Connections with Irreducible Holonomy Representations

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1 Introduction and history

An affine connection is one of the basic objects of interest in differential geometry. It provides a simple and invariant way of transferring information from one point of a connected manifold M to another and, not surprisingly, enjoys lots of applications in many branches of mathematics, physics and mechanics. Among the most informative characteristics of an affine connection is its (restricted) holonomy group which is defined, up to conjugacy, as the subgroup of $\mathrm{Aut}(T_pM)$ consisting of all automorphisms of the tangent space T_pM at $p \in M$ induced by parallel translations along p-based loops in M.

The *irreducible holonomy problem* which we shall investigate in this article is the following.

Given a finite dimensional vector space V, which are the irreducible (closed) Lie subgroups $\mathrm{H} \subset \mathrm{Aut}(V)$ that can occur as the holonomy group of a torsion free affine connection?

The condition of *torsion freeness* is an integrability condition which makes this problem non-trivial; namely, by a result of Hano and Ozeki [HO], *any* (closed) Lie subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ can be realized as the holonomy of an affine connection on some manifold M (with torsion, in general).

The notion of the holonomy group was introduced by É. Cartan in 1923 [Car2, Car4]. He used this invariant in order to investigate manifolds of dimensions 2 or 3 with a prescribed holonomy group. Also, in [Car3], he showed that for a *symmetric space*, the holonomy and the isotropy group coincide up to connected components. Thus, the holonomy problem contains the classification of irreducible symmetric spaces as a "sub-problem". This classification has been completed by Cartan in the Riemannian [Car3] and by Berger in the general case [Ber2].

In the 1950's, the concept of holonomy became the subject of further investigation. Following the work of Borel, Lichnerowicz [BL] and Nijenhuis [N1, N2], an important result, the *Ambrose-Singer Holonomy Theorem*, characterized the Lie algebra of the holonomy group in terms of the curvature of the connection [AS].

Using this result, Berger established a purely algebraic necessary condition which the Lie algebra of the holonomy group must satisfy [Ber1]. This condition is called *Berger's criterion*, and a subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ satisfying this criterion is called a *Berger subgroup*. Therefore, the holonomy problem splits into two parts:

1. Classify all irreducible Berger subgroups $\mathrm{H} \subset \mathrm{Aut}(V)$.

2. Decide for each Berger subgroup if it can occur as a holonomy group.

While the first problem is purely algebraic, the second is analytic in nature. Berger then proceeded to classify all (pseudo-)Riemannian Berger algebras, i.e. the holonomies of Levi-Civita connections of (pseudo-)Riemannian metrics. (In the non-definite case, there were some slight errors which were later corrected by Bryant [Br4].) Berger also gave a list of further Berger algebras; this final part of his classification, however, turned out to be incomplete.

It was in particular the list of possible *Riemannian* connections which received a tremendous amount of attention during the following decades. First, it turns out that the list of non-symmetric Riemannian holonomies is contained (in fact, is almost equal to) the list of transitive group actions on spheres [MoSa1, MoSa2, Bo1, Bo2]. This was later shown directly by Simons [Si].

The solution of problem 2, i.e. the existence of torsion free connections, for all Riemannian Berger algebras was finally settled in 1986. As it turns out, *all* Riemannian Berger algebras do occur as holonomies on some Riemannian manifold M – in fact, on some *closed* M. These results are due to the efforts of many mathematicians, e.g. Calabi [Cal], Yau [Y], Alekseevskii [A], Bryant [Br1, Br2], Joyce [J]. For surveys on the holonomies of Riemannian manifolds and many interesting interrelations between the holonomy and the geometry and topology of the underlying manifold M, see the books by Besse [Bes] and Salamon [Sa].

One of the most effective methods to solve problem 2 was Bryant's approach to describe torsion free connections with given holonomy group as solutions to an Exterior Differential System [Br2], and then to use Cartan-Kähler theory [BCG^3] to prove the local existence of such connections. This method turned out to be applicable for many holonomy groups and enabled Bryant to show that all pseudo-Riemannian Berger groups do occur as holonomies at least locally [Br4].

Bryant also found several new examples of Berger groups, called *exotic holonomies*, and showed the local existence of connections with these holonomies [Br3, Br4, Br5]. Global properties of some of these exotic holonomies are discussed in [Sc1, Sc2]. Further exotic holonomies were found in [CS].

An important application of connections with irreducible holonomies was given by Merkulov [Me1, Me2, Me3, Me4]. He showed that certain moduli of compact complex homogeneous Legendre manifolds of a complex contact manifold carry a natural torsion free connection. In fact, in the holomorphic category, every torsion free connection can be realized canonically as such a moduli. Moreover, this approach gave a new and efficient way to determine if a given subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ is Berger. Indeed, several new Berger groups were determined by that method [CMS1, CMS2, MeSc1]. While the occurance of these groups as holonomies can be shown in principle using Exterior Differential Systems as well, the proofs in [CMS1, CMS2] rely on a different method using certain quadratic deformations of Poisson structures on some Lie algebra. This method also reveals some more global properties of these connections.

Finally, in [MeSc1, MeSc2], a complete classification of irreducible Berger groups was given. That is, the new examples discovered there complete the list of Berger groups. Thus, the holonomy problem for irreducible connected holonomy groups is completely solved.

A Berger subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ is called *symmetric* if every torsion free connection with holonomy H is locally symmetric; otherwise, it is called *non-symmetric*. Since the classification of symmetric spaces is classically known [Car3, Ber2], we shall state the classification of *non-symmetric* Berger algebras only.

Moreover, the behaviour of Berger groups under complexification is well understood (cf. section 3.1); thus, it is not hard to obtain the list of all *real* Berger groups from the list of complex ones. The latter can be characterized as follows.

Theorem 1.1 *Let V be a finite dimensional complex vector space, let $H_{\mathbb{C}} \subset Aut(V)$ be an irreducible semi-simple complex connected Lie subgroup and let $K \subset H_{\mathbb{C}}$ be a maximal compact subgroup. Then the following holds.*

1. *If there is an irreducible hermitean symmetric space of the form $M = G/(U(1) \cdot K)$, then both $H_{\mathbb{C}}$ and $(\mathbb{C}^* Id_V) \cdot H_{\mathbb{C}}$ are non-symmetric Berger groups.*
2. *If there is an irreducible quaternionic symmetric space of the form $M = G/(Sp(1) \cdot K)$, then $H_{\mathbb{C}}$ is a non-symmetric Berger group. If $\dim V = 4$ then $(\mathbb{C}^* Id_V) \cdot H_{\mathbb{C}}$ is also a non-symmetric Berger group.*
3. *1. and 2. yield all complex non-symmetric Berger groups, with the following exceptions:*
   - *(a)* $H_{\mathbb{C}} = SL(2, \mathbb{C}) \cdot Sp(n, \mathbb{C}) \subset Aut(\mathbb{C}^2 \otimes \mathbb{C}^{2n})$, $n \geq 2$,
   - *(b)* $H_{\mathbb{C}} = G_2^{\mathbb{C}} \subset Aut(\mathbb{C}^7)$,
   - *(c)* $H_{\mathbb{C}} = Spin(7, \mathbb{C}) \subset Aut(\mathbb{C}^8)$.

Here, we use the standard notation $\mathrm{G} \cdot \mathrm{H} = (\mathrm{G} \times \mathrm{H})/\Gamma$ for some finite group Γ.

The original classification proof in [MeSc1] was based on the combination of two quite different methods. One of them relied on classical representation theory, using root and weight arguments, similar to Berger's original treatment of the problem in [Ber1]; the other used the twistor construction from [Me2] to determine whether or not certain subgroups are Berger.

The main purpose of this article is to give a simplified proof of the classification which relies on the use of classical representation theory only, and to present the various methods that are involved in the construction of these connections in a uniform way.

In the category of holomorphic connections, it seems also possible to obtain a proof which relies solely on the twistor theoretic approach and which might thus be complementary to the present paper.

While the classification in [MeSc1] was stated in terms of explicit lists, it was W.Ziller who noticed the close relation between these lists and the isotropies of symmetric spaces which allows us to state the classification result in the more elegant form of Theorem 1.1.

We list the irreducible non-symmetric *complex* Berger groups in Table 1 and the remaining irreducible non-symmetric *real* Berger groups in Table 2. Also, for the sake of completeness, we shall list the complex *symmetric* Berger subgroups in Table 3. These are those Berger groups for which there is a symmetric space G/K such that the complexification of K is not on the previous lists. In fact, our method also yields a new classification proof of symmetric spaces with simple holonomy.

Let us conclude this introduction with some remarks. First, it would be highly desirable to find a direct proof of Theorem 1.1. One direction, namely that to each such symmetric space there is a corresponding holonomy group, is relatively easy to see (cf. section 3.2.3 for the holonomies corresponding to quaternionic symmetric spaces). However, there is no known conceptual proof of the converse, i.e. of the fact that to each holonomy representation there is a corresponding symmetric space.

Second, there is another remarkable property of the *complex* holonomy representations. Namely, all of these have a hermitean symmetric space as their sky. However, there is no conceptual proof of this fact, either, though this observation was important for the discovery of a number of irreducible holonomy groups via the twistor approach [CMS1, CMS2, MeSc1].

Table 1 LIST OF IRREDUCIBLE COMPLEX NON-SYMMETRIC BERGER SUBGROUPS

NOTATIONS: $Z_{\mathbb{C}}$ denotes either the trivial group or $\mathbb{C}^* Id_V$.
$\odot^p V$ denotes the symmetric tensors of V of degree p.

No.	irreducible hermitean symmetric space $G/(U(1)\cdot K)$				corresponding Berger groups $H_{\mathbb{C}} \subset \mathrm{Aut}(V)$	V
	G	U(1)· K	K	restrictions[1]		
1	$SU(n+m)$	$S(U(n)U(m))$	$SU(n)\cdot SU(m)$	$n \geq m \geq 2$, $nm \neq 4$	$Z_{\mathbb{C}}\cdot SL(n,\mathbb{C})\cdot SL(m,\mathbb{C})$	$\mathbb{C}^n \otimes \mathbb{C}^m$
2	$SU(n+1)$	$S(U(1)U(n))$	$SU(n)$	$n \geq 1$	$Z_{\mathbb{C}}\cdot SL(n,\mathbb{C})$	$\mathbb{C}^n$
3	$SO(2n)$	$U(n)$	$SU(n)$	$n \geq 5$	$Z_{\mathbb{C}}\cdot SL(n,\mathbb{C})$	$\Lambda^2\mathbb{C}^n$
4	$Sp(n)$	$U(n)$	$SU(n)$	$n \geq 3$	$Z_{\mathbb{C}}\cdot SL(n,\mathbb{C})$	$\odot^2\mathbb{C}^n$
5	$SO(n+2)$	$SO(2)\cdot SO(n)$	$SO(n)$	$n \geq 3$	$Z_{\mathbb{C}}\cdot SO(n,\mathbb{C})$	$\mathbb{C}^n$
6	E_6	$U(1)\cdot Spin(10)$	$Spin(10)$		$Z_{\mathbb{C}}\cdot Spin(10,\mathbb{C})$	$(\Delta_{10}^+)^{\mathbb{C}}$
7	E_7	$U(1)\cdot E_6$	E_6		$Z_{\mathbb{C}}\cdot E_6^{\mathbb{C}}$	$\mathbb{C}^{27}$
	irreducible quaternionic symmetric space $G/(Sp(1)\cdot K)$					
	G	$Sp(1)\cdot K$	K	restrictions[1]		
8	$SO(n+4)$	$SO(n)\cdot SO(4)$	$SO(n)\cdot Sp(1)$	$n \geq 3$	$SO(n,\mathbb{C})\cdot SL(2,\mathbb{C})$	$\mathbb{C}^n \otimes \mathbb{C}^2$
9	$Sp(n+1)$	$Sp(n)\cdot Sp(1)$	$Sp(n)$	$n \geq 1$	$Sp(n,\mathbb{C})$ $Z_{\mathbb{C}}\cdot Sp(2,\mathbb{C})$	$\mathbb{C}^{2n}$ $\mathbb{C}^4$
10	G_2	$SO(4)$	$Sp(1)$		$Z_{\mathbb{C}}\cdot SL(2,\mathbb{C})$	$\odot^3\mathbb{C}^2$
11	F_4	$Sp(3)\cdot Sp(1)$	$Sp(3)$		$Sp(3,\mathbb{C})$	$\mathbb{C}^{14} \subset \Lambda^3\mathbb{C}^6$
12	E_6	$SU(6)\cdot Sp(1)$	$SU(6)$		$SL(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6$
13	E_7	$Spin(12)\cdot Sp(1)$	$Spin(12)$		$Spin(12,\mathbb{C})$	$(\Delta_{12}^+)^{\mathbb{C}}$
14	E_8	$E_7\cdot Sp(1)$	E_7		$E_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
15				$n \geq 2$	$SL(2,\mathbb{C})\cdot Sp(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^{2n}$
16					$G_2^{\mathbb{C}}$	$\mathbb{C}^7$
17					$Spin(7,\mathbb{C})$	$\mathbb{C}^8$

[1] Note that for an irreducible symmetric space $G/(U(1)\cdot K)$ or $G/(Sp(1)\cdot K)$ the representation of K is not necessarily irreducible. Indeed, irreducibility fails for the symmetric spaces $SU(n+2)/SU(2)\cdot SU(n)$ and $SO(6)/SO(4)\cdot SO(2)$. Also, the representations of K may be equivalent for different symmetric spaces. Thus, we consider only those hermitean and quaternionic symmetric spaces which are relevant for our purposes.

Table 2 LIST OF IRREDUCIBLE REAL NON-SYMMETRIC BERGER SUBGROUPS

NOTATIONS: $T_{\mathbb{F}}$ denotes any connected subgroup of $\mathbb{F}^*$.
$H_\lambda = \left\{e^{t(\lambda+i)} \mid t \in \mathbb{R}\right\} \subset \mathbb{C}^*$ for $\lambda > 0$.
$\odot^p V$ denotes the symmetric tensors of V of degree p.

