Exotic holonomies $E_7^{(a)}$

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Abstract

It is proved that the Lie groups $E_7^{(5)}$ and $E_7^{(7)}$ represented in $\mathbb{R}^{56}$ and the Lie group $E_7^{\mathbb{C}}$ represented in $\mathbb{R}^{112}$ occur as holonomies of torsion-free affine connections. It is also shown that the moduli spaces of torsion-free affine connnections with these holonomies are finite dimensional, and that every such connection has a local symmetry group of positive dimension.

§1 Introduction

The notion of the *Holonomy* of an affine connection was introduced originally by Élie Cartan in the 1920s who used it as an important tool in his attempt to classify all locally symmetric manifolds. Over time, the holonomy group proved to be one of the most informative and useful characteristics of an affine connection and found many applications in both mathematics and physics.

By definition, the holonomy of an affine connection on a connected manifold M is the subgroup of all linear automorphisms of T_pM which are induced by parallel translation along p-based loops.

In 1955, Berger [4] showed that the list of irreducibly acting matrix Lie groups which can, in principle, occur as the holonomy of a torsion-free affine connection is very restricted. Berger presented his classification of all possible candidates for irreducible holonomies in two parts. The first part contains all possible groups preserving a non-degenerate symmetric bilinear form, the second part consists of those groups which do *not* preserve such a form; the latter part was stated to be complete up to a finite number of missing terms and was given without a proof.

Bryant [5] was the first to discover the incompleteness of the second part of Berger's list, and referred to the missing entries as *exotic holonomies*. Since then, several other families of exotic holonomies have been found [6, 7, 8]. In this paper we present one more family of exotic holonomies associated with various real forms of the complex 56-dimensional representation of $E_7^{\mathbb{C}}$.

Main Theorem.

(i) All representations in the following table

Group:	$E_7^{(5)}$	$E_7^{(7)}$	$E_7^{\mathbb{C}}$
Representation space:	$\mathbb{R}^{56}$	$\mathbb{R}^{56}$	$\mathbb{R}^{112}$

occur as holonomies of torsion-free affine connections.

(ii) Any torsion-free affine connection with one of these holonomies is analytic.

(iii) The moduli space of torsion-free affine connections with one of these holonomies is finite dimensional.

(iv) Any such connection has a (local) symmetry group of positive dimension.

This theorem is proved by combining twistor techniques of [10] used to compute all the necessary E_7-modules $K(\mathfrak{e}_7)$, $K^1(\mathfrak{e}_7)$ and $\mathcal{P}^{(1)}(\mathfrak{e}_7)$ with the construction of torsion-free affine connections with prescribed holonomy via deformations of a certain linear Poisson structure [8].

§2 Borel-Weil approach to $E_7^{(a)}$

Let V be a vector space and $\mathfrak{g}$ an irreducible Lie subalgebra of $\mathfrak{gl}(V) \simeq V \otimes V^*$. In the holonomy group context, one is interested in the following three $\mathfrak{g}$-modules:

(i) $\mathfrak{g}^{(1)} := (\mathfrak{g} \otimes V^*) \cap (V \otimes \odot^2 V^*)$,

(ii) the *curvature space* $K(\mathfrak{g}) := \ker i_1$, where i_1 is the composition

$$i_1 : \mathfrak{g} \otimes \Lambda^2 V^* \longrightarrow V \otimes V^* \otimes \Lambda^2 V^* \longrightarrow V \otimes \Lambda^3 V^*,$$

(iii) the *2nd curvature space* $K^1(\mathfrak{g}) := \ker i_2$, where i_2 is the composition

$$i_2 : K(\mathfrak{g}) \otimes V^* \longrightarrow \mathfrak{g} \otimes \Lambda^2 V^* \otimes V^* \longrightarrow V \otimes \Lambda^3 V^*.$$

Note that if ∂ is the composition

$$\partial : \mathfrak{g}^{(1)} \otimes V^* \to \mathfrak{g} \otimes V^* \otimes V^* \to \mathfrak{g} \otimes \Lambda^2 V^*$$

then $\partial(\mathfrak{g}^{(1)} \otimes V^*) \subset K(\mathfrak{g})$.

