The classification of holonomies of torsion free connections

Lorenz Schwachhöfer

Let M be a smooth connected manifold and let ∇ be a torsion free connection on TM. Such a connection provides a way of parallel translating tangent vectors along piecewise smooth paths; indeed, if γ is such a path from p to q then the parallel translation along γ induces a linear isomorphism $P_\gamma : T_pM \to T_qM$. For any point $p \in M$ we define the *holonomy group of* ∇ *at* p by

$$\mathcal{H}_p := \{P_\gamma \mid \gamma \text{ a } p\text{-based loop in } M\} \subset \mathrm{Aut}(T_pM).$$

It is well-known that the identity component of $\mathcal{H}_p$ is a closed Lie subgroup of $\mathrm{Aut}(T_pM)$ and that moreover, $\mathcal{H}_p \subset \mathrm{Aut}(T_pM)$ and $\mathcal{H}_q \subset \mathrm{Aut}(T_qM)$ are isomorphic subgroups where the isomorphism is induced by conjugation with some P_γ where γ is any path from p to q. Thus, if we fix a linear isomorphism $\imath : V \to T_pM$ where V is a vector space of the appropriate dimension, then the subgroup $\mathcal{H} := \imath(\mathcal{H}_p) \subset \mathrm{Aut}(V)$ is well-defined up to conjugation, independent of the choice of $\imath$ or p. $\mathcal{H}$ is called the *holonomy* of ∇.

Next, if $\pi : (\tilde{M}, \tilde{\nabla}) \to (M, \nabla)$ is the universal cover of M, then $\mathcal{H}_{\tilde{p}}^{\tilde{\nabla}} \cong (\mathcal{H}_p^{\nabla})_0$ and thus, after passing to the universal cover if necessary, we may assume that $\mathcal{H}$ is connected and hence a closed Lie subgroup. In this talk, we shall only be concerned with this case.

The holonomy problem which was posed by Cartan and Lichnerowicz, is then the following question:

Which irreducible connected closed Lie subgroups $H \subset \mathrm{Aut}(V)$ can occur as the holonomy group of a torsion free affine connection?

The notion of the holonomy group was introduced in the 1926 by Élie Cartan who used it to classify all Riemannian locally symmetric spaces. In fact, each symmetric space can be represented as $M = G/H$, and the isotropy group H coincides with the holonomy group $\mathcal{H}$. Thus, as a sub-problem to the holonomy problem, we get the classification of symmetric spaces. This task has been achieved by Cartan in 1926 for Riemannian symmetric spaces, and by Berger in 1957 for general symmetric spaces.

The next step was the *Ambrose-Singer Holonomy Theorem* which describes the Lie algebra $\mathfrak{hol}_p$ of $\mathcal{H}_p$ in terms of the curvature endomorphisms:

$$\mathfrak{hol}_p = \langle\{(P_\gamma R)(x_q, y_q) \mid x_q, y_q \in T_qM,\ \gamma \text{ a path from } q \text{ to } p.\}\rangle \subset \mathrm{End}(V).$$

This motivated Berger in 1955 to pose the following criterion. If $\mathfrak{h} \subset \mathrm{End}(V)$ is an irreducible Lie subalgebra then we define the space of formal curvatures

$$K(\mathfrak{h}) := \{R : \Lambda^2 V \to \mathfrak{h} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0\}$$

and

$$\underline{\mathfrak{h}} := \{R(x,y) \mid x, y \in V, R \in K(\mathfrak{h})\}.$$

Thus, by the Ambrose-Singer Theorem, $\mathfrak{h}$ can occur as the Lie algebra of a holonomy group only if $\underline{\mathfrak{h}} = \mathfrak{h}$. A Lie subgroup whose algebra satisfies this criterion is called a *Berger group*. This splits the holonomy problem into two parts.

Problem A Classify all irreducible Berger groups.

Problem B For the irreducible Berger groups, decide if they can occur as holonomies.

Berger himself gave a complete solution to problem A in the case of metric representations, i.e. for those subgroups $H \subset O(V, < , >)$ where $< , >$ is a non-degenerate symmetric bilinear form on V. These correspond to the holonomies of (pseudo-)Riemannian connections. He also gave a partial classification of the remaining Berger groups. For all these entries, problem B was solved affirmatively in the following decades (until 1986). This is due to the efforts of many mathematicians, e.g. Calabi, Alekseevski, Bryant etc.

In the early 1990s, Bryant found several new Berger groups and was also able to solve problem B for these new holonomies. Other examples of Berger groups were found in joint work with Q.-S. Chi and S. Merkulov, and problem B was solved for these problems as well.

Finally, in joint work with S. Merkulov, we classified all possible Berger groups, and found yet some other new entries. This classification was obtained by direct methods; however, it was pointed out by W. Ziller that the list of possible holonomies is related to the classically known list of symmetric spaces. More precisely, the classification result of *complex* Berger groups may be stated as follows.

Classification Theorem Let $H_{\mathbb{C}} \subset \mathrm{Aut}(V_{\mathbb{C}})$ be a semi-simple irreducible complex Lie subgroup, and let $K \subset H_{\mathbb{C}}$ be the maximal compact subgroup.