complexification No.	real form H of $Z_{\mathbb{C}} \cdot \mathrm{H}_{\mathbb{C}}$	V	restrictions remarks
1	$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\{A \in M_n(\mathbb{C}) \mid A = A^*\}$	$n \geq 3$
	$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{R}) \cdot \mathrm{SL}(m,\mathbb{R})$	$\mathbb{R}^n \otimes \mathbb{R}^m$	$n \geq m \geq 2, nm \neq 4$
	$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{H}) \cdot \mathrm{SL}(m,\mathbb{H})$	$\mathbb{H}^n \otimes_{\mathbb{H}} \mathbb{H}^m$	$n \geq m \geq 1, nm \neq 1$
	$T_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C}) \cdot \mathrm{SL}(m,\mathbb{C})$	$\mathbb{C}^n \otimes \mathbb{C}^m$	$n \geq m \geq 2, nm \neq 4$
2	$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{R})$	$\mathbb{R}^n$	$n \geq 2$
	$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{H})$	$\mathbb{H}^n$	$n \geq 1$
	$T_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 2$
	$\mathrm{U}(p,q)$	$\mathbb{C}^{p+q}$	$p+q \geq 2$
	$\mathrm{SU}(p,q)$	$\mathbb{C}^{p+q}$	$p+q \geq 2,\ pq \neq 1$
	$\mathbb{C}^* \cdot \mathrm{SU}(p,q)$	$\mathbb{C}^2$	$p+q = 2$
2	$H_\lambda \cdot \mathrm{SU}(p,q)$	$\mathbb{C}^2$	$p+q=2$ existence unknown
3	$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{R})$	$\Lambda^2\mathbb{R}^n$	$n \geq 5$
	$T_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\Lambda^2\mathbb{C}^n$	$n \geq 5$
	$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{H})$	$\{A \in M_n(\mathbb{H}) \mid A = A^*\}$	$n \geq 3$
4	$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{R})$	$\odot^2\mathbb{R}^n$	$n \geq 3$
	$T_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\odot^2\mathbb{C}^n$	$n \geq 3$
	$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{H})$	$\{A \in M_n(\mathbb{H}) \mid A = -A^*\}$	$n \geq 2$
5	$T_{\mathbb{R}} \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^{p+q}$	$p+q \geq 3$
	$T_{\mathbb{C}} \cdot \mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 3$
6	$T_{\mathbb{R}} \cdot \mathrm{Spin}(5,5)$	$\Delta^+_{(5,5)}$	
	$T_{\mathbb{R}} \cdot \mathrm{Spin}(1,9)$	$\Delta^+_{(1,9)}$	
	$T_{\mathbb{C}} \cdot \mathrm{Spin}(10,\mathbb{C})$	$(\Delta^+_{10})^{\mathbb{C}}$	
7	$T_{\mathbb{R}} \cdot \mathrm{E}_6^1$	$\mathbb{R}^{27}$	
	$T_{\mathbb{R}} \cdot \mathrm{E}_6^4$	$\mathbb{R}^{27}$	
	$T_{\mathbb{C}} \cdot \mathrm{E}_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	
8	$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q}$	$p+q \geq 3$
	$\mathrm{Sp}(1) \cdot \mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n$	$n \geq 2$
9	$\mathrm{Sp}(n,\mathbb{R})$	$\mathbb{R}^{2n}$	$n \geq 2$
	$\mathbb{R}^* \cdot \mathrm{Sp}(2,\mathbb{R})$	$\mathbb{R}^4$	
	$\mathrm{Sp}(p,q)$	$\mathbb{H}^{p+q}$	$p+q \geq 2$
10	$T_{\mathbb{R}} \cdot \mathrm{SL}(2,\mathbb{R})$	$\odot^3\mathbb{R}^2$	

complexification No.	real form H of $Z_{\mathbb{C}} \cdot \mathrm{H}_{\mathbb{C}}$	V	restrictions remarks
11	$\mathrm{Sp}(3,\mathbb{R})$	$\mathbb{R}^{14} \subset \Lambda^3\mathbb{R}^6$	
12	$\mathrm{SL}(6,\mathbb{R})$ $\mathrm{SU}(1,5)$ $\mathrm{SU}(3,3)$	$\Lambda^3\mathbb{R}^6$ $\{\omega \in \Lambda^3\mathbb{C}^6 \mid *\omega = \omega\}$ $\{\omega \in \Lambda^3\mathbb{C}^6 \mid *\omega = \omega\}$	
13	$\mathrm{Spin}(2,10)$ $\mathrm{Spin}(6,6)$ $\mathrm{Spin}(6,\mathbb{H})$	$\Delta^+_{(2,10)}$ $\Delta^+_{(6,6)}$ $\Delta^{\mathbb{H}}_6$	
14	E_7^5 E_7^7	$\mathbb{R}^{56}$ $\mathbb{R}^{56}$	
15	$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{Sp}(n,\mathbb{R})$ $\mathrm{Sp}(1) \cdot \mathrm{Sp}(p,q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{2n}$ $\mathbb{H}^{p+q}$	$n \geq 2$ $p+q \geq 2$
16	G_2 G_2'	$\mathbb{R}^7$ $\mathbb{R}^7$	
17	$\mathrm{Spin}(7)$ $\mathrm{Spin}(4,3)$	$\mathbb{R}^8$ $\mathbb{R}^8$	

The structure of this paper is as follows. In chapter 2, we give some preliminary facts on representation theory and Spencer cohomology which will be needed in the following chapters. In chapter 3, the core of this paper, we discuss several examples of Berger groups and proceed to give the new proof of the classification. In chapter 4, we briefly summarize two methods to construct torsion free connection with prescribed holonomy, namely the method of Bryant via Exterior Differential Systems, and the method from [CMS1, CMS2] which is universal for *symplectic* holonomies and which relies on deformations of Poisson structures. Finally, in chapter 5, we briefly describe the twistor construction of Merkulov which realizes any holomorphic torsion free connection with irreducible holonomy group as the moduli of compact complex Legendre submanifolds of a complex contact manifold [Me2].

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2 Preliminary facts and results

2.1 Holonomy groups and holonomy algebras

Let M be a smooth connected n-manifold and let ∇ be an affine connection on M, i.e. a connection on the tangent bundle TM. Fix a point $p \in M$ and let

$$\mathcal{L}_p = \{\gamma : [0,1] \to M \mid \gamma(0) = \gamma(1) = p\}$$

Table 3 LIST OF IRREDUCIBLE COMPLEX SYMMETRIC BERGER SUBGROUPS

No.	irreducible symmetric space G/K			symmetric Berger groups $H_{\mathbb{C}} \subset \mathrm{Aut}(V)$	V
	G	K	restrictions		
1	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	$n \geq 3$	$\mathrm{Sp}(n, \mathbb{C})$	$\Lambda^2\mathbb{C}^{2n}$ mod Ω
2	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	$n \geq 3, \quad n \neq 4$	$\mathrm{SO}(n, \mathbb{C})$	$(\odot^2\mathbb{C}^n)_0$
3	$\mathrm{K} \times \mathrm{K}$	$\Delta\mathrm{K}$	K simple	$Ad_{\mathfrak{k}\otimes\mathbb{C}}$	$\mathfrak{k} \otimes \mathbb{C}$
4	F_4	$\mathrm{Spin}(9)$		$\mathrm{Spin}(9, \mathbb{C})$	$(\Delta_9)^{\mathbb{C}}$
5	E_6	$\mathrm{Sp}(4)$		$\mathrm{Sp}(4, \mathbb{C})$	$\Lambda^4\mathbb{C}^8$ mod $(\Omega \wedge \Lambda^2\mathbb{C}^8)$
6	E_6	F_4		$\mathrm{F}_4^{\mathbb{C}}$	$\mathbb{C}^{26}$
7	E_7	$\mathrm{SU}(8)$		$\mathrm{SL}(8, \mathbb{C})$	$\Lambda^4\mathbb{C}^8$
8	E_8	$\mathrm{Spin}(16)$		$\mathrm{Spin}(16, \mathbb{C})$	$(\Delta_{16}^+)^{\mathbb{C}}$
9	$\mathrm{SO}(p+q)$	$\mathrm{SO}(p) \cdot \mathrm{SO}(q)$	$p \geq q \geq 3$	$\mathrm{SO}(p, \mathbb{C}) \cdot \mathrm{SO}(q, \mathbb{C})$	$\mathbb{C}^p \otimes \mathbb{C}^q$
10	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p) \cdot \mathrm{Sp}(q)$	$p \geq q \geq 2$	$\mathrm{Sp}(p, \mathbb{C}) \cdot \mathrm{Sp}(q, \mathbb{C})$	$\mathbb{C}^{2p} \otimes \mathbb{C}^{2q}$
11	G_2	$\mathrm{SO}(4)$		$\mathrm{SL}(2, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$	$\mathbb{C}^2 \otimes \odot^3\mathbb{C}^2$
12	F_4	$\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$		$\mathrm{Sp}(3, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$	$(\Lambda^3\mathbb{C}^6$ mod $(\Omega \wedge \mathbb{C}^6)) \otimes \mathbb{C}^2$
13	E_6	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$		$\mathrm{SL}(6, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$	$\Lambda^3\mathbb{C}^6 \otimes \mathbb{C}^2$
14	E_7	$\mathrm{Spin}(12) \cdot \mathrm{Sp}(1)$		$\mathrm{Spin}(12, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$	$(\Delta_{12}^+)^{\mathbb{C}} \otimes \mathbb{C}^2$
15	E_8	$\mathrm{E}_7 \cdot \mathrm{Sp}(1)$		$\mathrm{E}_7^{\mathbb{C}} \cdot SL(2, \mathbb{C})$	$\mathbb{C}^{56} \otimes \mathbb{C}^2$

be the set of piecewise smooth loops based at p, and let $\mathcal{L}_p^0 \subset \mathcal{L}_p$ be those loops which are homotopic to the trivial loop.

For $\gamma \in \mathcal{L}_p$, denote by $P_\gamma : T_pM \longrightarrow T_pM$ the linear automorphism induced by ∇-parallel translations along γ. The *holonomy of* ∇ *at* $p \in M$ is defined as the subset

$$Hol_p := \{P_\gamma \mid \gamma \in \mathcal{L}_p\} \subset \mathrm{Aut}(T_pM),$$

and the *restricted holonomy* is given by

$$Hol_p^0 := \left\{P_\gamma \mid \gamma \in \mathcal{L}_p^0\right\} \subset Hol_p.$$

Some of the basic properties of these groups are (see, e.g., [Bes, KoNo])

1. Hol_p^0 is the connected component of Hol_p.
2. If $\pi : \tilde{M} \to M$ is the universal cover and $\tilde{\nabla}$ is the lift of ∇ to $\tilde{M}$, then $Hol_{\tilde{p}} \cong Hol_p^0$, where $\pi(\tilde{p}) = p$. Thus, by lifting the connection to the universal cover, we may assume that the holonomy group is connected.

3. Hol_p^0 is a closed Lie subgroup of $\mathrm{Aut}(T_pM)$; its Lie algebra $\mathfrak{hol}_p \subset \mathrm{End}(T_pM)$ is called the holonomy algebra at p.

4. $Hol_p \cong Hol_q$, with an isomorphism being induced by parallel translation along any path from p to q. Thus, if one fixes a linear isomorphism $\imath : T_pM \to V$, where V is a fixed vector space of the appropriate dimension, then the conjugacy class of $\imath(Hol_p) \subset \mathrm{Aut}(V)$ does not depend on the choice of $p \in M$ or $\imath$.

By a slight abuse of terminology, we refer to the conjugacy class of $Hol := \imath(Hol_p) \subset \mathrm{Aut}(V)$ (respectively, $Hol^0 := \imath(Hol_p^0) \subset \mathrm{Aut}(V)$) as the holonomy group (respectively, restricted holonomy group) of ∇. The Lie algebra $\mathfrak{hol} \subset \mathrm{End}(V)$ of $Hol \subset \mathrm{Aut}(V)$ is called the *holonomy algebra* of ∇.

To an affine connection ∇ we can associate two tensors, the *torsion* and the *curvature*, which are given by the formulae

$$Tor_p(x, y) = \nabla_X Y - \nabla_Y X - [X, Y], \text{ and} \tag{1}$$

$$R_p(x, y)z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2}$$

Here, $x, y, z \in T_pM$, and X, Y, Z are vector fields with $X_p = x, Y_p = y$ and $Z_p = z$.

We shall from now on assume that ∇ is *torsion free*, i.e. $Tor \equiv 0$. Then it is easy to show that the curvature satisfies the *first and second Bianchi identity*, i.e.

$$R(x, y)z + R(y, z)x + R(z, x)y = 0, \text{ and} \tag{3}$$

$$(\nabla_x R)(y, z) + (\nabla_y R)(z, x) + (\nabla_z R)(x, y) = 0 \tag{4}$$

for all $x, y, z \in T_pM$.

A remarkable link between the curvature and the holonomy algebra has been given by the following

Ambrose-Singer Holonomy Theorem [AS] *Let ∇ be an affine connection on M and let $p \in M$. Then the holonomy algebra at p is given by*

$$\mathfrak{hol}_p = \langle \{ (P_\gamma R)(x, y) \mid x, y \in T_pM, \gamma \text{ a path with end point } p \} \rangle ,$$

where $(P_\gamma R)(x, y) := P_\gamma \cdot R(P_\gamma^{-1} x, P_\gamma^{-1} y) \cdot P_\gamma^{-1}$.

It is obvious that $P_\gamma R$ also satisfies the first Bianchi identity (3). This algebraic description of the holonomy algebra was used by Berger [Ber1] to develop the following necessary condition for a Lie subalgebra to be the holonomy of a torsion free connection.

Let V be a vector space and $\mathfrak{h} \subset \mathrm{End}(V)$ a Lie subalgebra. We define the *space of formal curvature maps*

$$K(\mathfrak{h}) := \{ R \in \Lambda^2 V^* \otimes \mathfrak{h} \mid R(x, y)z + R(y, z)x + R(z, x)y = 0 \text{ for all } x, y, z \in V \},$$

and the *space of formal curvature derivatives*

$$K^1(\mathfrak{h}) := \{ \phi \in V^* \otimes K(\mathfrak{h}) \mid \phi(x)(y, z) + \phi(y)(z, x) + \phi(z)(x, y) = 0 \text{ for all } x, y, z \in V \}.$$

We also let $\underline{\mathfrak{h}} := \{R(x,y) \mid R \in K(\mathfrak{h}),\ x,y \in V\} \subset \mathfrak{h}$. Evidently, $\underline{\mathfrak{h}} \lhd \mathfrak{h}$. Note that $K(\mathfrak{h})$ and $K^1(\mathfrak{h})$ are defined by the exact sequences

$$0 \longrightarrow K(\mathfrak{h}) \longrightarrow \Lambda^2 V^* \otimes \mathfrak{h} \longrightarrow \Lambda^3 V^* \otimes V \tag{5}$$

and

$$0 \longrightarrow K^1(\mathfrak{h}) \longrightarrow V^* \otimes K(\mathfrak{h}) \longrightarrow \Lambda^3 V^* \otimes \mathfrak{h}, \tag{6}$$

where in each case, the last map is given by the composition of the natural inclusion and the skew-symmetrization map, i.e. $\Lambda^2 V^* \otimes \mathfrak{h} \hookrightarrow \Lambda^2 V^* \otimes V^* \otimes V \to \Lambda^3 V^* \otimes V$ in the first and $V^* \otimes K(\mathfrak{h}) \hookrightarrow V^* \otimes \Lambda^2 V^* \otimes \mathfrak{h} \to \Lambda^3 V^* \otimes \mathfrak{h}$ in the second case.

From (3) it follows that $P_\gamma R \in K(\mathfrak{hol}_p)$ for all path γ with end point p; hence the Ambrose-Singer Holonomy Theorem implies that $\underline{\mathfrak{hol}}_p = \mathfrak{hol}_p$. Moreover, from (4) it follows that the map $x \mapsto \nabla_x R$ lies in $K^1(\mathfrak{hol}_p)$. Thus, if $K^1(\mathfrak{hol}_p) = 0$ then $\nabla R \equiv 0$, i.e. the connection is locally symmetric. These facts motivate the following definition.

Definition 2.1 *A Lie subalgebra $\mathfrak{h} \subset End(V)$ is called a* Berger algebra *if $\underline{\mathfrak{h}} = \mathfrak{h}$. A Berger algebra $\mathfrak{h} \subset End(V)$ is called* symmetric *if $K^1(\mathfrak{h}) = 0$ and* non-symmetric *otherwise.*

A Lie subgroup $H \subset Aut(V)$ is called a (symmetric respectively non-symmetric) Berger group *if its Lie algebra $\mathfrak{h} \subset End(V)$ is a (symmetric respectively non-symmetric) Berger algebra.*

In the literature, the two criteria for a non-symmetric Berger algebra are usually referred to as *Berger's first and second criterion.* Our discussion from above now yields the following.

Proposition 2.2 *[Ber1] Let $H \subset Aut(V)$ be an irreducible Lie subgroup which occurs as the holonomy group of a torsion free affine connection on some manifold M. Then H must be a Berger group. If the connection is not locally symmetric, then H must be a non-symmetric Berger group.*

We shall often utilize the following simple

Lemma 2.3 *If $\mathfrak{h} \subset End(V)$ is an irreducible Berger algebra, and if $K(\mathfrak{h})$ is a trivial $\mathfrak{h}$-module, then $\mathfrak{h}$ is symmetric.*

Proof. W.l.o.g. we may assume that $\dim V > 2$. Suppose $K(\mathfrak{h})$ is a trivial $\mathfrak{h}$-module. Then $K^1(\mathfrak{h}) \subset V^* \otimes K(\mathfrak{h})$ is a submodule and thus, since V is irreducible, we have $K^1(\mathfrak{h}) = V^* \otimes W$ for some subspace $W \subset K(\mathfrak{h})$. Suppose there is a $0 \neq R \in W$. Pick independent elements $x, y, z \in V$ such that $R(x,y) \neq 0$, and define $\phi : V \to W$ such that $\phi(x) = \phi(y) = 0$ and $\phi(z) = R$. Then it follows that $\phi \notin K^1(\mathfrak{h})$ which is a contradiction.

Therefore, $W = 0$, i.e. $K^1(\mathfrak{h}) = 0$, and thus $\mathfrak{h}$ is symmetric. ∎

2.2 Spencer cohomology

We shall briefly summarize the construction of the Spencer complex for a Lie subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$. For a more detailed exposition, we refer the interested reader to [G, O] and [Br4].