The geometric meaning of $\mathfrak{g}^{(1)}$ is that if there exists a (local) torsion-free affine connection ∇ on a manifold M with holonomy algebra $\mathfrak{g}$ then, for any (local) function $\Gamma : M \to \mathfrak{g}^{(1)}$, the affine connection $\nabla + \Gamma$ is again torsion-free and has holonomy algebra $\mathfrak{g}$; thus, in some sense, $\mathfrak{g}^{(1)}$ measures the non-uniqueness of torsion-free affine connections with holonomy $\mathfrak{g}$ on a fixed manifold.

The significance of $K(\mathfrak{g})$ and $K^1(\mathfrak{g})$ is that the curvature tensor (the covariant derivative of the curvature tensor respectively) of a torsion-free affine connection ∇ with holonomy $\mathfrak{g}$ at a point $p \in M$ is represented by an element of $K(\mathfrak{g})$ ($K^1(\mathfrak{g})$ respectively).

Therefore, $\mathfrak{g}$ can be a candidate to the holonomy algebra of a torsion-free affine connection only if $K(\mathfrak{g}) \neq 0$. The question then remains how to compute $K(\mathfrak{g})$.

With any real irreducible representation of a real reductive Lie algebra one may associate an irreducible complex representation of a complex reductive Lie algebra. Since all the above $\mathfrak{g}$-modules behave reasonably well under this association, we may assume from now on that V is a finite dimensional complex vector space and $\mathfrak{g} \subset \mathfrak{g}l(V)$ is an irreducible representation of a complex reductive Lie algebra. Clearly, $G = \exp(\mathfrak{g})$ acts irreducibly in V^* via the dual representation. Let $\tilde{X}$ be the G-orbit of a highest weight vector in $V^* \setminus 0$. Then the quotient $X := \tilde{X}/\mathbb{C}^*$ is a compact complex homogeneous-rational manifold canonically embedded into $\mathbb{P}(V^*)$, and there is a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & V^* \setminus 0 \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{P}(V^*) \end{array}$$

In fact, $X = G_s/P$, where G_s is the semisimple part of G and P is the parabolic subgroup of G_s leaving a highest weight vector in V^* invariant up to a scalar. Let L be the restriction of the hyperplane section bundle $\mathcal{O}(1)$ on $\mathbb{P}(V^*)$ to the submanifold X. Clearly, L is an ample homogeneous line bundle on X. We call (X, L) the *Borel-Weil data* associated with $(\mathfrak{g}, V)$.

According to Borel-Weil, the representation space V can be easily reconstructed from (X, L) as $V = \mathrm{H}^0(X, L)$. What about $\mathfrak{g}$? The Lie algebra of the Lie group of all global biholomorphisms of the line bundle L which commute with the projection $L \to X$ is isomorphic to $H^0(X, L \otimes (J^1 L)^*)$ — a central extension of the Lie algebra $\mathrm{H}^0(X, TX)$. Whence, as a complex Lie algebra, $\mathrm{H}^0(X, L \otimes (J^1 L)^*)$ has a natural complex irreducible representation in $\mathrm{H}^0(X, L) = V$; with very few (and well studied in the holonomy context) exceptions [2], this representation is, up to a central extension, isomorphic to the original $\mathfrak{g}$.

Remarkably enough, the basic $\mathfrak{g}$-modules defined above fit nicely into the Borel-Weil paradigm as well.

Proposition 1 [10] *For a compact complex homogeneous-rational manifold X and an ample line bundle $L \to X$, there is an isomorphism*

$$\mathfrak{g}^{(1)} = \mathrm{H}^0\left(X, L \otimes \odot^2 N^*\right),$$

and an exact sequence of $\mathfrak{g}$-modules,

$$0 \longrightarrow \frac{K(\mathfrak{g})}{\partial(\mathfrak{g}^{(1)} \otimes V^*)} \longrightarrow \mathrm{H}^1\left(X, L \otimes \odot^3 N^*\right) \longrightarrow \mathrm{H}^1\left(X, L \otimes \odot^2 N^*\right) \otimes V^*,$$

where $\mathfrak{g}$ is $\mathrm{H}^0(X, L \otimes N^*)$ represented in $V = \mathrm{H}^0(X, L)$.