1. If there is an irreducible real symmetric space of the form G/K, then $H_{\mathbb{C}}$ is a Berger group.
2. If there is a irreducible hermitian symmetric space of the form $G/(U(1) \cdot K)$, then both $H_{\mathbb{C}}$ and $\mathbb{C}^* \cdot H_{\mathbb{C}}$ are Berger groups.
3. If there is a irreducible quaternionic symmetric space of the form $G/(Sp(1) \cdot K)$, then $H_{\mathbb{C}}$ is a Berger group.
4. The items 1. – 3. yield a complete list of Berger groups, except for the following:
   (a) $G_2^{\mathbb{C}} \subset \mathrm{Aut}(\mathbb{C}^7)$,
   (b) $Spin(7, \mathbb{C}) \subset \mathrm{Aut}(\mathbb{C}^8)$,
   (c) $\mathbb{C}^* Sp(2, \mathbb{C}) \subset \mathrm{Aut}(\mathbb{C}^4)$.

Since real Berger subgroups remain Berger subgroups after complexification, it is not hard to obtain a complete list of real Berger subgroups from this list as well. Thus, this completely answers problem A.

There are several holonomy groups in this classification that had been previously unknown. These are precisely those which correspond to the quaternionic symmetric spaces. Thus, for these entries, we still need to solve problem B. This is done by the following method which had been established in joint work with Q.-S. Chi and S. Merkulov.

Let V be a finite dimensional vector space, $\mathfrak{h} \subset \mathrm{End}(V)$ a Lie sub-algebra, and let $H \subset \mathrm{Aut}(V)$ be the corresponding connected Lie group.

Let $W := \mathfrak{h} \oplus V$. Denote elements of $\mathfrak{h}$ and V by $A, B, \dots$ and $x, y, \dots$, respectively, and elements of W by $w, w', \dots$. We may regard W as the semi-direct product of Lie algebras, i.e. we define a Lie algebra structure on W by the equation

$$[A + x, B + y] := [A, B] + A \cdot y - B \cdot x.$$

It is well-known that this induces a natural Poisson structure on the dual space W^* (the so-called *Kirillov bracket*) which we denote by $\{\ ,\ \}_K$. Now, we wish to perturb this Poisson structure as follows. Regarding elements $A + x, B + y \in W$ as linear functions on W^*, we define

$$\{A + x, B + y\}(p) := \{A + x, B + y\}_K(p) + \Phi(p)(x, y). \tag{1}$$

Here, $\Phi := \phi \circ pr$, where $pr : W^* \to \mathfrak{h}^*$ is the natural projection, and where $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ is a smooth map satisfying

1. ϕ is H-equivariant,
2. for every $p \in \mathfrak{h}^*$, the dual map $(d\phi_p)^* : \Lambda^2 V \to \mathfrak{h}$ is contained in $K(\mathfrak{h})$.

It is easy to check that these conditions on ϕ are equivalent to saying that the bracket in (1) is indeed Poisson.

Let $\pi : S \to U$ be a *symplectic realization* of an open subset $U \subset W^*$, i.e. S is a symplectic mainfold, π is a submersion which is compatible with the Poisson structures on S and P. For each $w \in W$, we define the vector fields

$$\xi_w := \#(\pi^*(w)) \in \mathfrak{X}(S),$$

where $w \in W \cong T^*W^*$ is regarded as a 1-form on W^*. Then the map $w \mapsto \xi_w$ is *pointwise injective*, i.e. $\Xi := \{\xi_w \mid w \in W\} \subset TS$ is a distribution on S whose rank equals the dimension of W. For the bracket relations, we compute

$$\begin{array}{lcl} [\xi_A, \xi_B] & = & \xi_{[A,B]} \\ [\xi_A, \xi_x] & = & \xi_{A \cdot x} \\ [\xi_x, \xi_y](s) & = & \xi_{d\Phi(p)^*(x,y)} \quad \text{where } p = \pi(s). \end{array} \tag{2}$$

This implies, of course, that the distribution ξ on S is *integrable*. Moreover, the first equation in (2) implies that the flow along the vector fields ξ_A induces a local H-action on S. Let $F \subset S$ be a maximal integral leaf of ξ. Clearly, F is H-invariant, and after shrinking F, we may assume that $M := F/H$ is a mainfold. Then F can be extended to a principal H-bundle over M, and the vector fields $\{\xi_x, x \in V\}$ define a connection on M whose holonomy is contained in H. In fact, for the exapmles we consider, we can always achieve that the holonomy is *all* of H.

We then get the following remarkable result.

Theorem Let $\mathfrak{h}$ and V be as before, and suppose there is a *quadratic polynomial map* $\phi : \mathfrak{h}^* \to \Lambda^2 V$ satisfying the conditions from above and such that its (linear) differential

$$d\phi : \mathfrak{h}^* \to K(\mathfrak{h})$$

is a linear isomorphism. Then every torsion free connection whose holonomy algebra is contained in $\mathfrak{h}$ comes from the above construction.

In particular, it turns out that this theorem applies to all the newly found holonomies. This has some remarkable consequences. For example, it means that the moduli space of connections with one of these holonomies is finite dimensional. Moreover, in many cases, there exist local symmetries, i.e. vector fields on M whose flow preserves the connections.