Let V be a finite dimensional vector space over $\mathbb{F}$. We let $A^{p,q}(V) := \odot^p V^* \otimes \Lambda^q V^*$. This space can be thought of as the space of q-forms on V with values in the space of homogeneous polynomials on V of degree p. Exterior differentiation thus yields a map $\delta : A^{p,q}(V) \to A^{p-1,q+1}(V)$, which makes $A^{*,*}(V) = \bigoplus_{p,q\geq 0} A^{p,q}(V)$ into a bigraded complex. Likewise, $\bigoplus_{p,q\geq 0}(V \otimes A^{p,q}(V))$ becomes a bigraded complex by the maps $\delta_V := Id_V \otimes \delta$.

Let $\mathfrak{h} \subset \mathrm{End}(V) \cong V^* \otimes V$ be a subalgebra. The *k-th prolongation of* $\mathfrak{h}$, denoted by $\mathfrak{h}^{(k)}$ for an integer k, is defined by the formulae $\mathfrak{h}^{(-1)} = V$, $\mathfrak{h}^{(0)} = \mathfrak{h}$, and

$$\mathfrak{h}^{(k)} = \delta_V^{-1}(\mathfrak{h}^{(k-1)} \otimes V^*).$$

That is,

$$\mathfrak{h}^{(k)} = (\mathfrak{h} \otimes \odot^k V^*) \cap (V \otimes \odot^{k+1} V^*),$$

where we use exterior differentiation $\delta : \odot^{k+1} V^* \to V^* \otimes \odot^k V^*$ to regard both $\mathfrak{h} \otimes \odot^k V^*$ and $V \otimes \odot^{k+1} V^*$ as subspaces of $V \otimes V^* \otimes \odot^k V^*$. Alternatively, we can define $\mathfrak{h}^{(k)}$ inductively by $\mathfrak{h}^{(-1)} = V$, $\mathfrak{h}^{(0)} = \mathfrak{h}$ and the exact sequence

$$0 \longrightarrow \mathfrak{h}^{(k)} \longrightarrow \mathfrak{h}^{(k-1)} \otimes V^* \longrightarrow \mathfrak{h}^{(k-2)} \otimes \Lambda^2 V^*. \tag{7}$$

For example,

$$\mathfrak{h}^{(1)} = \{\alpha \in V^* \otimes \mathfrak{h} \mid \alpha(x)y = \alpha(y)x \text{ for all } x, y \in V\}.$$

Furthermore, we define the *Spencer complex of* $\mathfrak{h}$ to be $(C^{p,q}(\mathfrak{h}), \delta)$ with

$$C^{p,q}(\mathfrak{h}) = \mathfrak{h}^{(p-1)} \otimes \Lambda^q(V^*) \subset V \otimes \odot^p V^* \otimes \Lambda^q V^* = V \otimes A^{p,q}(V).$$

It is not hard to see that $\delta_V(C^{p,q}(\mathfrak{h})) \subset C^{p-1,q+1}(\mathfrak{h})$, and thus, $(C^{p,q}(\mathfrak{h}), \delta)$ is indeed a complex. Its cohomology groups $H^{p,q}(\mathfrak{h})$ are called the *Spencer cohomology groups of* $\mathfrak{h}$. The lower corner of this bigraded complex takes the form

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & & & & & \\
\mathfrak{h}^{(2)} & & \mathfrak{h}^{(2)} \otimes V^* & & \cdots & & & & \\
& \searrow & & \searrow & & & & & \\
\mathfrak{h}^{(1)} & & \mathfrak{h}^{(1)} \otimes V^* & & \mathfrak{h}^{(1)} \otimes \Lambda^2 V^* & & \cdots & & \\
& \searrow & & \searrow & & \searrow & & & \\
\mathfrak{h} & & \mathfrak{h} \otimes V^* & & \mathfrak{h} \otimes \Lambda^2 V^* & & \mathfrak{h} \otimes \Lambda^3 V^* & & \cdots \\
& \searrow & & \searrow & & \searrow & & \searrow & \\
V & & V \otimes V^* & & V \otimes \Lambda^2 V^* & & V \otimes \Lambda^3 V^* & & \cdots
\end{array}$$

It is worth pointing out that all of these spaces are $\mathfrak{h}$-modules in an obvious way, and that all maps are $\mathfrak{h}$-equivariant. Thus, the Spencer cohomology groups are $\mathfrak{h}$-modules as well.

Table 4: List of irreducible complex matrix Lie groups H with $\mathfrak{h}^{(1)} \neq 0$

	group H	representation V		$\mathfrak{h}^{(1)}$	$\mathfrak{h}^{(2)}$	$H^{1,2}(\mathfrak{h})$
1	$\mathrm{SL}(n,\mathbb{C})$	$\mathbb{C}^n$,	$n \geq 2$	$(V \otimes \odot^2 V^*)_0$	$(V \otimes \odot^3 V^*)_0$	$\odot^2 V^*$
2	$\mathrm{GL}(n,\mathbb{C})$	$\mathbb{C}^n$,	$n \geq 1$	$V \otimes \odot^2 V^*$	$V \otimes \odot^3 V^*$	0
3	$\mathrm{GL}(n,\mathbb{C})$	$\odot^2\mathbb{C}^n$,	$n \geq 2$	V^*	0	0
4	$\mathrm{GL}(n,\mathbb{C})$	$\Lambda^2\mathbb{C}^n$,	$n \geq 5$	V^*	0	0
5	$\mathrm{GL}(m,\mathbb{C}) \cdot \mathrm{GL}(n,\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^n$,	$m, n \geq 2$	V^*	0	0
6	$\mathrm{Sp}(n,\mathbb{C})$	$\mathbb{C}^{2n}$,	$n \geq 2$	$\odot^3 V^*$	$\odot^4 V^*$	0
7	$\mathbb{C}^* \cdot \mathrm{Sp}(n,\mathbb{C})$	$\mathbb{C}^{2n}$,	$n \geq 2$	$\odot^3 V^*$	$\odot^4 V^*$	0
8	$\mathrm{CO}(n,\mathbb{C})$	$\mathbb{C}^n$,	$n \geq 3$	V^*	0	$\mathcal{W}$[1]
9	$\mathbb{C}^* \cdot \mathrm{Spin}(10,\mathbb{C})$	$\mathbb{C}^{16}$		V^*	0	0
10	$\mathbb{C}^* \cdot \mathrm{E}_6^{\mathbb{C}}$	$\mathbb{C}^{27}$		V^*	0	0

[1] $\mathcal{W}$ denotes the space of *formal Weyl curvatures* (see e.g. [Bes]).

Also, note that $K(\mathfrak{h})$ is the kernel of the map $\delta : C^{1,2}(\mathfrak{h}) \to C^{0,3}(\mathfrak{h})$, and hence, we have the exact sequence

$$0 \longrightarrow \mathfrak{h}^{(2)} \longrightarrow \mathfrak{h}^{(1)} \otimes V^* \longrightarrow K(\mathfrak{h}) \longrightarrow H^{1,2}(\mathfrak{h}) \longrightarrow 0, \tag{8}$$

where the second map is given by $R_{\alpha\otimes\phi}(x,y) = \phi(x)\alpha(y) - \phi(y)\alpha(x)$ for $\alpha \otimes \phi \in \mathfrak{h}^{(1)} \otimes V^*$.

If we assume that $\mathfrak{h} \subset \mathrm{End}(V)$ acts *irreducibly*, then there are only very few possibilities for which $\mathfrak{h}^{(1)} \neq 0$. These subalgebras have been classified by Cartan [Car1] and Kobayashi and Nagano [KoNa]. The result is listed in Table 4 for *complex* Lie algebras. The Spencer cohomologies $H^{1,2}(\mathfrak{h})$ of these Lie algebras are well-known. (See e.g. [Br4] and [MeSc1] who use considerably different techniques for the calculations).

2.3 H-structures, intrinsic torsion and intrinsic curvature

As before, let M be a smooth connected (real or complex) manifold of dimension n. Let $\pi : \mathfrak{F} \to M$ be the *coframe bundle* of M, i.e. each $u \in \mathfrak{F}$ is a linear isomorphism $u : T_{\pi(u)}M \xrightarrow{\sim} V$, where V is a fixed n-dimensional (real or complex) vector space. Then $\mathfrak{F}$ is naturally a principal right $\mathrm{Aut}(V)$-bundle over M, where the right action $R_g : \mathfrak{F} \to \mathfrak{F}$ is defined by $R_g(u) = g^{-1} \circ u$. The *tautological* 1*-form* θ on $\mathfrak{F}$ with values in V is defined by $\theta(\xi) = u(\pi_*(\xi))$ for $\xi \in T_u\mathfrak{F}$. For θ, we have the $\mathrm{Aut}(V)$-equivariance

$$R_g^*(\theta) = g^{-1}\theta. \tag{9}$$

Let $\mathrm{H} \subset \mathrm{Aut}(V)$ be a closed Lie subgroup and let $\mathfrak{h} \subset \mathrm{End}(V)$ be the Lie algebra of H. An H-structure on M is, by definition, an H-subbundle $F \subset \mathfrak{F}$. For any H-structure, we will denote the restrictions of π and θ to F by the same letters. Given $A \in \mathfrak{h}$ we define the vector field A_* on F by

$$(A_*)_u = \frac{d}{dt}\left(R_{exp(tA)}(u)\right)|_{t=0}.$$

The vector fields A_* are called the *fundamental vertical vector fields* on F. It is evident that $\pi_*(A_*) = 0$ and thus $\theta(A_*) = 0$ for all $A \in \mathfrak{h}$; in fact, $\{A_* \mid A \in \mathfrak{h}\} = \ker(\pi_*)$. Moreover, for $A, B \in \mathfrak{h}$ we have $[A_*, B_*] = [A, B]_*$.

For a given H-structure $\pi : F \to M$, we define the vector bundles $\mathfrak{h}_F^{(k)} := F \times_H \mathfrak{h}^{(k)}$, $C_F^{p,q} := F \times_H C^{p,q}(\mathfrak{h})$ and $H_F^{p,q} := F \times_H H^{p,q}(\mathfrak{h})$. Note that $\mathfrak{h}_F := \mathfrak{h}_F^{(0)}$ is a subbundle of $T^*M \otimes TM$, and that $\mathfrak{h}_F^{-1} = TM$. The boundary maps of the Spencer complex induce bundle maps $\delta_F^{p,q} : C_F^{p,q} \to C_F^{p-1,q+1}$ whose kernels we denote by $\mathcal{Z}_F^{p,q}$. In particular, we let $K(\mathfrak{h}_F) := \mathcal{Z}_F^{1,2}$.

A *connection* on F is a $\mathfrak{h}$-valued 1-form ω on F satisfying the conditions

$$\begin{array}{ll} \omega(A_*) = A & \text{for all } A \in \mathfrak{h}, \text{ and} \\ R_h^*(\omega) = h^{-1}\omega h & \text{for all } h \in \mathrm{H}. \end{array} \tag{10}$$

Given a connection ω, its *torsion* Θ is the V-valued 2-form given by

$$\Theta = d\theta + \omega \wedge \theta. \tag{11}$$

From (9), (10) and (11) it follows that

$$R_h^*\Theta = h^{-1}\Theta, \tag{12}$$

and hence, Θ induces a section Tor of the bundle $\mathcal{Z}_F^{0,2} = \Lambda^2 T^*M \otimes TM$. Note that Tor coincides with the torsion tensor given in (1). ω is called *torsion free* if $\Theta = 0$. Using the natural projection map $p : \mathcal{Z}_F^{0,2} \to H_F^{0,2}$, we obtain a section $\tau := p(\Theta)$ of $H_F^{0,2}$.

Now let ω' be another connection on F with torsion Θ'. From (10) it follows that $\alpha := \omega' - \omega$ is an $\mathfrak{h}$-valued 1-form with $\alpha(A_*) = 0$ and $R_h^*\alpha = h^{-1}\alpha h$, and hence, α induces a section $\underline{\alpha}$ of $\mathfrak{h}_F \otimes T^*M$. Note that the section $\delta_F^{1,1}(\underline{\alpha})$ of $\mathcal{Z}_F^{0,2} = \Lambda^2 T^*M \otimes TM$ is induced by the section $\alpha \wedge \theta$. But for the torsion, we have $\Theta' = \Theta + \alpha \wedge \theta$, and hence $p(\Theta - \Theta') = p(\delta_F^{1,1}(\underline{\alpha})) = 0$, i.e. the section $\tau = p(\Theta)$ is independent of the choice of ω. This motivates the following terminology.

Definition 2.4 *Let $\pi : F \to M$ be an H-structure. Then the vector bundle $H_F^{0,2}$ is called the* intrinsic torsion bundle *of F, and the section τ of $H_F^{0,2}$ defined by any connection is called the* intrinsic torsion *of F. Moreover, F is called* torsion free *or* 1-flat *if its intrinsic torsion τ vanishes.*

It is then obvious that F admits a torsion free connection iff F is torsion free, and moreover, that the difference of two torsion free connections is given by a section of $\mathfrak{h}_F^{(1)}$. In particular, if $\mathfrak{h}^{(1)} = 0$ then F admits at most one torsion free connection.

Suppose now that F is torsion free and let ω be a torsion free connection on F, i.e.

$$d\theta + \omega \wedge \theta = 0.$$

Exterior differentiation yields the *first Bianchi identity*

$$\Omega \wedge \theta = 0, \tag{13}$$

where

$$\Omega := d\omega + \omega \wedge \omega$$

is the *curvature 2-form of* ω. Then $R_h^*\Omega = h^{-1}\Omega h$ for all $h \in H$, and hence Ω induces a section R of $\Lambda^2 T^*M \otimes \mathfrak{h}_F$. Note that R coincides with the curvature tensor given in (2). Moreover, (13) implies that $\delta^{1,2}(R) = 0$. Therefore, R is a section of $K(\mathfrak{h}_F) = \mathcal{Z}_F^{1,2}$ and thus induces a section $\rho := p(R)$ of $H_F^{1,2}$ where again, $p : K(\mathfrak{h}_F) \to H_F^{1,2}$ is the natural projection.

Now let ω' be another torsion free connection on F, i.e. $\alpha := \omega - \omega'$ satisfies $\alpha \wedge \theta = 0$ or, equivalently, the induced section $\underline{\alpha}$ of $T^*M \otimes \mathfrak{h}_F$ satisies $\delta^{1,1}(\underline{\alpha}) = 0$. If we denote the curvature sections of ω and ω' by R and R' respectively, then an easy calculation shows that

$$R' = R + d\underline{\alpha} + \underline{\alpha} \wedge \underline{\alpha}.$$

It is now straightforward to verify that the map

$$\begin{array}{rlll} \phi : & TM & \longrightarrow & \mathfrak{h}^{(1)} \\ & X & \longmapsto & \nabla_X \underline{\alpha} + \underline{\alpha}(X)\underline{\alpha} \end{array} \tag{14}$$

is well defined and satisfies

$$\delta^{2,1}(\phi) = d\underline{\alpha} + \underline{\alpha} \wedge \underline{\alpha}, \tag{15}$$

and thus the section $\rho := pr(R)$ of $H_F^{1,2}$ is independent of the choice of the torsion free connection.

Definition 2.5 *Let $\pi : F \to M$ be a torsion free H-structure. The section ρ of $H_F^{1,2}$ defined above is called the* intrinsic curvature *of F. Moreover, if $\rho \equiv 0$ then F is called* 2-flat*. F is called* locally flat *if there exists a torsion free connection on F whose curvature vanishes.*

Evidently, local flatness implies 2-flatness. The converse is not true in general; indeed, F is 2-flat iff for any $p \in M$, there exists a torsion free connection on F whose curvature vanishes *at* p.

In general, an H-structure F is called k-flat if for every $p \in M$ there is a torsion free connection on F whose curvature vanishes at p up to $(k-1)$-st order. One can show that the obstruction for F to be k-flat is represented by a section of $H_F^{k,2}$. We shall not give the precise definition, but refer the interested reader to [Br2] for details.

2.4 A brief review of representation theory

In this section, we shall give a brief outline of standard facts of representation theory of complex semi-simple Lie algebras. For a more detailed exposition, see e.g. [FH] or [Hu].

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and G the associated simply connected Lie group, and let $\mathfrak{t} \subset \mathfrak{g}$ be a *Cartan subalgebra*, i.e. a maximal abelian self-normalizing subalgebra. The *rank of* $\mathfrak{g}$ is by definition $\mathrm{rk}(\mathfrak{g}) := \dim \mathfrak{t}$.