Proof. The result follows easily from the exact sequences

$$0 \longrightarrow L \otimes \odot^2 N^* \longrightarrow L \otimes N^* \otimes V^* \longrightarrow L \otimes \Lambda^2 V^*$$

and

$$0 \longrightarrow L \otimes \odot^3 N^* \longrightarrow L \otimes \odot^2 N^* \otimes V^* \longrightarrow L \otimes N^* \otimes \Lambda^2 V^* \longrightarrow L \otimes \Lambda^3 V^*,$$

where arrows are a combination of a natural monomorphism $N^* \longrightarrow V^* \otimes \mathcal{O}_X$ (which holds due to ampleness of L) with the antisymmetrization. □

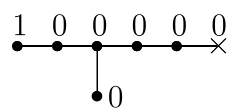
It is well known that the complex exceptional Lie algebra $\mathfrak{e}_7^{\mathbb{C}}$ has four real forms $\mathfrak{e}_7^{(4)}$, $\mathfrak{e}_7^{(5)}$, $\mathfrak{e}_7^{(6)}$ and $\mathfrak{e}_7^{(7)}$ with signatures 0, 54, 64 and 70 respectively (see, e.g., [9, 12]). Two of these, $\mathfrak{e}_7^{(5)}$ and $\mathfrak{e}_7^{(6)}$, can be irreducibly represented in $\mathbb{R}^{56}$. Let ρ denote the irreducible real representation $\mathfrak{e}_7^{(a)} \to \mathfrak{gl}(V)$, where V is $\mathbb{R}^{56}$ for $a = 5, 7$ and $V = \mathbb{C}^{56} \simeq \mathbb{R}^{112}$ for $(a) = \mathbb{C}$. Let $\mathrm{Ad} : \mathfrak{e}_7^{(a)} \to \mathfrak{gl}(\mathfrak{e}_7^{(a)})$ denote the adjoint representation.

Theorem 1

$$K(\rho(\mathfrak{e}_7^{(a)})) \simeq \mathrm{Ad}(\mathfrak{e}_7^{(a)}), \qquad K^1(\rho(\mathfrak{e}_7^{(a)})) \simeq V^*.$$

Proof. We shall prove this statement for the complex representation only. That it is true for real representations as well will follow from the invariance of all the constructions under the associated real structures in $e_7^{\mathbb{C}}$.

Let (X, L) be the Borel-Weil data associated to $\rho : \mathfrak{e}_7^{\mathbb{C}} \to \mathfrak{gl}(V)$, $V \simeq \mathbb{C}^{56}$. Then $X = \mathrm{E}_7^{\mathbb{C}}/P$ is a 27-dimensional compact complex homogeneous-rational manifold whose tangent bundle has, as an irreducible homogeneous vector bundle, the Dynkin diagram representation [3]

$TX =$ 

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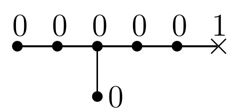
while $L \to X$ is given by

$L =$ 

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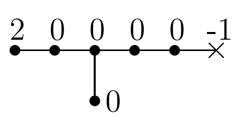
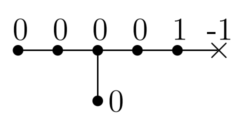
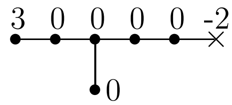
Here and below the weights of irreducible homogeneous vector bundles are given in the basis of fundamental weights.

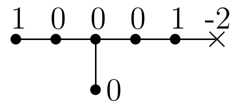
Using Konstant's formula and Table 5 in the reference chapter of [12] to find irreducible decompositions of tensor powers of the simplest 27-dimensional irreducible representation of $E_6^{\mathbb{C}}$ (which, in our case, is isomorphic to the semisimple part of the parabolic P), one obtains

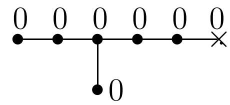
$\odot^2 TX \otimes L^* =$ 

$+$ 

and

$$\odot^3 TX \otimes L^{*2} =$$ 

$+$ 

$+$ 

Then, using the long exact sequences of the extensions

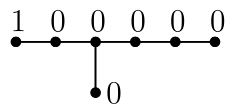
$$0 \longrightarrow N^* \longrightarrow L \otimes \odot^2 N^* \longrightarrow \odot^2 \, TX \otimes L^* \longrightarrow 0,$$

$$0 \longrightarrow \odot^2 \, N^* \longrightarrow L \otimes \odot^3 N^* \longrightarrow \odot^3 \, TX \otimes L^{2*} \longrightarrow 0,$$

the Bott-Borel-Weil theorem and Proposition 1, one easily finds

$$\mathrm{H}^0(X, L \otimes \odot^2 N^*) = \mathrm{H}^1(X, L \otimes \odot^2 N^*) = 0$$

and

$$K(\rho(\mathfrak{e}_7^{\mathbb{C}} + \mathbb{C})) \simeq \mathrm{H}^1(X, L \otimes \odot^3 N^*) =$$ 

$\simeq \mathrm{Ad}(\mathfrak{e}_7^{\mathbb{C}}).$

Let us find next the explicit form of $K(\rho(\mathfrak{e}_7^{\mathbb{C}} + \mathbb{C}))$ as a subset of all elements in $\rho(\mathfrak{e}_7^{\mathbb{C}} + \mathbb{C})) \otimes \Lambda^2 V^*$ satisfying the first Bianchi identities.

