{14} \subset \Lambda^3\mathbb{F}^6$
(ix)	E_6		$\mathfrak{sl}(6, \mathbb{F})$	$\Lambda^3\mathbb{F}^6$
(x)			$\mathfrak{su}(1, 5)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$
(xi)			$\mathfrak{su}(3, 3)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$
(xii)	E_7		$\mathfrak{spin}(12, \mathbb{C})$	$\Delta^{\mathbb{C}} \cong \mathbb{C}^{32}$
(xiii)			$\mathfrak{spin}(6, 6)$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xiv)			$\mathfrak{spin}(2, 10)$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xv)			$\mathfrak{spin}(6, \mathbb{H})$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xvi)	E_8		$E_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
(xvii)			$\mathfrak{e}_7^5$	$\mathbb{R}^{56}$
(xviii)			$\mathfrak{e}_7^7$	$\mathbb{R}^{56}$

Theorem 5.6. [CMS1, CMS2, MS] *Let $G \subset Sp(V, \omega)$ be a special symplectic subgroup. Then every special symplectic connection is locally equivalent to a Poisson type connection.*

In particular, this applies to all torsion free connections whose holonomy is contained in one of the special symplectic subalgebras corresponding to entries (iv) – $(xviii)$ in Table 1, and to all symplectic connections of Ricci type.

From this characterization, we can now deduce the following properties.

Corollary 5.7. *Let M be a manifold which carries a special symplectic connection. Then the following hold.*

(1) The connection is analytic.
(2) There is a map $\pi : F \to P = (\mathfrak{g} \oplus V)^*$ which has constant even rank $2k$ which we shall call the rank of the connection. $k = 0$ iff the connection is flat.
(3) $\pi(F)$ is contained in a $2k$-dimensional symplectic leaf Σ of the Poisson structure on P induced by Φ^c from (6) and (9) for some $c \in \mathbb{R}$. In particular, $\pi : F \to \Sigma$ is a submersion onto its image.
(4) Conversely, every symplectic leaf $\Sigma \subset P$ can be covered by open neighborhoods $\{U_\alpha\}$ such that there is a special symplectic connection with $\pi(F_\alpha) = U_\alpha$.
(5) The moduli space of special symplectic connections associated to any of the special symplectic subgroups is finite dimensional. Indeed, the 2nd derivative of the curvature at a single point in M completely determines the connection on all of M.
(6) Let $\mathfrak{s}$ be the Lie algebra of infinitesimal symmetries, i.e. of vector fields X on F which commute with the fundamental horizontal and vertical vector fields. Then $\dim \mathfrak{s} = \dim P - 2k = \dim \mathfrak{g} + \dim V - 2k$ where k is the rank of the connection. In particular, $\dim \mathfrak{s} \geq rk(\mathfrak{g}) > 0$.

6. GLOBAL SYMPLECTIC CONNECTIONS

By Theorem 5.6, we know that locally all special symplectic connections are constructed in a canonical way. However, it is not clear if there exist *global* special symplectic connections. Let us make this more precise.

Definition 6.1. *Let* (M, ω, ∇) *be a triple consisting of a symplectic manifold* (M, ω) *and a special symplectic connection. We call* ∇ maximal *if* M *is not equivalent to a proper open subset of another manifold with a special symplectic connection, i.e. if* (M', ω', ∇') *is another symplectic manifold with a special symplectic connection and* $\dim M' = \dim M$*, and if there is a connection preserving embedding* $\imath : (M, \omega, \nabla) \hookrightarrow (M', \omega', \nabla')$*, then we must have* $\imath(M) = M'$*.*

The problem in constructing maximal connections is that (4) in Corollary 5.7 is not an optimal statement. One would like to show that there are connections such that $\pi(F) \subset P$ is an *entire symplectic leaf*.

This would be the case if we knew that the Poisson structures on $P = (\mathfrak{g} \oplus V)^*$ had a *complete* symplectic realization. In this case, maximal connections would exist, and the moduli space of maximal connections would also be finite dimensional.

However, it is in general difficult to decide whether or not a given Poisson manifold admits a complete symplectic realization. There are, in principle, invariants which guarantee the existence of complete realizations. But they involve the calculation of the cohomology groups of the symplectic leaves, and this seems to be an impossible task for the Poisson structures in question here.

One can describe the difficulty to go from local to global structures in another way. Namely, if we let

$$\Sigma = \bigcup_{\alpha} U_\alpha$$

be the open covering from (4) in Corollary 5.7 with the associated submersions $\pi_\alpha : F_\alpha \to U_\alpha$, then there is a connection preserving diffeomorphism

$$\varphi_{\alpha\beta} : \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \leftrightarrow \pi_\beta^{-1}(U_\alpha \cap U_\beta)$$

by the local unicity of the symplectic realization. Also, the composition $\psi_{\alpha\beta\gamma} := \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} \circ \varphi_{\gamma\alpha}$ must be a connection preserving map, and – again by the unicity of the symplectic realization – must be an element of the symmetry group S, which is a Lie group whose Lie algebra equals $\mathfrak{s}$. In other words, $\{\psi_{\alpha\beta\gamma}\}$ is a cocycle which represents an element $[\psi] \in H^2(\Sigma, S)$, and there exists a maximal symplectic connection with $\pi(F) = \Sigma$ iff $[\psi] = 0$. Thus, $[\psi] \in H^2(\Sigma, S)$ represents the obstruction to the existence of a maximal connection corresponding to the symplectic leaf $\Sigma \subset P$.

But the calculation of $H^2(\Sigma, S)$ seems to be completely out of reach, especially since S may be non-abelian. Therefore, this approach to decide whether or not maximal special symplectic connections exist does not seem to be very promising.

In summary, one can say that while the picture of local special symplectic connections is quite clear, the understanding of *global* special symplectic connections is still unsatisfactory.

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