If $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$ on a complex vector space V, then for any $\lambda \in \mathfrak{t}^*$ we define the *weight space* V_λ by

$$V_\lambda = \{v \in V \mid \rho(h)v = \lambda(h)v \text{ for all } h \in \mathfrak{t}\}.$$

An element $\lambda \in \mathfrak{t}^*$ is called a *weight* of V if $V_\lambda \neq 0$. We let $\Phi \subset \mathfrak{t}^*$ be the set of weights of ρ, and thus have the decomposition

$$V = \bigoplus_{\lambda \in \Phi} V_\lambda.$$

In particular, if $V = \mathfrak{g}$ and ρ is the adjoint representation, then we get the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

i.e. $\mathfrak{t}$ is the weight space of weight 0, and $\Delta \subset \mathfrak{t}^*$ is the set of non-zero weights. Δ is called the *set of roots* or the *root system* of $\mathfrak{g}$. It is well known that $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$.

For each root system Δ, there is a subset $S = \{\alpha_1, \ldots, \alpha_r\} \subset \Delta$ where $r = \mathrm{rk}(\mathfrak{g})$, called a *system of simple roots*, with the property that every $\alpha \in \Delta$ may be expressed as a linear combination $\alpha = \sum_{i=1}^r a_i \alpha_i$ with either $a_i \geq 0$ for all i, or $a_i \leq 0$ for all i. Then α is called a *positive* respectively a *negative root*, and the sets of positive and negative roots are denoted by $\Delta^\pm$. Thus, $\Delta = \Delta^+ \cup \Delta^-$.

For any root $\alpha \in \Delta$, there is a unique element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}$ such that $\alpha(H_\alpha) = 2$. If $S = \{\alpha_1, \ldots, \alpha_r\}$ is the set of simple roots, then the associated set $\{H_{\alpha_1}, \ldots, H_{\alpha_r}\}$ forms a basis of $\mathfrak{t}$. Its dual basis $\{\lambda_1, \ldots, \lambda_r\}$ of $\mathfrak{t}^*$ is called the set of *fundamental weights.* The lattice $\Lambda \subset \mathfrak{t}^*$ generated by this basis is called the *(integral) weight lattice.* It is well known that $\Phi \subset \Lambda$ for any representation ρ. The lattice Π generated by Δ is called the *root lattice.* Evidently, $\Pi \subset \Lambda$, and moreover, the quotient Λ/Π is isomorphic to the center of the simply connected Lie group G associated to $\mathfrak{g}$.

Let $\Lambda^+ := \{\lambda \in \Lambda \mid \lambda = \sum_{i=1}^r a_i \lambda_i \text{ with } a_i \geq 0\}$ be the set of *dominant weights.* Note that $a_i = \lambda_i(H_{\alpha_i})$. If $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is an irreducible representation then there exists a unique weight $\lambda_0 \in \Lambda^+$, called the *dominant weight* of ρ, such that $\dim V_{\lambda_0} = 1$ and $\rho(\mathfrak{g}_\alpha) V_{\lambda_0} = 0$ for all $\alpha \in \Delta^+$. Any non-zero element of V_{λ_0} is called a *dominant weight vector.* In fact, the dominant weight determines the representation ρ, and thus establishes a one-to-one correspondence between finite-dimensional irreducible representations of $\mathfrak{g}$ and the set Λ^+.

Given an $\lambda \in \Lambda$ and a root α, we let

$$\langle \lambda, \alpha \rangle := \lambda(H_\alpha) \in \mathbb{Z}.$$

Note that $\langle\ ,\ \rangle$ is linear in the first entry only. There is a $\mathrm{ad}(\mathfrak{g})$-invariant symmetric bilinear form B on $\mathfrak{g}$, the so-called *Killing form*, which is given by $B(x, y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$ for all $x, y \in \mathfrak{g}$. We shall use it to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. With this, we have

$$\langle \lambda, \alpha \rangle = \frac{2B(\lambda, \alpha)}{B(\alpha, \alpha)}. \tag{16}$$

The significance of $\langle \lambda, \alpha \rangle$ is the following. If λ occurs as the weight of an irreducible representation of $\mathfrak{g}$ and $\langle \lambda, \alpha \rangle > 0$ ($\langle \lambda, \alpha \rangle < 0$, respectively) then $\lambda - k\alpha$ ($\lambda + k\alpha$, respectively) is also a weight of that representation for $k = 1, \ldots, |\langle \lambda, \alpha \rangle|$.

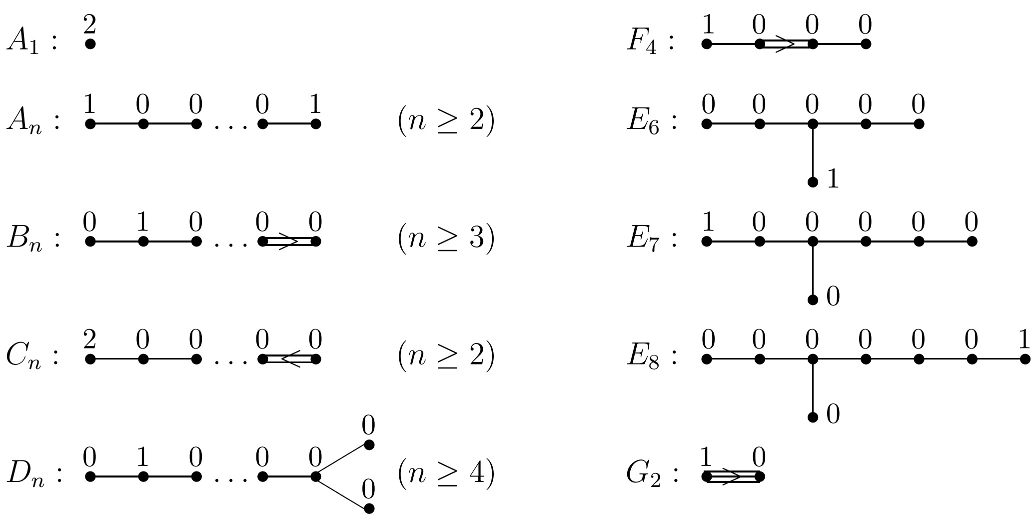
For any root $\alpha \in \Delta$, denote by σ_α the orthogonal reflection of $\mathfrak{t}^*$ in the hyperplane perpendicular to α. The *Weyl group* W of $\mathfrak{g}$ is the group generated by all σ_α. W is always finite. If $\mathfrak{g}$ is simple then W acts irreducibly on $\mathfrak{t}^*$. Moreover, W acts transitively on the set of roots of equal length, and the set of weights Φ of any irreducible representation is W-invariant.

A weight $\lambda \in \Phi$ of an irreducible representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is called *extremal* if it lies in the W-orbit of the dominant weight. Two weights $\lambda, \mu \in \Phi$ are said to have *opposite sign* if for all roots α we have $\langle \lambda, \alpha \rangle \langle \mu, \alpha \rangle \le 0$. It is known that for every extremal weight λ there is always an extremal weight μ of opposite sign.

For any two simple roots $\alpha_i, \alpha_j \in S$, it turns out that $\langle \alpha_i, \alpha_j \rangle \le 0$. To a simple basis S, we associate the *Dynkin diagram of* $\mathfrak{g}$ by representing each $\alpha_i \in S$ as a node, and to join the nodes of α_i and α_j by $|\langle \alpha_i, \alpha_j \rangle|$ edges. If $|\langle \alpha_i, \alpha_j \rangle| > 1$ then α_i, α_j have different lengths, and we draw an arrow from the longer to the shorter root.

Any integral weight λ of $\mathfrak{g}$ can be graphically represented by inscribing the integer $\langle \lambda, \alpha_i \rangle$ over the node of the Dynkin diagram corresponding to α_i. In particular, we can represent any irreducible representation ρ of $\mathfrak{g}$ by inscribing the integers of the dominant weight on the nodes of the Dynkin diagram of $\mathfrak{g}$.

If $\mathfrak{g}$ is simple, then the adjoint representation $\rho : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is irreducible. Its dominant weight is called the *maximal root* of $\mathfrak{g}$. The following is the list of all Dynkin diagrams of simple Lie algebras, together with their maximal roots:



(17)

It is worth pointing out that from this list it follows that $|\langle \alpha, \beta \rangle| \le 3$ for all roots $\alpha, \beta \in \Delta$, and $|\langle \alpha, \beta \rangle| = 3$ occurs iff $\mathfrak{g}$ contains $\mathfrak{g}_2$ as a direct summand. If this is *not* the case, then the following conditions hold for all $\alpha, \beta \in \Delta$:

$$\alpha + 3\beta \text{ is not a root.} \tag{18}$$

$$|\langle \beta, \alpha \rangle| \le 2; \text{ if } \alpha \text{ is a long root then equality holds iff } \alpha = \pm\beta. \tag{19}$$

$$\text{if } \alpha \text{ is a long root then } 2\alpha + \beta \text{ is a root iff } \beta = -\alpha. \tag{20}$$

Finally, we shall need the following definition.

Definition 2.6 *Two representations $\rho_1, \rho_2 : \mathfrak{g} \to End(V)$ are called* conjugate *if their images $\rho_i(\mathfrak{g}) \subset End(V)$ are conjugate to each other.*

It is then well known that two representations are conjugate to each other iff there is an isomorphism $\imath : \mathfrak{g} \to \mathfrak{g}$ such that ρ_1 and $\rho_2 \circ \imath$ are equivalent representations. In terms of the Dynkin diagram notation this means that two representations are conjugate if their coefficients coincide after possibly applying a symmetry of the corresponding Dynkin diagram.

Thus, in the context of the holonomy problem we only need to classify the representations up to conjugacy.

Definition 2.7 *Let V be a complex vector space and let $G \subset Aut(V)$ be an irreducible complex Lie subgroup with corresponding Lie algebra $\mathfrak{g} \subset End(V)$. Then the* sky *of G is $\tilde{X} := G \cdot x_0 \subset V$ where x_0 is a dominant weight vector. The* projectivized sky *is the subset $X := \pi(\tilde{X}) \subset \mathbb{P}(V)$, where $\pi : V \backslash \{0\} \to \mathbb{P}(V)$ is the natural projection.*

It is well known that for any irreducible complex $\mathrm{G} \subset \mathrm{Aut}(V)$ the projectivized sky is a compact complex homogeneous space and can be written as $X = \mathrm{G}/P$ where $P \subset \mathrm{G}$ is a parabolic subgroup. [BasE]

3 Irreducible Berger algebras

3.1 Real Berger algebras

In this subsection we shall use the following notation: if W is a *complex* vector space, then we denote the Lie algebras of real and complex endomorphisms of W by $\mathrm{End}_{\mathbb{R}}(W)$ and $\mathrm{End}_{\mathbb{C}}(W)$, respectively.

Let V be a finite dimensional real vector space, and let $\mathfrak{h} \subset \mathrm{End}_{\mathbb{R}}(V)$ be a real Lie subalgebra. We denote their complexifications by $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{h}_{\mathbb{C}} := \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$. Then obviously, $\mathfrak{h}_{\mathbb{C}} \subset \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$, and by complexifying the exact sequences (5) and (6), we obtain

$$K(\mathfrak{h}_{\mathbb{C}}) = K(\mathfrak{h}) \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad K^1(\mathfrak{h}_{\mathbb{C}}) = K^1(\mathfrak{h}) \otimes_{\mathbb{R}} \mathbb{C}.$$

In particular, $\mathfrak{h} \subset \mathrm{End}_{\mathbb{R}}(V)$ is a (symmetric respectively non-symmetric) Berger algebra iff $\mathfrak{h}_{\mathbb{C}} \subset \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ is.

Let us now assume that $\mathfrak{h} \subset \mathrm{End}_{\mathbb{R}}(V)$ is *irreducible*. Then there are two cases to be distinguished.

If $\mathfrak{h}$ is *absolutely irreducible*, i.e. $\mathfrak{h}_{\mathbb{C}} \subset \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ is also irreducible, then from the above it follows that $\mathfrak{h}$ is a Berger algebra iff $\mathfrak{h}_{\mathbb{C}}$ is an irreducible Berger algebra.

Next, suppose that $\mathfrak{h}$ is *not* absolutely irreducible, i.e. $\mathfrak{h}_{\mathbb{C}} \subset \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ is *not* irreducible. This happens iff there is a complex structure J on V which commutes with the elements of $\mathfrak{h}$. That is, $\mathfrak{h} \subset \mathrm{End}_{\mathbb{C}}(V)$ w.r.t. this complex structure J. In this case, $V_{\mathbb{C}} = W \oplus \overline{W}$ decomposes into two irreducible $\mathfrak{h}_{\mathbb{C}}$-submodules of equal dimension given by

$$W = \{x + iJx \mid x \in V\} \;\text{ and }\; \overline{W} = \{x - iJx \mid x \in V\}.$$

Let $\mathfrak{h}_1 := \{A \in \mathfrak{h} \mid JA \in \mathfrak{h}\}$. Then $\mathfrak{h}_1 \lhd \mathfrak{h}$, and J induces a complex Lie algebra structure on $\mathfrak{h}_1$; $(\mathfrak{h}_1)_{\mathbb{C}}$ can be written as the direct sum of complex Lie algebras $(\mathfrak{h}_1)_{\mathbb{C}} = \mathfrak{h}_1^+ \oplus \mathfrak{h}_1^-$ with

$$\mathfrak{h}_1^+ = \{A + iJA \mid A \in \mathfrak{h}_1\} \quad \text{and} \quad \mathfrak{h}_1^- = \{A - iJA \mid A \in \mathfrak{h}_1\}.$$

Let $R \in K(\mathfrak{h}_{\mathbb{C}})$. Then for $u, v \in W$ and $\overline{w} \in \overline{W}$ the first Bianchi identity implies that $R(u,v)\overline{w} = 0$. Since this is true for all $\overline{w} \in \overline{W}$, it follows that $R(u,v) \in \mathfrak{h}_1^+$. Likewise, for $\overline{u}, \overline{v} \in \overline{W}$, we have $R(\overline{u}, \overline{v}) \in \mathfrak{h}_1^-$.

Next, for any $R \in K(\mathfrak{h}_{\mathbb{C}})$ the first Bianchi identity also implies that $R(\overline{u}, v)w = R(\overline{u}, w)v$ for all $\overline{u} \in \overline{W}$, $v, w \in W$. Thus, we have a map

$$\overline{W} \longrightarrow (\mathfrak{h}_{\mathbb{C}}|_W)^{(1)}, \quad \overline{u} \longmapsto R(\overline{u}, \_).$$

If $(\mathfrak{h}_{\mathbb{C}}|_W)^{(1)} = 0$ then this implies that $R(W, \overline{W}) = 0$, and hence by the above, $R(V_{\mathbb{C}}, V_{\mathbb{C}}) \subset \mathfrak{h}_1^+ \oplus \mathfrak{h}_1^- = (\mathfrak{h}_1)_{\mathbb{C}}$ for all $R \in K(\mathfrak{h}_{\mathbb{C}})$, that is, $\underline{\mathfrak{h}}_{\mathbb{C}} \subset (\mathfrak{h}_1)_{\mathbb{C}}$. Hence $\mathfrak{h}_{\mathbb{C}}$ is not Berger unless $\mathfrak{h}_1 = \mathfrak{h}$, i.e. $\mathfrak{h}$ is a complex Lie algebra which acts irreducibly on the complex vector space V.

We define a map $\imath : \mathfrak{h}_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(V)$ by

$$\imath(A + iB) := A + JB. \tag{21}$$

In fact, it is easy to see that $\imath(\mathfrak{h}_{\mathbb{C}}) \subset \mathrm{End}_{\mathbb{C}}(V)$ is congruent to $(\mathfrak{h}_{\mathbb{C}})|_W \subset \mathrm{End}_{\mathbb{C}}(W)$, and hence $(\mathfrak{h}_{\mathbb{C}}|_W)^{(1)} = 0$ iff $(\imath(\mathfrak{h}_{\mathbb{C}}))^{(1)} = 0$. Thus, we obtain the following.