Recall [1] that $\rho : \mathfrak{e}_7^{\mathbb{C}} \to \mathfrak{gl}(V)$ enjoys a non-zero invariant skew symmetric invariant product

$$\begin{array}{rcl} \Lambda^2 V & \longrightarrow & \mathbb{C} \\ u \otimes v & \longrightarrow & < u, v >, \end{array}$$

and a non-zero invariant symmetric map

$$\begin{array}{rcl} \odot^2 V & \longrightarrow & e_7^{\mathbb{C}} \\ u \otimes v & \longrightarrow & u \circ v, \end{array}$$

which are unique up to non-zero scalar factor and satisfy

$$\begin{array}{rcl} < \rho(A)u, v > & = & \lambda B(A, u \circ v) \\ B(u \circ v, s \circ t) - B(u \circ t, s \circ v) & = & \mu \, (2 <u,s><v,t> - <u,t><v,s> - <u,v><s,t>) \end{array}$$

for all $A \in e_7^{\mathbb{C}}$ and $u, v, s, t \in V$. Here λ and μ are fixed non-zero constants and $B(\ , \ )$ is the Killing form.

Then it is not hard to check that, for any fixed $A \in e_7^{\mathbb{C}}$, the following map

$$\begin{array}{rcl} \Lambda^2 V & \longrightarrow & \rho(e_7^{\mathbb{C}} + \mathbb{C}) \\ u \otimes v & \longrightarrow & 2\lambda\mu <u,v> \rho(A) + \rho(u \circ \rho(A)v) - \rho(v \circ \rho(A)u) \end{array} \quad (1)$$

defines an element of $\rho(e_7^{\mathbb{C}} + \mathbb{C}) \otimes \Lambda^2 V^*$ which lies in the kernel of the composition

$$\rho(e_7^{\mathbb{C}} + \mathbb{C}) \otimes \Lambda^2 V^* \longrightarrow V \otimes V^* \otimes \Lambda^2 V^* \longrightarrow V \otimes \Lambda^3 V^*.$$

Thus, the above formula gives an explicit realization of the isomorphism $K(\rho(e_7^{\mathbb{C}} + \mathbb{C})) = \mathrm{Ad}(e_7^{\mathbb{C}})$. In particular, it shows that $K(\rho(e_7^{\mathbb{C}} + \mathbb{C})) = K(\rho(e_7^{\mathbb{C}}))$.

Having obtained an explicit structure of $K(\rho(e_7^{\mathbb{C}}))$, it is straightforward to show that a generic element of $K^1(\rho(e_7^{\mathbb{C}})) \subset \rho(e_7^{\mathbb{C}}) \otimes V^* \otimes \Lambda^2 V^*$ is of the form

$$\begin{array}{ccc} V \otimes \Lambda^2 V & \longrightarrow & \rho(e_7^{\mathbb{C}}) \\ s \otimes u \otimes v & \longrightarrow & 2\lambda\mu < u, v > \rho(s \circ w) + \rho(u \circ \rho(s \circ w) v) - \rho(v \circ \rho(s \circ w) u) \end{array}$$

for some fixed $w \in V \simeq V^*$. This establishes the isomorphism $K^1(\rho(e_7^{\mathbb{C}})) = V^*$. □

§3 A construction of torsion-free connections

We briefly describe here the construction of torsion-free connections with prescribed holonomy which was presented in [8].

Let $\mathfrak{g} \subset \mathfrak{g}l(V)$ a Lie sub-algebra where V is a finite-dimensional vector space.

A G-equivariant C^∞-map $\phi : \mathfrak{g}^* \to \Lambda^2 V^*$ is called *admissible* if for every $p \in \mathfrak{g}^*$, the map $d\phi_p^* : \Lambda^2 V \longrightarrow \mathrm{T}_p^* \mathfrak{g}^* \simeq \mathfrak{g}$ lies in $K(\mathfrak{g})$.

For a given admissible map ϕ, one may define the following Poisson structure on the dual W^* of the semi-direct Lie algebra $W = \mathfrak{g} \oplus V$:

$$\{f, g\}_{p+\nu} = p([A, B]) + \nu(A \cdot y - B \cdot x) + \phi(p)(x, y), \tag{2}$$

where $df = A + x$ and $dg = B + y$ are the decompositions of $df, dg \in T^*W^* \simeq \mathfrak{g} \oplus V$, and $p \in \mathfrak{g}^*, \nu \in V^*$. This Poisson structure may be regarded as a deformation of the natural linear Poisson structure on W^*.

Let $\pi : S \to U$ be a *symplectic realization* of an open subset $U \subset W^*$, i.e. π is a submersion from a symplectic manifold S with symplectic 2-form Ω such that

$$\{\pi^*(f), \pi^*(g)\}_S = \pi^*(\{f, g\}) \quad \text{for all } f, g \in C^\infty(P, \mathbb{R}),$$

where $\{\ ,\ \}_S$ is the Poisson structure on S induced by the symplectic structure. At those points where the rank of the Poisson structure is maximal, such a symplectic realization exists at least locally.

Regarding each element $w \in W \simeq T^*W^*$ as a 1-form on W, we define the distribution

$$\mathcal{D} = \{\xi_w := \#(\pi^*(w)) \mid w \in W\} \subset TS$$

on S, where $\#$ is the index-raising map induced by Ω. Since Ω is non-degenerate, rank $\mathcal{D} = \dim W$. Moreover, for the bracket relations one calculates

$$[\xi_A, \xi_B] = \xi_{[A,B]}, \quad [\xi_A, \xi_x] = \xi_{A \cdot x}, \quad \text{and} \quad [\xi_x, \xi_y](s) = \xi_{d\phi_p^*(x,y)}, \tag{3}$$

where $A, B \in \mathfrak{g}$, $x, y \in V$ and $p = \pi(s)$.

Let $F \subset S$ be an integral leaf of $\mathcal{D}$. By the very definition of $\mathcal{D}$, F comes equipped with a W-valued coframe $\theta + \omega$, where θ and ω take values in V and $\mathfrak{g}$ respectively, defined by the equation

$$(\omega + \theta)(\xi_w) = w.$$

Note that by the first equation in (3), the vector fields ξ_A, $A \in \mathfrak{g}$, induce a free local group action of G on F, where $G \subset Gl(V)$ is the connected Lie subgroup corresponding to $\mathfrak{g} \subset \mathfrak{gl}(V)$. After shrinking F as necessary, we may assume that $M := F/G$ is a manifold. Standard arguments then imply that there is a unique embedding of $\imath : F \hookrightarrow \mathfrak{F}_V$, where $\mathfrak{F}_V$ denotes the V-valued coframe bundle of M and a torsion-free connection on M such that $\imath^*(\underline{\theta} + \underline{\omega}) = \theta + \omega$, where $\underline{\theta}$ and $\underline{\omega}$ denote the tautological and the connection 1-form on $\mathfrak{F}_V$, respectively. Clearly, the holonomy of this connection is contained in G; in fact, by the *Ambrose-Singer-Holonomy Theorem*, the holonomy algebra is generated by $\{d\phi_p^*(x, y) \mid x, y \in V, p \in \pi(F)\}$. A connection which comes from this construction is called a *Poisson connection.* This leads to the following

Theorem 2 [8] *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie sub-algebra where V is a finite-dimensional vector space, and let*

$$K_0(\mathfrak{g}) = \{R \in K(\mathfrak{g}) \mid \operatorname{span}\{R(x, y),\ \text{all } x, y \in V\} = \mathfrak{g}\}.$$

If $\phi : \mathfrak{g}^* \to \Lambda^2 V^*$ is admissible, and if the open set $U_0 \subset \mathfrak{g}^*$ given by

$$U_0 := (d\phi^*)^{-1}(K_0(\mathfrak{g}))$$

is non-empty, then there exist Poisson connections induced by ϕ whose holonomy representations are equivalent to $\mathfrak{g}$. Moreover, if $\phi|_{U_0}$ is not affine, then not all of these connections are locally symmetric.