Proposition 3.1 *Let V be a finite dimensional real vector space, and let $\mathfrak{h} \subset End_{\mathbb{R}}(V)$ be an irreducible real subalgebra with complexification $\mathfrak{h}_{\mathbb{C}} \subset End_{\mathbb{C}}(V_{\mathbb{C}})$.*

1. *If $\mathfrak{h}$ is absolutely irreducible, i.e. if there is no complex structure on V which commutes with the elements of $\mathfrak{h}$, then $\mathfrak{h}$ is a Berger algebra iff $\mathfrak{h}_{\mathbb{C}} \subset End_{\mathbb{C}}(V_{\mathbb{C}})$ is an irreducible Berger algebra.*

2. *If $\mathfrak{h}$ is not absolutely irreducible, i.e. if there is a complex structure J on V which commutes with the elements of $\mathfrak{h}$, and if the subalgebra $\imath(\mathfrak{h}_{\mathbb{C}}) \subset End_{\mathbb{C}}(V)$ given by (21) satisfies $(\imath(\mathfrak{h}_{\mathbb{C}}))^{(1)} = 0$, then $\mathfrak{h}$ is a Berger algebra iff $J\mathfrak{h} = \mathfrak{h}$ and $\mathfrak{h} \subset End_{\mathbb{C}}(V)$ is a complex irreducible Berger algebra.*

Thus, in order to classify all Berger algebras we need to classify all irreducible *complex* Berger subalgebras $\mathfrak{h}_{\mathbb{C}} \subset \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$, add all their absolutely irreducible real forms, and finally, to investigate the real algebras $\mathfrak{h} \subset \mathrm{End}_{\mathbb{C}}(V)$ for which $\imath(\mathfrak{h}_{\mathbb{C}}) \subset \mathrm{End}_{\mathbb{C}}(V)$ is one of the entries of Table 4. The latter task has been completed by Bryant [Br4]; hence, we shall mainly concern ourselves with the investigation of complex Berger algebras.

3.2 Examples of Berger algebras

3.2.1 Conformal Lie algebras

Let $(V, \langle\ ,\ \rangle)$ be a real or complex vector space with the symmetric bilinear form $\langle\ ,\ \rangle$, let $\mathfrak{so}(V)$ be the Lie algebra of endomorphisms preserving $\langle\ ,\ \rangle$ and $\mathfrak{co}(V) := span(Id_V, \mathfrak{so}(V))$. We have $\mathfrak{so}(V) \cong \Lambda^2 V$, with an isomorphism given by

$$(x \wedge y) \cdot z := \langle x, z\rangle\, y - \langle y, z\rangle\, x.$$

We use $\langle\ ,\ \rangle$ to identify V and V^*. With this, an element of $K(\mathfrak{so}(V))$ may be regarded as a map $R : \Lambda^2 V \to \Lambda^2 V$, and an easy calculation involving the first Bianchi identity shows that each such $R \in K(\mathfrak{so}(n, \mathbb{C}))$ is *symmetric* w.r.t. the inner product on $\Lambda^2 V$ induced by $\langle\ ,\ \rangle$, i.e. $K(\mathfrak{so}(V)) \subset \odot^2 \mathfrak{so}(V) \subset \Lambda^2 V \otimes \mathfrak{so}(V)$. But the image of the restriction $\delta^{1,2} : \odot^2 \mathfrak{so}(V) \to \Lambda^3 V \otimes V$ equals $\Lambda^4 V$, and hence we have

$$K(\mathfrak{so}(V)) \cong (\odot^2 \Lambda^2 V)/\Lambda^4 V.$$

We define the map $\tau : K(\mathfrak{co}(V)) \to \mathfrak{so}(V)$ by the equation $tr(R(x, y)) = \langle \tau(R)x, y\rangle$ for all $x, y \in V$ and $R \in K(\mathfrak{co}(V))$. Clearly, the kernel of τ is $K(\mathfrak{so}(V))$. Moreover, one checks that for each $A \in \mathfrak{so}(V)$, the map

$$R_A(x, y) := \langle Ax, y\rangle\, Id_V + \frac{1}{2}(Ax \wedge y - Ay \wedge x)$$

lies in $K(\mathfrak{co}(V))$, and $\tau(R_A) = nA$. Therefore, τ is surjective, and if we let $K^c(V) := \{R_A \mid A \in \mathfrak{so}(V)\}$, then

$$K(\mathfrak{co}(V)) \cong K(\mathfrak{so}(V)) \oplus K^c(V).$$

Proposition 3.2 *Let $\mathfrak{h} \subset \mathfrak{so}(V, \langle\ ,\ \rangle)$ be a proper irreducible Lie subalgebra where V is an n-dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with $n \geq 3$, $n \neq 4$. Then $K(\mathfrak{h} \oplus \mathbb{F} Id_V) = K(\mathfrak{h})$. In particular, $\mathfrak{h} \oplus \mathbb{F} Id_V$ is not a Berger algebra.*

For the proof, we shall need the following Lemma.

Lemma 3.3 *Let $\mathfrak{g}$ be a simple Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a proper semi-simple subalgebra. Moreover, let $W \subset \mathfrak{g}$ be a linear subspace such that $[\mathfrak{h}, W] \subset W$ and $[\mathfrak{h}^\perp, W] \subset \mathfrak{h}$. Then either $W = 0$ or $W = \mathfrak{h}^\perp$ in which case $(\mathfrak{g}, \mathfrak{h})$ is an irreducible symmetric pair.*

Proof. Let $h + v \in W$ with $h \in \mathfrak{h}$ and $v \in \mathfrak{h}^\perp$, and let $h' \in \mathfrak{h}$. Consider the map $\tau := ad(v) \circ ad(h') : \mathfrak{g} \to \mathfrak{g}$. By definition of the Killing form, we have $tr(\tau) = B(v, h') = 0$. Clearly, $\tau(\mathfrak{h}) \subset \mathfrak{h}^\perp$, and hence $tr(\tau) = tr(\sigma)$ with $\sigma = pr_{\mathfrak{h}^\perp} \circ ad(v)|_{\mathfrak{h}^\perp} \circ ad(h')|_{\mathfrak{h}^\perp}$ and where $pr_{\mathfrak{h}^\perp} : \mathfrak{g} \to \mathfrak{h}^\perp$ is the orthogonal projection. Now, for $v' \in \mathfrak{h}^\perp$, we have

$$\sigma(v') = pr_{\mathfrak{h}^\perp}([(h + v) - h, [h', v']]) = -[h, [h', v']],$$

since $[h + v, [h', v']] \in [W, \mathfrak{h}^\perp] \subset \mathfrak{h}$ and $[h, [h', v']] \in \mathfrak{h}^\perp$. Therefore, $\sigma = -ad(h)|_{\mathfrak{h}^\perp} \circ ad(h')|_{\mathfrak{h}^\perp}$, and thus, $tr(\sigma) = -cB_{\mathfrak{h}}(h, h')$ for some constant $c > 0$ and where $B_{\mathfrak{h}}$ is the Killing form on $\mathfrak{h}$. Thus, $B_{\mathfrak{h}}(h, h') = 0$ for all $h' \in \mathfrak{h}$, and hence $h = 0$, i.e. $W \subset \mathfrak{h}^\perp$.

Suppose that $W \neq 0$. Then there is an $\mathfrak{h}$-invariant decomposition $\mathfrak{h}^\perp = V_1 \oplus V_2$ such that $0 \neq V_1 \subset W$ and V_1 is irreducible. Thus, $[V_1, V_2] \subset [W, \mathfrak{h}^\perp] \subset \mathfrak{h}$. On the other hand, for $v_i \in V_i$ and $h \in \mathfrak{h}$, we have $B([v_1, v_2], h) = B(v_1, [v_2, h]) = 0$, since $[v_2, h] \in V_2$. Therefore, $[V_1, V_2] = 0$.

Also, $[V_1, V_1] \subset [W, \mathfrak{h}^\perp] \subset \mathfrak{h}$, and from there it follows that $[V_1, V_1] \oplus V_1 \lhd \mathfrak{g}$. Since $\mathfrak{g}$ is simple and $V_1 \neq 0$, this implies that $W = V_1 = \mathfrak{h}^\perp$ is $\mathfrak{h}$-irreducible and $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] = \mathfrak{h}$. ∎

Proof of Proposition 3.2. We have $K(\mathfrak{h} \oplus \mathbb{F}Id_V) \subset K(\mathfrak{co}(V))$, and we let $W \subset \mathfrak{so}(V)$ be the image of $K(\mathfrak{h} \oplus \mathbb{F}Id_V)$ under the natural projection $K(\mathfrak{co}(V)) \to K^c(V) \cong \mathfrak{so}(V)$. Clearly, W is $\mathfrak{h}$-invariant, i.e. $[\mathfrak{h}, W] \subset W$. We need to show that $W = 0$.

We identify $\Lambda^2 V$ and $\mathfrak{so}(V)$ as before, and denote the induced inner product on $\Lambda^2 V$ by $(\ , \ )$. Then every $\underline{R} \in K(\mathfrak{h} \oplus \mathbb{F}Id)$ can be written as $\underline{R}(\alpha) = (A, \alpha)Id + \frac{1}{2}[A, \alpha] + R(\alpha)$ for all $\alpha \in \mathfrak{so}(V)$, where $A \in W$, $R \in K(\mathfrak{so}(V)) \subset \odot^2 \mathfrak{so}(V)$ and where $\frac{1}{2}[A, \alpha] + R(\alpha) \in \mathfrak{h}$ for all $\alpha \in \mathfrak{so}(V)$.

Let $\alpha, \beta \in \mathfrak{h}^\perp \subset \mathfrak{so}(V)$. Then since $R \in \odot^2 \mathfrak{so}(V)$, we have $0 = (R(\alpha), \beta) - (\alpha, R(\beta)) = \frac{1}{2}(-([A, \alpha], \beta) + (\alpha, [A, \beta])) = -([A, \alpha], \beta)$, and hence, $[\mathfrak{h}^\perp, W] \subset \mathfrak{h}$.

Since $\mathfrak{so}(V)$ is simple, Lemma 3.3 implies that either $W = 0$, or $W = \mathfrak{h}^\perp$ and $(\mathfrak{so}(V), \mathfrak{h})$ is a symmetric pair. If the latter is the case, then the symmetric reflection map $\sigma : \mathfrak{so}(V) \to \mathfrak{so}(V)$ with $\sigma|_\mathfrak{h} = Id_\mathfrak{h}$ and $\sigma|_{\mathfrak{h}^\perp} = -Id_{\mathfrak{h}^\perp}$ is an automorphism of $\mathfrak{so}(V)$ of order 2. It is known that any such automorphism is of the form $\sigma = Ad_g$ for some $g \in O(V)$. Since $\mathfrak{h}$ acts irreducibly on V and $\sigma|_\mathfrak{h} = Id_\mathfrak{h}$, Schur's Lemma implies that either $g = \lambda Id_V$, some $\lambda \in \mathbb{F}$, or V is real and g an orthogonal complex structure on V.

In the first case, $\sigma = Id_{\mathfrak{so}(V)}$ and hence $\mathfrak{h} = \mathfrak{so}(V)$ which was excluded. In the second case, $\mathfrak{h} = \mathfrak{u}(V, g) \subset \mathfrak{sp}(V, \Omega)$, where $\Omega(x, y) := \langle x, gy \rangle$. But we shall see in Lemma 3.5 that $\mathfrak{h} \subset \mathfrak{sp}(V, \Omega)$ implies that $K(\mathfrak{h} \oplus \mathbb{F}Id) = K(\mathfrak{h})$, thus $W = 0$. ∎

3.2.2 Symmetric connections

In this section, we want to discuss the existence of $\mathfrak{h}$-invariant elements of $K(\mathfrak{h})$. As it turns out, any such element can be realized as the holonomy of a symmetric connection. More precisely, we have the following result.

Proposition 3.4 *[He] Let V be a complex vector space with $\dim V > 2$, and let $\mathfrak{h} \subset End(V)$ be an irreducible complex subalgebra with semi-simple part $\mathfrak{h}_s$. Suppose there is an $\mathfrak{h}_s$-invariant element $0 \neq R \in K(\mathfrak{h})$. Then the following hold.*

1. *$\mathfrak{h}_s \subset \mathfrak{so}(V, \langle \ , \ \rangle)$ and $\mathfrak{h} \subset \mathfrak{co}(V, \langle \ , \ \rangle)$ for some symmetric bilinear form $\langle \ , \ \rangle$ on V.*

2. *$\{R(x, y) \mid x, y \in V\} = \mathfrak{h}_s$.*

3. *There is an irreducible symmetric pair $(\mathfrak{g}, \mathfrak{h}_s)$ whose curvature is given by R.*

4. *If $\mathfrak{h}_s$ is simple then R is unique up to scalar multiples.*

Proof. Let $0 \neq R \in K(\mathfrak{h})$ be $\mathfrak{h}_s$-invariant. Then the 2-form $\Omega(x, y) := trR(x, y)$ is also $\mathfrak{h}_s$-invariant. By Schur's Lemma, if $\Omega \neq 0$, then Ω is non-degenerate and $\mathfrak{h}_s \subset \mathfrak{sp}(V, \Omega)$. But by Lemma 3.5, this implies that $\Omega \wedge \Omega = 0$ which is impossible since $\dim V > 2$.

Therefore, $\Omega = 0$ and thus, $R(x, y) \in \mathfrak{h}_s$ for all $x, y \in V$. The direct sum $\mathfrak{g} := \mathfrak{h}_s \oplus V$ can be given a Lie algebra structure by the bracket

$$[h_1 + x, h_2 + y] := ([h_1, h_2] + R(x, y)) + (h_1 y - h_2 x) \quad \text{for all } h_1, h_2 \in \mathfrak{h}_s \text{ and } x, y \in V.$$

Indeed, it is straightforward to verify that this bracket satisfies the Jacobi identity iff R is $\mathfrak{h}_s$-invariant. Thus, for the bracket on $\mathfrak{g}$ the following holds:

$$[\mathfrak{h}_s, \mathfrak{h}_s] \subset \mathfrak{h}_s, \quad [\mathfrak{h}_s, V] \subset V, \quad [V, V] \subset \mathfrak{h}_s. \tag{22}$$

Let $\mathfrak{h}_s = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$ be the decomposition of $\mathfrak{h}_s$ into its simple components, and let $\imath : \mathfrak{h}_s \hookrightarrow \mathrm{End}(V)$ be the inclusion map. We define a symmetric bilinear form on $\mathfrak{h}_s$ by the formula

$$(h_1, h_2) := tr(\imath(h_1) \circ \imath(h_2)) \;\text{ for all } h_1, h_2 \in \mathfrak{h}_s.$$

Clealry, $(\ ,\ )$ is $ad_{\mathfrak{h}_s}$-invariant, and it is not hard to show that

$$(h_1, h_2) = c_1 B_1(h_1, h_2) + \ldots + c_k B_k(h_1, h_2)$$

for some constants $c_i > 0$ and where B_i denotes the Killing form of $\mathfrak{h}_i$. If $B_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$, then from (22) we get for all $h_1, h_2 \in \mathfrak{h}_s$

$$\begin{aligned} B_{\mathfrak{g}}(h_1, h_2) &= B_{\mathfrak{h}}(h_1, h_2) + (h_1, h_2) \\ &= (c_1 + 1) B_1(h_1, h_2) + \ldots + (c_k + 1) B_k(h_1, h_2), \end{aligned}$$

and

$$B_{\mathfrak{g}}(\mathfrak{h}_s, V) = 0.$$

Thus, in particular, the restriction of $B_{\mathfrak{g}}$ to $\mathfrak{h}_s \subset \mathfrak{g}$ is non-degenerate. Therefore, if $B_{\mathfrak{g}}|_V = 0$ then V is the null-space of $B_{\mathfrak{g}}$, and hence $V \lhd \mathfrak{g}$. However, (22) then would imply that $R = 0$.

Thus, the restriction $B_{\mathfrak{g}}|_V$ yields a non-vanishing $\mathfrak{h}_s$-invariant symmetric bilinear form on V, and hence Schur's Lemma implies that $B_{\mathfrak{g}}|_V$ is non-degenerate and $\mathfrak{h}_s \subset \mathfrak{so}(V, B_{\mathfrak{g}})$. Also, $B_{\mathfrak{g}}$ is non-degenerate which means that $\mathfrak{g}$ is semi-simple.