It is not clear at present how general the class of Poisson connections is, nor how many irreducible Lie subalgebras $\mathfrak{g} \subset \mathfrak{gl}(V)$ admit admissible maps $\phi : \mathfrak{g}^* \to \Lambda^2 V^*$ which are not affine. However, there is a class of Lie subalgebras for which the above construction exhausts *all possible torsion-free connections* with this holonomy. Namely, we define the $\mathfrak{g}$-module

$$\mathcal{P}^{(1)}(\mathfrak{g}) = (\odot^2 \mathfrak{g} \otimes \Lambda^2 V^*) \cap (\mathfrak{g} \otimes K(\mathfrak{g})) \subset V \otimes K^1(\mathfrak{g})$$

and regard elements $\phi_2 \in \mathcal{P}^{(1)}(\mathfrak{g})$ as polynomial maps $\mathfrak{g}^* \to \Lambda^2 V^*$ of degree 2. It is then obvious that each G-invariant $\phi_2 \in \mathcal{P}^{(1)}(\mathfrak{g})$ is admissible, and we have the following result.

Theorem 3 [8] *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be an irreducibly acting subalgebra, and suppose that there is an invariant element $\phi_2 \in \mathcal{P}^{(1)}(\mathfrak{g})$ such that the associated G-equivariant linear maps*

$$\begin{array}{rccc} \phi_2' : & \mathfrak{g}^* & \longrightarrow & K(\mathfrak{g}) \\ \phi_2'' : & V^* & \longrightarrow & K^1(\mathfrak{g}). \end{array} \qquad (4)$$

are isomorphisms. Then every torsion-free affine connection whose holonomy algebra is contained in $\mathfrak{g}$ is a Poisson connection induced by an admissible map

$$\phi = \phi_2 + \tau,$$

where $\tau \in \Lambda^2 V^*$ is a (possibly vanishing) $\mathfrak{g}$-invariant 2-form. In particluar, the moduli space of such connections is finite dimensional, and each such connection is analytic. Also, the dimension of the symmetry group of this connection equals $\dim W^* - 2k$ where k is the half-rank of the Poisson structure on W^* induced by ϕ in (2).

At first sight, the premise that the maps (4) be isomorphisms looks like an unreasonably strong condition in order to utilize this Theorem. Nevertheless, this premise *does* hold for the exotic holonomies $\mathrm{SO}(p,q)\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}(n,\mathbb{C})\mathrm{SL}(2,\mathbb{C})$ which were discovered in [8]. Also, we will show in §4 that it also holds for the representations $E_7^{(a)}$ from the main theorem.

For the proof, we shall need the following version of Schur's Lemma:

Lemma 4 *Let $\mathfrak{g}$ be a reductive Lie algebra, and suppose that $\mathfrak{g}$ acts irreducibly on the finite dimensional vector spaces V and W. If $\rho : V \to W$ is a linear map satisfying*

$$A\ \rho\ B = B\ \rho\ A \quad \text{for all } A, B \in \mathfrak{g}, \tag{5}$$

then $\rho = 0$.

Proof of Theorem 3 Let $F \subset \mathfrak{F}_V$ be a G-structure on the manifold M where $\mathfrak{F}_V \to M$ is the V-valued coframe bundle of M, and denote the tautological V-valued 1-form on F by θ. Suppose that F is equipped with a torsion-free connection, i.e. a $\mathfrak{g}$-valued 1-form ω on F. Since ϕ_2' is an isomorphism, the *first and second structure equations* read

$$\begin{aligned} d\theta &= -\omega \wedge \theta \\ d\omega &= -\omega \wedge \omega - 2(\phi_2'(\mathbf{a})) \circ (\theta \wedge \theta), \end{aligned} \tag{6}$$

where $\mathbf{a} : F \to \mathfrak{g}^*$ is a G-equivariant map. Differentiating (6) and using that ϕ_2'' is an isomorphism yields the *third structure equation* for the differential of $\mathbf{a}$:

$$d\mathbf{a} = -\omega \cdot \mathbf{a} + \jmath(\mathbf{b} \otimes \theta), \tag{7}$$

for some G-equivariant map $\mathbf{b} : F \to V^*$, where $\jmath : V^* \otimes V \to \mathfrak{g}^*$ is the natural projection. The multiplication in the first term refers to the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