Let $\mathfrak{h}' := \{R(x, y) \mid x, y \in V\} \lhd \mathfrak{h}_s$ and hence there is a decomposition $\mathfrak{h}_s = \mathfrak{h}' \oplus \mathfrak{h}''$. But then, it is obvious from (22) that $(\mathfrak{h}' \oplus V) \lhd \mathfrak{g}$ which implies $[\mathfrak{h}'', V] = 0$, and therefore, $\mathfrak{h}'' = 0$, and $(\mathfrak{g}, \mathfrak{h}_s)$ is an irreducible symmetric pair whose curvature is given by R.

The last assertion follows since $R \in K(\mathfrak{h}_s \cap \mathfrak{so}(V)) \subset \odot^2 \mathfrak{h}_s$, and if $\mathfrak{h}_s$ is simple then the only $\mathfrak{h}_s$-invariant elements of $\odot^2 \mathfrak{h}_s$ are the multiples of the Killing form. ∎

3.2.3 Symplectic Lie algebras

Let Ω be a non-degenerate 2-form on V, let $\mathfrak{sp}(V, \Omega)$ be the Lie algebra of linear endomorphisms of V preserving Ω, and let $\mathfrak{csp}(V, \Omega) = span(Id_V, \mathfrak{sp}(V, \Omega))$. We have $\mathfrak{sp}(V, \Omega) \cong \odot^2 V$, with an isomorphism given by

$$(xy) \cdot z := \Omega(x, z) y + \Omega(y, z) x. \tag{23}$$

We use Ω to identify V and V^*.

For $\mathfrak{h} = \mathfrak{sp}(V, \Omega)$, it is known that $H^{1,2}(\mathfrak{h}) = 0$ [Br4, p.37], and hence the map $\mathfrak{h}^{(1)} \otimes V^* \to K(\mathfrak{h})$ from (8) is surjective. From Table 4 we see that $K(\mathfrak{sp}(V, \Omega)) \cong (\odot^3 V \otimes V)/\odot^4 V$, with an explicit isomorphism being induced by

$$\begin{aligned} \odot^3 V \otimes V &\longrightarrow K(\mathfrak{sp}(V, \Omega)) \\ \tau &\longmapsto R_\tau, \end{aligned}$$

where R_τ is determined by $\Omega(R_\tau(x, y) z, w) = \tau(xzw, y) - \tau(yzw, x)$.

Lemma 3.5 *Let $R \in K(\mathfrak{csp}(V,\Omega))$ be given by $R(x,y) = \rho(x,y)Id_V + \underline{R}(x,y)$ for some $\rho \in \Lambda^2 V^*$ and $\underline{R} \in \Lambda^2 V^* \otimes \mathfrak{sp}(n,\mathbb{C})$. Then $\rho \wedge \Omega = 0$.*

If $\dim V \geq 6$ then $K(\mathfrak{csp}(V,\Omega)) = K(\mathfrak{sp}(V,\Omega))$ and hence, $\mathfrak{csp}(V,\Omega)$ is not a Berger algebra. If $\dim V = 4$ then $K(\mathfrak{csp}(V,\Omega)) = K(\mathfrak{sp}(V,\Omega)) \oplus (\Lambda^2 V)/\Omega$.

Proof. Let $R \in K(\mathfrak{csp}(V,\Omega))$ be given as above, and let $\tau(x,y,z,w) := \Omega(R(x,y)z,w) - \Omega(R(x,y)w,z)$. Then $\tau(x,y,z,w) = 2\rho(x,y)\Omega(z,w)$, and the first Bianchi identity implies that $\rho \wedge \Omega = 0$ as claimed. The second assertion follows immediately.

Finally, one verifies that for each $\rho \in \Lambda^2 V^*$ with $\rho \wedge \Omega = 0$, the element R_ρ given by

$$R_\rho(x,y) = 4\rho(x,y)Id_V + \underline{R}(x,y),$$

$$\Omega(\underline{R}(x,y)z,w) = \rho(x,z)\Omega(y,w) + \rho(x,w)\Omega(y,z) - \rho(y,z)\Omega(x,w) - \rho(y,w)\Omega(x,z),$$

lies in $K(\mathfrak{csp}(V))$, and this shows the last assertion. ∎

The following gives a construction of symplectic Berger groups and algebras.

Theorem 3.6 *Let $\mathfrak{h} \subset End(V)$ be an irreducible semi-simple Lie subalgebra where V is a finite dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $W := \mathbb{F}^2 \otimes V$ and consider the induced tensor representation of $\mathfrak{h}^+ := \mathfrak{sl}(2,\mathbb{F}) \oplus \mathfrak{h} \subset End(W)$. Then the following are equivalent.*

1. *There is an irreducible symmetric pair $(\mathfrak{g}, \mathfrak{h}^+)$ whose isotropy representation is equivalent to the representation of $\mathfrak{h}^+$ on W.*

2. *There is an $\Omega \in \Lambda^2 V$ such that $\mathfrak{h} \subset \mathfrak{sp}(V,\Omega)$, and an $\mathfrak{h}$-equivariant map $\circ : \odot^2 V \to \mathfrak{h}$ which satisfies for all $x,y,z \in V$*

$$(x \circ y)z - (x \circ z)y = 2\ \Omega(y,z)x + \Omega(x,z)y - \Omega(x,y)z. \tag{24}$$

3. *There is an $\Omega \in \Lambda^2 V$ such that $\mathfrak{h} \subset \mathfrak{sp}(V,\Omega)$, and an $\mathfrak{h}$-equivariant map $\circ : \odot^2 V \to \mathfrak{h}$ such that for $A \in \mathfrak{h}$ the map $R_A : \Lambda^2 V \to \mathfrak{h}$ given by*

$$R_A(x,y) = 2\ \Omega(x,y)\ A + x \circ (Ay) - y \circ (Ax)$$

lies in $K(\mathfrak{h})$.

If these conditions are satisfied then the map $\mathfrak{h} \to K(\mathfrak{h})$, $A \mapsto R_A$ is injective, thus $\mathfrak{h}$ is an irreducible Berger algebra.

Proof. Let $R : \Lambda^2 W \to \mathfrak{h}^+ = \mathfrak{sl}(2,\mathbb{F}) \oplus \mathfrak{h}$. Since $\Lambda^2 W = \odot^2 V \oplus \mathfrak{sl}(2,\mathbb{F}) \otimes \Lambda^2 V$ as an $\mathfrak{h}^+$-module, it follows that R is $\mathfrak{h}^+$-equivariant iff there is an $\mathfrak{h}$-invariant $\Omega \in \Lambda^2 V$ and an $\mathfrak{h}$-equivariant map $\circ : \odot^2 V \to \mathfrak{h}$ such that

$$R(e \otimes x, f \otimes y) = \Omega(x,y)ef + \langle e,f \rangle\, x \circ y$$

for all $e, f \in \mathbb{F}^2$ and $x, y \in V$, where $\langle \_, \_ \rangle$ is the determinant on $\mathbb{F}^2$. Moreover, (24) is equivalent to the first Bianchi identity for R, i.e. to $R \in K(\mathfrak{h}^+)$. Thus, the first and second statements of the Theorem are equivalent by Proposition 3.4.

The equivalence of the second and third statement follows from an easy calculation, and evidently, $R_A = 0$ only if $A = 0$. ∎

From the classification of irreducible symmetric spaces [Ber2], we immediately get the following

Corollary 3.7 *The images of the following representations are Berger subgroups:*

Group H	Representation space	Group H	Representation space
$\mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^4 \simeq \odot^3\mathbb{R}^2$	E_7^5	$\mathbb{R}^{56}$
$\mathrm{SL}(2,\mathbb{C})$	$\mathbb{C}^4 \simeq \odot^3\mathbb{C}^2$	E_7^7	$\mathbb{R}^{56}$
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^{2(p+q)}$, $(p+q) \geq 3$	$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
$\mathrm{SL}(2,\mathbb{C}) \cdot \mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^{2n}$, $n \geq 3$	$\mathrm{Spin}(2,10)$	$\mathbb{R}^{32}$
$\mathrm{Sp}(1)\mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n \simeq \mathbb{R}^{4n}$, $n \geq 2$	$\mathrm{Spin}(6,6)$	$\mathbb{R}^{32}$
$\mathrm{SL}(6,\mathbb{R})$	$\mathbb{R}^{20} \simeq \Lambda^3\mathbb{R}^6$	$\mathrm{Spin}(6,\mathbb{H})$	$\mathbb{R}^{32}$
$\mathrm{SU}(1,5)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$	$\mathrm{Spin}(12,\mathbb{C})$	$\mathbb{C}^{32}$
$\mathrm{SU}(3,3)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$	$\mathrm{Sp}(3,\mathbb{R})$	$\mathbb{R}^{14} \subset \Lambda^3\mathbb{C}^6$
$\mathrm{SL}(6,\mathbb{C})$	$\mathbb{C}^{20} \simeq \Lambda^3\mathbb{C}^6$	$\mathrm{Sp}(3,\mathbb{C})$	$\mathbb{C}^{14} \subset \Lambda^3\mathbb{C}^6$

Historically, these holonomy groups were first discovered via the twistor approach which we shall explain in more detail in section 5. From these calculations, one can also determine the space $K(\mathfrak{h})$ and obtains the following result (cf. [MeSc1, ch.4]).

Proposition 3.8 *For all Berger algebras listed in Corollary 3.7 we have $K(\mathfrak{h}) \cong \mathfrak{h}$, i.e. the injective map $\mathfrak{h} \to K(\mathfrak{h})$ from Theorem 3.6 is an isomorphism.*

3.2.4 Complex Lie algebras with $\mathfrak{h}^{(1)} \neq 0$

These are the entries of Table 4. The entries $6, 7$ and 8 have been discussed in the previous sections already.

Throughout this section, we write $\mathfrak{gl}(W)$ for $\mathrm{End}(W)$, and let $\mathfrak{sl}(W) \subset \mathfrak{gl}(W)$ be the Lie algebra of traceless endomorphisms.

The representations corresponding to entries $3, 4, 5, 9$ and 10 of Table 4

For all these, the exact sequence (8) implies that $K(\mathfrak{h}) \cong V^* \otimes \mathfrak{h}^{(1)} \cong V^* \otimes V^*$. We shall prove in each case that $K(\mathfrak{h} \cap \mathfrak{sl}(V)) \cong \odot^2 V^* \subset K(\mathfrak{h})$.

Item 3 corresponds to the action of $\mathfrak{h} = \mathfrak{gl}(W)$ on $V := \odot^2 W$. An explicit isomorphism $K(\mathfrak{gl}(W)) \to V^* \otimes \mathfrak{gl}(W)^{(1)} \cong V^* \otimes V^*$ is given by

$$R_\tau(rs, tu) \cdot x := \tau(rx, tu)s + \tau(sx, tu)r - \tau(tx, rs)u - \tau(ux, rs)t$$

for all $r, s, t, u, x \in W$ and where $\tau \in V^* \otimes V^*$. In particular, since $tr\ R(rs, tu) = 2(\tau(rs, tu) - \tau(tu, rs))$, the claim for $K(\mathfrak{sl}(W))$ follows.

Likewise, we get for item 4 which is the representation of $\mathfrak{h} = \mathfrak{gl}(W)$ on $V := \Lambda^2 W$ the explicit isomorphism $K(\mathfrak{gl}(W)) \to V^* \otimes (\mathfrak{gl}(W))^{(1)} \cong V^* \otimes V^*$ by the explicit isomorphism

$$R_\tau(r \wedge s, t \wedge u) \cdot x := \tau(r \wedge x, t \wedge u)s - \tau(s \wedge x, t \wedge u)r - \tau(t \wedge x, r \wedge s)u + \tau(u \wedge x, r \wedge s)t$$

for all $r, s, t, u, x \in W$ and $\tau \in V^* \otimes V^*$. Again, $tr\ R_\tau(r \wedge s, t \wedge u) = 2(\tau(r \wedge s, t \wedge u) - \tau(t \wedge u, r \wedge s))$, thus $K(\mathfrak{sl}(W)) \cong \odot^2 V^*$.

In item 5, we consider the tensor representation of $\mathfrak{h} := \mathfrak{gl}(V_1) \oplus \mathfrak{gl}(V_2)$ on $V := V_1 \otimes V_2$. Then $K(\mathfrak{h}) \cong V^* \otimes V^*$, with an explicit isomorphism given by $\tau \in V^* \otimes V^* \mapsto \phi^\tau \in K(\mathfrak{h})$ with

$$\begin{aligned} &\phi^\tau = \phi_1^\tau + \phi_2^\tau \\ &\phi_1^\tau(e_1 \otimes u_1, e_2 \otimes u_2)\ e_3 = \tau(e_1, u_1, e_3, u_2)e_2 - \tau(e_2, u_2, e_3, u_1)e_1 \\ &\phi_2^\tau(e_1 \otimes u_1, e_2 \otimes u_2)\ u_3 = \tau(e_1, u_1, e_2, u_3)u_2 - \tau(e_2, u_2, e_1, u_3)u_1. \end{aligned} \qquad (25)$$

Moreover, $K(\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)) \cong \odot^2 V^*$.

Similar calculations can be performed for the representations in items 9 and 10. We omit the details.

The representations corresponding to entries 1 and 2 of Table 4

These are the standard representations of $\mathfrak{gl}(V)$ and $\mathfrak{sl}(V)$, respectively, on V. Consider the following part of the Spencer complex of $\mathfrak{gl}(V)$:

$$0 \longrightarrow \mathfrak{gl}(V)^{(2)} \longrightarrow \mathfrak{gl}(V)^{(1)} \otimes V^* \longrightarrow \mathfrak{gl}(V) \otimes \Lambda^2 V^* \longrightarrow V \otimes \Lambda^3 V^*,$$

i.e. the sequence

$$0 \longrightarrow \odot^3 V^* \otimes V \longrightarrow \odot^2 V^* \otimes V^* \otimes V \longrightarrow \Lambda^2 V^* \otimes V^* \otimes V \longrightarrow \Lambda^3 V^* \otimes V \longrightarrow 0, \qquad (26)$$

where all maps are symmetrizations and skew-symmetrizations. It is not hard to see that this is an exact sequence, i.e. all cohomologies vanish. In particular, we have the exact sequence

$$0 \longrightarrow \odot^3 V^* \otimes V \longrightarrow \odot^2 V^* \otimes V^* \otimes V \longrightarrow K(\mathfrak{gl}(V)) \longrightarrow 0,$$

that is, we have

$$K(\mathfrak{gl}(V)) \cong (V^* \otimes \mathfrak{gl}(V)^{(1)})/\mathfrak{gl}(V)^{(2)},$$

with an explicit isomorphism being induced by

$$R_\tau(x, y)z := \tau(x, yz) - \tau(y, xz), \quad \tau \in \odot^2 V^* \otimes V^* \otimes V.$$

Next, for $\mathfrak{h} = \mathfrak{sl}(V)$, we see that $R_\tau(x, y) \in \mathfrak{sl}(V)$ for all $x, y \in V$ iff $\sigma(xy) := tr\ \tau(x, y\_)$ is symmetric.

Conversely, given $\sigma \in \odot^2(V^*)$, we let $\tau_\sigma(x, yz) := \frac{1}{n-1}(\sigma(xy)z + \sigma(xz)y - 2\sigma(yz)x)$. Then $tr\ \tau_\sigma(x, y\_) = \sigma(xy)$, and hence we have

$$K(\mathfrak{sl}(V)) = \odot^2 V^* \oplus (V^* \otimes \mathfrak{sl}(V)^{(1)})/\mathfrak{sl}(V)^{(2)},$$

which illustrates that $H^{1,2}(\mathfrak{sl}(V)) \cong \odot^2 V^*$.

3.3 Irreducible complex Berger algebras

Throughout this section, all Lie algebras and vector spaces are understood to be complex. Let $\mathfrak{g} \subset \mathrm{End}(V)$ be an irreducible complex representation, and let $\mathfrak{g}_s$ denote the semi-simple part of $\mathfrak{g}$. That is, $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_s$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$, and $\dim \mathfrak{z} \leq 1$. If $\mathfrak{t} \subset \mathfrak{g}_s$ is a Cartan subalgebra, we let $\mathfrak{t}_0 := \mathfrak{z} \oplus \mathfrak{t}$. As usual, we denote the set of roots of $\mathfrak{g}_s$ by Δ and the set of weights of the embedding $\mathfrak{g} \hookrightarrow \mathrm{End}(V)$ by Φ. We also let $\Delta_0 := \Delta \cup \{0\}$. For each root α of $\mathfrak{g}_s$, we fix $0 \neq A_\alpha \in \mathfrak{g}_\alpha$, and let

$$\Phi_\alpha := \{\text{weights of } A_\alpha V\} \subset \Phi.$$

Definition 3.9 *With $\mathfrak{g} \subset End(V)$ as above, we call $(\lambda_0, \lambda_1, \alpha)$ with $\lambda_i \in \Phi$ and $\alpha \in \Delta$ a* spanning triple *if*

$$\Phi_\alpha \subset \{\lambda_0 + \beta, \lambda_1 + \beta \mid \beta \in \Delta_0\}. \tag{27}$$

A spanning triple $(\lambda_0, \lambda_1, \alpha)$ is called extremal *if λ_0, λ_1 are extremal weights; it is said to be* of opposite sign *if λ_0, λ_1 are extremal weights of opposite sign.*

Note that the Weyl group W acts on (extremal) spanning triples. As a consequence, if a root $\alpha \in \Delta$ occurs in a (extremal) spanning triple, then all roots of the same length as α occur in such a triple.

Proposition 3.10 *Let $\mathfrak{g} \subset End(V)$ be an irreducible Berger algebra. Then for every root $\alpha \in \Delta$ there is a spanning triple $(\lambda_0, \lambda_1, \alpha)$.*

In fact, if $R \in K(\mathfrak{g})$ is a weight element and if there are weight vectors $x_i \in V$ of weights λ_i for $i = 0, 1$ such that $R(x_0, x_1) = A_\alpha$, then $(\lambda_0, \lambda_1, \alpha)$ is a spanning triple.

Proof. We first show the second assertion. Let $R \in K(\mathfrak{g})$ and $x_i \in V$ as required. Then, for any $y \in V$, the first Bianchi identity of $R \in K(\mathfrak{g})$ reads

$$A_\alpha y = R(y, x_1)x_0 - R(y, x_0)x_1 \in span\{\mathfrak{g}x_0, \mathfrak{g}x_1\},$$

i.e. $A_\alpha V \subset span\{\mathfrak{g}x_0, \mathfrak{g}x_1\}$. Then (27) holds since both $A_\alpha V$ and $span\{\mathfrak{g}x_0, \mathfrak{g}x_1\}$ are a direct sum of weight spaces, and the weights of the latter are contained in the right hand side of (27).

To show that such an R exists for all roots, let

$$D := \left\{ \alpha \in \Delta \;\middle|\; \begin{array}{l} \text{there are weight elements } R \in K(\mathfrak{g}),\ x_0, x_1 \in V \\ \text{such that } R(x_0, x_1) = A_\alpha \end{array} \right\}.$$

Since $K(\mathfrak{g})$ and V are spanned by their weight vectors, it follows that

$$\underline{\mathfrak{g}} \subset \mathfrak{t}_0 \oplus \bigoplus_{\alpha \in D} \mathfrak{g}_\alpha.$$

Then, since $\mathfrak{g}$ is Berger, it follows that $D = \Delta$. ∎

Lemma 3.11 *Let $\mathfrak{g} \subset End(V)$ be an irreducible Lie subalgebra with $K(\mathfrak{g}) \neq 0$. Then there are extremal weight vectors x_0, x_1 of weights λ_0, λ_1 of opposite sign such that $R(x_0, x_1) \neq 0$ for some $R \in K(\mathfrak{g})$.*

Proof. Suppose that $R(x_0, x_1) = 0$ for all $R \in K(\mathfrak{g})$ and all such extremal weight vectors x_0, x_1.

We write the sky and the projectivized sky as $\tilde{X}$ and $X = \mathrm{G}/P$, respectively, where $P \subset \mathrm{G}$ is the isotropy group of $\mathbb{C}x_0$, i.e. $gx_0 = c_g x_0$, some scalar $c_g \neq 0$, for all $g \in P$. It follows that for $g \in P$ and $R \in K(\mathfrak{g})$ we have $R(x_0, gx_1) = c_{g^{-1}} Ad_{g^{-1}}((gR)(x_0, x_1)) = 0$. Since the Lie algebra $\mathfrak{p} \subset \mathfrak{g}$ contains all positive root elements and λ_0, λ_1 have opposite signs, it follows that $\mathfrak{p} \cdot x_1 = T_{x_1}\tilde{X}$, hence $P \cdot x_1$ contains an open neighborhood of x_1 in $\tilde{X}$. But since every open subset of $\tilde{X}$ spans all of V, it follows that $R(x_0, V) = 0$ for all $R \in K(\mathfrak{g})$. Since $x_0 \in \tilde{X}$ is arbitrary and $\tilde{X}$ spans all of V, this implies that $K(\mathfrak{g}) = 0$. ∎

We now will show a further result which allows us to reduce to a more restrictive criterion than the one in Proposition 3.10.

Theorem 3.12 *Let $\mathfrak{g} \subset End(V)$ be an irreducible Berger algebra. Then either there is an extremal spanning triple $(\lambda_0, \lambda_1, \alpha)$, or $\mathfrak{g}$ is congruent to the representation of $\mathfrak{so}(n, \mathbb{C})$ on $(\odot^2 \mathbb{C}^n)_0$ for some $n \geq 3$. If the latter is the case then* $\dim K(\mathfrak{g}) = 1$ *and thus, $\mathfrak{g}$ is symmetric.*

Proof. Suppose that $\mathfrak{g} \subset \mathrm{End}(V)$ is an irreducible Berger algebra, but there are no extremal spanning triples for Φ. Then $K(\mathfrak{g}) \neq 0$, hence by Lemma 3.11, there is an element $R \in K(\mathfrak{g})$ and weight vectors $x_0, x_1 \in V$ of weight $\lambda_0, \lambda_1 \in \Phi$ such that $R(x_0, x_1) \neq 0$ where λ_0, λ_1 are extremal weights of opposite sign. W.l.o.g we may assume that R is a weight element and λ_1 is the dominant weight of Φ.

If $R(x_0, x_1) = A_\alpha$ for some root $\alpha \in \Delta$, then Proposition 3.10 implies that $(\lambda_0, \lambda_1, \alpha)$ is an extremal spanning triple which was excluded, hence

$$0 \neq R(x_0, x_1) = T + cId_V \in \mathfrak{t}_0,$$

where $T \in \mathfrak{t}$ and $c \in \mathbb{C}$. We define

$$K := \{\mu \in \mathfrak{t}^* \mid \mu(T) + c = 0\} \subset \mathfrak{t}^*.$$

Step 1: $T \neq 0$ and therefore, K is an affine hyperplane in $\mathfrak{t}^*$.

Proof of step 1: Let $\lambda \in \Phi$ be a weight such that $\lambda - \lambda_i \notin \Delta_0$ for $i = 1, 2$. Such a λ must exist since otherwise, $(\lambda_0, \lambda_1, \alpha)$ would be a spanning triple for *any* $\alpha \in \Delta$. For $x_\lambda \in V_\lambda$ we have $R(x_i, x_\lambda) \in \mathfrak{g}_{\lambda_i - \lambda} = 0$, thus by the Bianchi identity, $0 = R(x_0, x_1)x_\lambda = (\lambda(T) + c)x_\lambda$. If $T = 0$ this would imply that $c = 0$, contradicting that $R(x_0, x_1) \neq 0$.

Step 2: K contains all extremal weights $\lambda \in \Phi$ with $\lambda \neq \lambda_i$, $i = 0, 1$.

Proof of step 2: If $\lambda \in \Phi$ is as above, then $R(x_i, x_\lambda) \in \mathfrak{g}_{\lambda - \lambda_{i+1}}$ for weight reasons where we take indices mod 2. On the other hand, since $(\lambda_i, \lambda, \lambda - \lambda_{i+1})$ is not spanning by hypothesis, Proposition 3.10 implies that $R(x_i, x_\lambda) = 0$. Thus, from the Bianchi identity we get $0 = R(x_0, x_1)x_\lambda = (\lambda(T) + c)x_\lambda$ and hence, $\lambda \in K$.

Step 3: Let $\alpha \in \Delta$ be such that $\alpha(T) \neq 0$. Then $|\langle \lambda_i, \alpha\rangle| \geq 2$ for $i = 0, 1$, and $|\langle \lambda_1 - \lambda_0, \alpha\rangle| \geq 4$. Moreover, after changing α to $-\alpha$ if necessary, $(\lambda_0 + \alpha, \lambda_1, \alpha)$ and $(\lambda_0, \lambda_1 - \alpha, -\alpha)$ are spanning triples.

Proof of step 3: Let $\alpha \in \Delta^+$ and fix $0 \neq A_\alpha \in \mathfrak{g}_\alpha$. Then $(A_\alpha R)(x_0, x_1) \in \mathfrak{g}_\alpha$ for weight reasons, but since $(\lambda_0, \lambda_1, \alpha)$ is not spanning, Proposition 3.10 implies that

$$\begin{aligned} 0 &= (A_\alpha R)(x_0, x_1) \\ &= [A_\alpha, R(x_0, x_1)] + R(A_\alpha x_0, x_1) + R(x_0, A_\alpha x_1) \\ &= -\alpha(T)A_\alpha + R(A_\alpha x_0, x_1), \end{aligned}$$

because λ_1 is dominant and $\alpha \in \Delta^+$. Thus, $0 \neq R(A_\alpha x_0, x_1) \in \mathfrak{g}_\alpha$, and by Proposition 3.10 $(\lambda_0 + \alpha, \lambda_1, \alpha)$ is spanning. In particular, $\lambda_0 + \alpha \in \Phi$ is not extremal by our assumption, and hence, $\langle \lambda_0, \alpha\rangle \leq -2$. The remaining statements are shown analogously by replacing α by $-\alpha \in \Delta^-$.

Step 4: For all $\alpha \in \Delta$ we have $|\langle \lambda_i, \alpha\rangle| \neq 1$.

Proof of step 4: If, say, $\langle \lambda_0, \alpha\rangle = -1$ for some $\alpha \in \Delta$ then $\lambda_0 + \alpha$ is extremal. Since by step 3 $|\langle \lambda_1 - \lambda_0, \alpha\rangle| \geq 4$, it follows that $\lambda_0 + \alpha \neq \lambda_i$ and hence, $\lambda_0 + \alpha \in K$ by step 2. But now, by step 3, $\alpha(T) = 0$ so that $\lambda_0 \in K$ as well.

Let $w \in W$ be an element of the Weyl group for which $w \cdot \lambda_0 = \lambda_1$, and let $\beta := w \cdot \alpha$. Then $\langle \lambda_1, \beta\rangle = 1$, and the same argument implies that $\lambda_1 \in K$, i.e. K contains *all* extremal weights.

But now, the convex hull of the extremal weights must contain 0, and since K is convex it follows that $0 \in K$, i.e. K is a subspace. On the other hand, the weights linearly span all of V, and this yields a contradiction.

Step 5: If $rk(\mathfrak{g}) \geq 2$ then there exists a long root $\alpha \in \Delta$ with $\alpha(T) \neq 0$ and a root $\beta \in \Delta_\alpha^\perp =: \Delta \cap \alpha^\perp$ with $\langle \lambda_0, \beta\rangle \neq 0$.

Proof of step 4: By step 1, there is a long root $\alpha \in \Delta$ with $\alpha(T) \neq 0$. Suppose we have $\langle \lambda_0, \beta\rangle = 0$ for all $\beta \in \Delta_\alpha^\perp$. This implies that Δ is simple.

If Δ is not of type A_n then $\Delta_\alpha^\perp$ spans $\alpha^\perp$ so that $\lambda_1 = -\lambda_0 = c\alpha$ for some constant c. Then the extremal weights are of the form $c\beta$ for a long root $\beta \in \Delta$, and so, by step 2, K contains all long $\beta \neq \pm\alpha$ which implies that the root system of long roots contains $\{\pm\alpha\}$ as a summand. This happens only if Δ is of type C_n. Moreover, by step 3, $((-c + 1)\alpha, c\alpha, \alpha)$ is a spanning triple and this implies that $c \leq 1$. On the other hand, for $c \leq 1$, there are extremal spanning triples, violating our assumption.

If Δ is of type A_n, $n \geq 2$, and $\alpha = \theta_r - \theta_s$ then our assumption implies that $\lambda_0 = c_1\theta_r + c_2\theta_s$ for some $c_i \in \mathbb{Z}$. Moreover, since by step 3 $\langle \lambda_0, \alpha\rangle \neq 0$, we have $c_1 \neq c_2$. The extremal weights are of the form $c_1\theta_i + c_2\theta_j$ for all $i \neq j$. But then, since K contains all but two extremal weights by step 2, we must have $c_1 = 0$ or $c_2 = 0$. Since by step 3, $\langle \lambda_i, \alpha\rangle \neq 0$, it follows that – after possibly switching λ_0, λ_1 – we have $\lambda_1 = c\theta_r, \lambda_0 = c\theta_s$ for some $c \in \mathbb{Z}$. Also,

$(\lambda_0+\alpha, \lambda_1, \alpha)$ must be a spanning triple by step 3, and this implies that $|c| \leq 2$. However, if $|c| \leq 2$ then $(c\theta_1, c\theta_2, \theta_1 - \theta_2)$ is an extremal spanning triple which was excluded.

Step 5: Suppose that $\mathrm{rk}(\mathfrak{g}) \geq 2$ and let $\alpha, \beta \in \Delta$ as in step 4. Then $\{\pm\beta\}$ is a direct summand of $\Delta_\alpha^\perp$. Further, $|\langle\lambda_i, \alpha\rangle| = 2$ for $i = 0, 1$ and, after possibly replacing α, β by their negatives, $\lambda_1 = \lambda_0 + 2(\alpha + \beta)$.

Proof of step 5: After changing α, β to their negatives if necessary and using step 3, we may assume that $\langle\lambda_0, \alpha\rangle, \langle\lambda_0, \beta\rangle \leq -2$ and $\langle\lambda_1, \alpha\rangle \geq 2$. Thus, $\lambda_0 + \alpha + 2\beta \in \Phi_\alpha$.

By step 3, $(\lambda_0+\alpha, \lambda_1, \alpha)$ is a spanning triple. Since $2\beta \notin \Delta$, we have $\lambda_0+\alpha+2\beta = \lambda_1 - \gamma$ for some $\gamma \in \Delta_0$. Now, we calculate

$$2 \leq \langle\lambda_1, \alpha\rangle = \langle\lambda_0, \alpha\rangle + 2 + \langle\gamma, \alpha\rangle \leq \langle\gamma, \alpha\rangle,$$

and since α is a long root, (19) implies that $\alpha = \gamma$ and all inequalities in this chain are equalities. Thus, $\lambda_1 = \lambda_0 + 2(\alpha + \beta)$ which means that for given α, there is - up to sign - at most one $\beta \in \Delta_\alpha^\perp$ with $\langle\lambda_0, \beta\rangle \neq 0$ which shows that $\{\pm\beta\}$ is a direct summand of $\Delta_\alpha^\perp$.

Step 6: $\mathfrak{g} \cong \mathfrak{so}(n, \mathbb{C})$ for some $n \geq 3$, and λ_1 is the dominant weight of the representation of $\mathfrak{g}$ on $(\odot^2\mathbb{C}^n)_0$.

Proof of step 6: If $\mathrm{rk}(\mathfrak{g}) = 1$, then $\lambda_1 = -\lambda_0 = c\alpha$ for some $c \in \frac{1}{2}\mathbb{Z}^+$. By step 3, $(c\alpha, (-c+1)\alpha, \alpha)$ is spanning which implies that $c \leq 2$. On the other hand, $(c\alpha, -c\alpha, \alpha)$ is an extremal spanning triple if $c \leq \frac{3}{2}$, violating our assumption. Thus we must have $c = 2$ which yields the asserted representation for $n = 3$.

The only root systems Δ which satisfy that $\Delta_\alpha^\perp$ contains a direct summand of type A_1 for some long root α are A_3, $B_n, n \geq 2$, $D_n, n \geq 4$, G_2 and $A_1 + \Delta'$. Thus, if $\mathrm{rk}(\mathfrak{g}) \geq 2$, then Δ must be one of these root systems by step 5.

First, suppose that Δ is simple. Suppose that $\beta \in \Delta_\alpha^\perp$ from step 4 is short. Then Δ is of type B_3 or G_2. In both cases, $-Id_{\mathfrak{t}^*} \in W$ is an element of the Weyl group, so that $\lambda_0 + \lambda_1 = 0$, i.e. by step 5, $\lambda_1 = \alpha + \beta$. But now it is easy to see that in both cases, there is no affine hyperplane containing all extremal weights $\neq \pm\lambda_1$, contradicting step 2. Thus, Δ is not of type G_2 and α, β are long roots. This now implies that in this case, $\mathfrak{g} \cong \mathfrak{so}(n, \mathbb{C})$, $n \geq 5$, with the asserted representation.