Let us define the map $\mathbf{c} : F \to V^* \otimes V^*$ by

$$\mathbf{c}_p(x, y) := d\mathbf{b}(\xi_x)(y) - \phi_2(\mathbf{a}_p, \mathbf{a}_p, x, y). \tag{8}$$

Differentiation of (7) yields

$$\mathbf{c}_p(x, Ay) = \mathbf{c}_p(y, Ax) \quad \text{for all } x, y \in V \text{ and all } A \in \mathfrak{g}. \tag{9}$$

If we let $\rho : V \to V^*, x \mapsto \mathbf{c}_p(x, \_) + \mathbf{c}_p(\_, x)$, then (9) and Lemma 4 imply that $\rho = 0$, i.e. $\mathbf{c}_p$ must be skew-symmetric and G-invariant. This and differentiation of (8) implies that

$$d\mathbf{c} = 0,$$

i.e. $\mathbf{c}_p \equiv \tau \in \Lambda^2 V^*$ is *constant*. Thus, the G-equivariance of $\mathbf{b}$ and (8) yield

$$d\mathbf{b} = -\omega \cdot \mathbf{b} + \left(\mathbf{a}_p^2 \lrcorner \phi_2 + \tau\right) \circ \theta, \tag{10}$$

where $\lrcorner$ refers to the contraction of $\mathbf{a}_p^2 \in \odot^2(\mathfrak{g}^*)$ with $\phi_2 \in \odot^2(\mathfrak{g}) \otimes \Lambda^2 V^*$.

Let us now define the Poisson structure on $W^* = \mathfrak{g}^* \oplus V^*$ induced by $\phi := \phi_2 + \tau$, and let $\pi := \mathbf{a} + \mathbf{b} : F \to W^*$. From (7) and (10), one can now show that, at least locally, the connection is indeed a Poisson connection induced by ϕ.

Let $\mathfrak{s} \subset \mathfrak{X}(F)$ be the Lie algebra of infinitesimal symmetries. Let $f : W^* \supset U \to \mathbb{F}$ be a local function which is constant on the symplectic leaves. Then it is easy to see that $\#\pi^*(df)$ is an infinitesimal symmetry. It follows that $\dim \mathfrak{s} \geq \dim W^* - 2k$. On the other hand, if $X \in \mathfrak{s}$ then $\pi_*(X) = 0$, hence $\dim \mathfrak{s} \leq \dim W^* - 2k$.

The statements about analyticity and the moduli space are now immediate. ∎

Proof of Lemma 4 Throughout the proof, we make the simplifying assumption that $\operatorname{rank} \mathfrak{g} > 1$, as the case $\operatorname{rank} \mathfrak{g} = 1$ is straightforward. Let $P \subset V^* \otimes W$ be the subspace of all maps $\rho : V \to W$ satisfying (5). It is easy to verify that P is $\mathfrak{g}$-invariant. We complexify $\mathfrak{g}$, V and W and pick Cartan and weight space decompositions

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha, \quad V = \bigoplus_\mu V_\mu \quad \text{and} \quad W = \bigoplus_\mu W_\mu.$$

Let $\rho \in P$, and let $x_\mu \in V_\mu$ with $\mu \neq 0$. Then choosing $A, B \in \mathfrak{t}$, with $\mu(A) = 0,\ \mu(B) \neq 0$, (5) implies that $A\rho x_\mu = 0$, and therefore,

$$\rho x_\mu \in \sum_k W_{k\mu}, \tag{11}$$

where the sum is taken over all weights of W which are scalar multiples of μ. Now let $\rho_\lambda \in P$ be an element of weight $\lambda \neq 0$. Then $\rho_\lambda x_\mu \in W_{\lambda+\mu}$, and thus from (11) we conclude:

$$\rho_\lambda x_\mu = 0 \text{ whenever } \lambda, \mu \text{ are linearly independent.} \tag{12}$$

Let $x_{k\lambda} \in V_{k\lambda}$ be a weight vector with $k \neq 0$, and let $A_\alpha \in \mathfrak{g}_\alpha$ where α is a root independent of λ. Then, using (12) twice, we get

$$\begin{aligned} 0 &= (A_\alpha \cdot \rho_\lambda) x_{k\lambda} \\ &= A_\alpha(\rho_\lambda x_{k\lambda}) - \rho_\lambda(A_\alpha x_{k\lambda}) \\ &= A_\alpha(\rho_\lambda x_{k\lambda}) - 0. \end{aligned}$$

Next, note that V_0 is spanned by elements of the form $A_\alpha x_{-\alpha}$ with $A_\alpha \in \mathfrak{g}_\alpha$ and $x_{-\alpha} \in V_{-\alpha}$. If α, λ are independent, pick $A_0 \in \mathfrak{t}$ with $\alpha(A_0) = 0$ and $\lambda(A_0) \neq 0$. Then (5) implies that $\rho_\lambda(A_\alpha x_{-\alpha}) = 0$. Finally, if α, λ are dependent, then for $\beta \neq \pm\alpha$, we get from (5) and (12) that $A_\beta(\rho_\lambda(A_\alpha x_{-\alpha})) = 0$.

In either case, we get that for any μ, $A_\alpha(\rho_\lambda x_\mu) = 0$ whenever λ, α are independent, and hence

$$\mathfrak{g} \cdot (\rho_\lambda V) \subset \sum_k W_{k\lambda}.$$

Since there must be weights in W independent of λ, and since W is irreducible, we conclude that $\rho_\lambda = 0$, contradicting $\lambda \neq 0$.

Thus, P has no weights $\neq 0$, i.e. P is acted on trivially by $\mathfrak{g}$, and from there it is easy to conclude that $P = 0$. □

§4 Proof of the main theorem

Let $\mathfrak{g} \subset \mathfrak{g}l(V)$ be one of the representations in the Main Theorem. Evidently, the Main Theorem will follow from Theorems 2 and 3 if we can find an element $\phi_2 \in \mathcal{P}^{(1)}(\mathfrak{g})$ such that $K_0(\mathfrak{g})$ is dense in $K(\mathfrak{g})$, and the corresponding maps in (4) are isomorphisms. In particular, (iv) of the main Theorem follows since, in each case, $\dim W^* = \dim V + \dim \mathfrak{g} = 56 + 133$ is odd.

The density of $K_0(\mathfrak{g})$ in $K(\mathfrak{g})$ follows immediately from (1).

To compute $\mathcal{P}^{(1)}(\mathfrak{g})$, first note that the $\mathfrak{g}$-module $K(\mathfrak{g}) \otimes \mathfrak{g} \simeq \mathfrak{g} \otimes \mathfrak{g}$ has only one 1-dimensional $\mathfrak{g}$-submodule. So, if there is an invariant element ϕ_2 in $\mathcal{P}^{(1)}(\mathfrak{g})$, it is unique up to a non-zero scalar factor. Since $\phi_2' : \mathfrak{g}^* \longrightarrow K(\mathfrak{g})$ must be an isomorphism, the formula (1) leaves no choice but the following element of $\mathfrak{g} \otimes K(\mathfrak{g})$ as a candidate for ϕ_2:

$$\phi_2(C, D, u, v) = 2\lambda\mu < u, v > B(C, D) + B(u \circ \rho(C)v, D) - B(v \circ \rho(C)u, D),$$

where $C, D \in \mathfrak{g}$, $u, v \in V$ and where we identify $\mathfrak{g} = \mathfrak{g}^*$ via the Killing form. Clearly, this element is $\mathfrak{g}$-invariant. Since $B(u \circ \rho(C)v, D) = \lambda < \rho(D)u, \rho(C)v >$, we have

$$\begin{aligned} \phi_2(C, D, u, v) &= 2\lambda\mu < u, v > B(C, D) + \lambda < \rho(D)u, \rho(C)v > - \lambda < \rho(D)v, \rho(C)u > \\ &= 2\lambda\mu < u, v > B(C, D) - \lambda < \rho(C)v, \rho(D)u > - \lambda < \rho(D)v, \rho(C)u > \end{aligned}$$

which makes it evident that $\phi_2 \in (\odot^2\mathfrak{g} \otimes \Lambda^2 V^*) \cap (\mathfrak{g} \otimes K(\mathfrak{g}))$. That $\phi_2' : \mathfrak{g}^* \longrightarrow K(\mathfrak{g})$ is an isomorphism follows from the very definition of ϕ_2. Since $\phi_2'' : V^* \longrightarrow K^1(\mathfrak{g})$ is evidently non-zero, then, by Theorem 1 and the $\mathfrak{g}$-invariance of ϕ_2, ϕ_2'' must be an isomorphism as well. □

Acknowledgement. One of the authors (SM) would like to thank Andrew Swann for valuable discussions and for drawing his attention to the very informative lecture notes [1].

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