Next, suppose that $\Delta = A_1 + \Delta'$. Then by step 5, $\lambda_i = (-1)^i\alpha + \mu_i$ with μ_i in the weight lattice of Δ'. Now suppose that $rk(\Delta') \geq 2$. Then there are extremal weights of the form $\pm\alpha + \mu$ with $\mu \neq \mu_i$, and thus, by step 2, $\pm\alpha + \mu \in K$, which implies that $\alpha(T) = 0$, contradicting our assumption. Thus, $\Delta = A_1 + A_1$ which means that $\lambda_1 = -\lambda_0 = \alpha + \beta$ by step 5, and this yields the asserted representation for $n = 4$.

Step 7: Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be one of the representations in step 6, let $K_0 \subset K(\mathfrak{g})$ be the 0-weight space, and let $x_0, x_1 \in V$ be extremal weight vectors of opposite sign. Then the map $K_0 \to \mathfrak{t}_0$, $R \mapsto R(x_0, x_1)$ is injective.

Proof of step 7: Suppose that $0 \neq R \in K_0$ exists with $R(x_0, x_1) = 0$. W.l.o.g. we assume that λ_1 is the dominant weight. Then as in the proof of step 3, we have for all $\alpha \in \Delta^+$ that $0 = (A_\alpha R)(x_0, x_1) = R(A_\alpha x_0, x_1)$, and therefore, $R(u, x_1) = 0$ for all $u \in V$. Likewise, $R(u, x_0) = 0$ for all $u \in V$. Thus, from the Bianchi identity, we get that

$$R(u, v)x_i = 0 \quad \text{for } i = 0, 1 \text{ and all } u, v \in V. \tag{28}$$

Next, let $\alpha \in \Delta$ be such that $\langle \lambda_0, \alpha \rangle \neq 0$. Then if y_0, y_1 are extremal weight vectors of opposite sign, then $0 = (A_\alpha R)(y_0, y_1)$, and since for weight reasons, $R(A_\alpha y_i, y_j) \in \mathfrak{g}_\alpha$ for $\{i, j\} = \{0, 1\}$, (28) implies that $R(A_\alpha y_i, y_j) = 0$, and hence, $[A_\alpha, R(y_0, y_1)] = 0$, that is $\alpha(R(y_0, y_1)) = 0$ for all such α.

Note, however, that $\mathfrak{t}^*$ is linearly generated by all $\alpha \in \Delta$ with $\langle \lambda_0, \alpha \rangle \neq 0$ and thus, $R(y_0, y_1) = cId_V$. But then, by step 1, we must have $c = 0$, i.e. $R(y_0, y_1) = 0$. By the same argument as for x_i, we conclude that $R(u, v)y_i = 0$ for $i = 0, 1$ and all $u, v \in V$, i.e.

$$R(u, v)y_0 = 0 \quad \text{for all } u, v \in V \text{ and } y_0 \text{ an extremal weight vector,}$$

and this implies that $R = 0$, a contradiction.

Step 8: For the subalgebras $\mathfrak{g} \subset \mathfrak{gl}(V)$ from step 6, we have $\dim K(\mathfrak{g}) = 1$.

Proof of step 8: Let λ_1 be the dominant weight of one of these representations. Then for all extremal weights $\mu \neq \pm\lambda_1$ we have $(\mu, \lambda_1) = 0$. Thus, if $R \in K_0$ then by step 2 we have $R(x_0, x_1) = cT_{\lambda_1}$ for some $c \in \mathbb{C}$ where $T_{\lambda_1} \in \mathfrak{t}$ is the element characterized by $\mu(T_{\lambda_1}) = (\lambda_1, \mu)$ for all $\mu \in \mathfrak{t}^*$. Thus, by step 7, $\dim K_0 \leq 1$.

On the other hand, each irreducible submodule of $K(\mathfrak{g}) \subset \Lambda^2 V \otimes \mathfrak{g}$ has 0 as a weight, thus, $K(\mathfrak{g})$ must be irreducible. Finally, note that these representations are the isotropies of the symmetric spaces $SU(n)/SO(n)$, so that $K(\mathfrak{g})$ contains a trivial summand, and thus is one-dimensional. ∎

3.4 Simple complex Berger algebras

In this section, we assume that $\mathfrak{g} \subset \mathrm{End}(V)$ with $\mathfrak{g} \cong \mathfrak{z} \oplus \mathfrak{g}_s$ and $\mathfrak{g}_s$ simple. Again, both $\mathfrak{g}$ and V are understood to be complex. By Theorem 3.12, we need to classify those representations which admit extremal spanning triples.

Proposition 3.13 *Let $\mathfrak{g} \subset End(V)$ be an irreducible subalgebra with $\mathfrak{g}_s$ simple, let Δ and Φ as before, and suppose that $0 \in \Phi$. If Δ is not of type C_n then there is an extremal spanning triple only if the dominant weight is a root, i.e. $\Phi \subset \Delta_0$. In particular, this holds if Δ is of type G_2, F_4 or E_8.*

Proof. If $0 \in \Phi$ and Δ is not of type C_n then either the dominant weight is a short root or $\Delta_0 \subset \Phi$ which means that $0 \in \Phi_\alpha$ for any root $\alpha \in \Delta$. Thus, if there exists an extremal triple $(\lambda_0, \lambda_1, \alpha)$, then $0 = \lambda_0 + \gamma$, some root γ. Since λ_0 is extremal, $\Phi \subset \Delta_0$ follows.

Finally, if Δ is of type G_2, F_4 or E_8, then every representation has 0 as a weight. ∎

Proposition 3.14 *Let $\mathfrak{g} \subset End(V)$, $\mathfrak{g}_s$, Δ and Φ as before. If there is an extremal spanning triple, then $|\langle \lambda, \alpha \rangle| \leq 3$ for all $\lambda \in \Phi$ and $\alpha \in \Delta$.*

Proof. By Proposition 3.13, we may assume that Δ is not of type G_2. Suppose that there is an extremal spanning triple $(\lambda_0, \lambda_1, \beta)$. Let $0 \neq \lambda \in \Phi$ be a weight with $|\langle \lambda, \alpha \rangle| \leq 1$

for all roots α. After possibly applying an element of the Weyl group to λ, we may assume that $\langle \lambda, \beta \rangle > 0$, i.e. $\lambda \in \Phi_\beta$, hence $\lambda = \lambda_0 + \gamma$ for some $\gamma \in \Delta_0$. But then, for any $\alpha \in \Delta$, $|\langle \lambda_0, \alpha \rangle| \leq |\langle \lambda, \alpha \rangle| + |\langle \gamma, \alpha \rangle| \leq 3$ by (19). The claim then follows since λ_0 is extremal. ∎

Proposition 3.15 *Let $\mathfrak{g} \subset End(V)$ be as in Proposition 3.14 such that $rk\ \mathfrak{g}_s \geq 2$, and suppose there exists an extremal spanning triple. Then for every weight λ and every long root α, $|\langle \lambda, \alpha \rangle| \leq 2$.*

Proof. Suppose that there is a weight λ and a long root α with $\langle \lambda, \alpha \rangle = -3$.

Let us first consider the case where all roots have equal length. Let β be a root with $\langle \alpha, \beta \rangle = 1$. Then, after replacing β by $\alpha - \beta$ if necessary, we may assume that $\langle \lambda, \beta \rangle \leq -2$. It follows that $\lambda + k\alpha + l\beta \in \Phi_\alpha$ for $k = 1, 2, 3$ and $0 \leq l \leq 3 - k$.

By hypothesis, there is an extremal spanning triple $(\lambda_0, \lambda_1, \alpha)$. Then $\lambda + \alpha = \lambda_0 + \gamma$. Since λ_0 is extremal, $\gamma \neq -\alpha$ and thus, by (20), $\gamma + 2\alpha$ is not a root. Therefore,

$$\begin{aligned} \lambda + \alpha &= \lambda_0 + \gamma \\ \lambda + 3\alpha &= \lambda_1 + \delta \end{aligned} \quad \text{where } \gamma, \delta \in \Delta_0. \tag{29}$$

Now, $\Phi_\alpha \ni \lambda + \alpha + 2\beta = \lambda_0 + \gamma + 2\beta = \lambda_1 + \delta + 2(\beta - \alpha)$. But by (20), $\gamma + 2\beta$ or $\delta + 2(\beta - \alpha)$ are roots only if $\gamma = -\beta$ or $\delta = \alpha - \beta$, both of which contradict the extremality of λ_i.

Second, suppose that there are roots of different length. Since by Proposition 3.13 we may assume that Δ is not of type G_2, it follows that $\alpha = \alpha_1 + \alpha_2$ for short roots α_i with $\langle \alpha_1, \alpha_2 \rangle = 0$. Since $-3 = \langle \lambda, \alpha \rangle = \frac{1}{2}(\langle \lambda, \alpha_1 \rangle + \langle \lambda, \alpha_2 \rangle)$, Proposition 3.14 implies that $\langle \lambda, \alpha_i \rangle = -3$ for $i = 1, 2$.

By hypothesis, there is either an extremal spanning triple $(\lambda_0, \lambda_1, \alpha)$ or $(\lambda_0, \lambda_1, \alpha_1)$. It is then easy to check that $\{\lambda + k\alpha_1 + l\alpha_2 \mid 1 \leq k, l \leq 3\} \subset \Phi_\alpha \cap \Phi_{\alpha_1}$. Thus, we get as in the previous case that (29) holds, and from the extremality of λ_i, we have that $\gamma \neq -\alpha$ and $\delta \neq \alpha$. Using (19), we conclude that $\langle \lambda_0, \alpha \rangle \leq 0$ and $\langle \lambda_1, \alpha \rangle \geq 2$.

Next, we have $\langle \lambda + 2\alpha_1 + \alpha_2, \alpha \rangle = 0$, hence if $\lambda + 2\alpha_1 + \alpha_2 = \lambda_1 + \varepsilon$, some $\varepsilon \in \Delta_0$, then from $\langle \lambda_1, \alpha \rangle \geq 2$ and (19) it would follow that $\varepsilon = -\alpha$, contradicting the extremality of λ_1. Thus, $\lambda + 2\alpha_1 + \alpha_2 = \lambda_0 + \gamma + \alpha_1$ implies that $\gamma + \alpha_1 \in \Delta_0$, and likewise, $\gamma + \alpha_2 \in \Delta_0$.

If γ was long, then this would imply that $\langle \gamma, \alpha_i \rangle = -2$ for $i = 1, 2$, and hence $\langle \gamma, \alpha \rangle = \frac{1}{2}(\langle \gamma, \alpha_1 \rangle + \langle \gamma, \alpha_2 \rangle) = -2$, that is $\gamma = -\alpha$ which is impossible. Thus, γ is a short root.

Finally, for $\{i, j\} = \{1, 2\}$, consider the weights $\lambda + 3\alpha_i + \alpha_j = \lambda_0 + \gamma + 2\alpha_i = \lambda_1 + \delta - 2\alpha_j$. Since γ is short, $\gamma + 2\alpha_i$ is a root iff $\gamma = -\alpha_i$ which would contradict the extremality of λ_0. Thus, $\delta - 2\alpha_i \in \Delta$ for $i = 1, 2$. But $\delta - 2\alpha_2 = (\delta - 2\alpha_1) + 2(\alpha_1 - \alpha_2)$, and since $\alpha_1 - \alpha_2$ is a long root, (20) implies that $\delta = \alpha$, contradicting the extremality of λ_1. ∎

Proposition 3.16 *Let $\mathfrak{g} \subset End(V)$ be as in Proposition 3.15, and suppose that $|\langle \lambda, \alpha \rangle| = 2$ for some $\lambda \in \Phi$ and a long root α. Then for every long root $\beta \in \Delta$ with $\langle \alpha, \beta \rangle = 0$ we have $|\langle \lambda, \beta \rangle| \leq 1$.*

Proof. By contradiction, suppose that there is a long root β with $\langle \alpha, \beta \rangle = 0$ and $|\langle \lambda, \beta \rangle| \geq 2$. By Proposition 3.15, we may change α and β to their negatives if necessary and assume that $\langle \lambda, \alpha \rangle = \langle \lambda, \beta \rangle = -2$. Also, we may assume that Δ is not of type G_2.

If there are roots of different length, we write $\alpha = \alpha_1 + \alpha_2$ with short roots α_i. From the identity $2\langle \lambda, \alpha \rangle = \langle \lambda, \alpha_1 \rangle + \langle \lambda, \alpha_2 \rangle$ and Proposition 3.14 we may assume that w.l.o.g. $\langle \lambda, \alpha_1 \rangle \in \{-2, -3\}$ and $\langle \lambda, \alpha_2 \rangle \in \{-1, -2\}$. Then $\beta + 2\alpha_i$ is not a root, since otherwise $\langle \lambda, \beta + 2\alpha_i \rangle \leq -3$ which is impossible. Thus, $\langle \beta, \alpha_i \rangle \geq 0$, and then $\langle \beta, \alpha \rangle = 0$ implies that $\langle \beta, \alpha_i \rangle = 0$.

From this, it follows that $\lambda + \alpha_1 + l\beta \in \Phi$, and thus $\lambda + \alpha + l\beta \in \Phi_{\alpha_2}$ for $l = 0, 1, 2$. Also, $\langle \lambda + 2\alpha + l\beta, \alpha_2 \rangle \geq 2$, and so we get

$$\{\lambda + k\alpha + l\beta \mid k = 1, 2,\ l = 0, 1, 2\} \subset \Phi_\alpha \cap \Phi_{\alpha_2}.$$

By hypothesis, there must be extremal weights λ_0, λ_1 such that either $(\lambda_0, \lambda_1, \alpha)$ or $(\lambda_0, \lambda_1, \alpha_2)$ is spanning. Thus, we have $\lambda + \alpha = \lambda_0 + \gamma$ for some $\gamma \in \Delta_0$. Since $\lambda + \alpha$ is not extremal, we must have $\gamma \neq 0$ and $\lambda_0 + 2\gamma \in \Phi$.

Then, on the one hand, $-2 = \langle \lambda + \alpha, \beta \rangle = \langle \lambda_0, \beta \rangle + \langle \gamma, \beta \rangle \geq -2 + \langle \gamma, \beta \rangle$, i.e. $\langle \gamma, \beta \rangle \leq 0$. On the other hand, $-2 \leq \langle \lambda_0 + 2\gamma, \beta \rangle = \langle \lambda + \alpha + \gamma, \beta \rangle = -2 + \langle \gamma, \beta \rangle$, i.e. $\langle \gamma, \beta \rangle \geq 0$.

Thus, $\langle \gamma, \beta \rangle = 0$ and hence $\langle \lambda_0, \beta \rangle = -2$. Since $\gamma + 2\beta \notin \Delta_0$, it follows that $\lambda + \alpha + 2\beta = \lambda_1 + \delta$, some $\delta \in \Delta_0$, and in complete analogy we get $\delta \neq 0$, $\langle \lambda_1, \beta \rangle = 2$ and $\langle \delta, \beta \rangle = 0$.

But then, $\Phi_\alpha \cap \Phi_{\alpha_2} \ni \lambda + \alpha + \beta = \lambda_0 + \gamma + \beta = \lambda_1 + \delta - \beta$, and neither $\gamma + \beta$ nor $\delta - \beta$ are in Δ_0, which is impossible. ∎

Proposition 3.17 *Let $\mathfrak{g} \subset End(V)$, Φ and Δ be as in Proposition 3.15, and let us assume that all roots of Δ have equal length. Suppose that there are roots α, β with $\langle \alpha, \beta \rangle = 0$, $|\langle \lambda, \alpha \rangle| = 2$ and $|\langle \lambda, \beta \rangle| = 1$ for some $\lambda \in \Phi$. Then for every root $\gamma \in \Delta$ with $\langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle = 0$ we have $\langle \lambda, \gamma \rangle = 0$.*

Proof. Let $(\lambda_0, \lambda_1, \alpha)$ be an extremal spanning triple and suppose there is a weight $\lambda \in \Phi$ and roots $\beta, \gamma \in \Delta$ with $\langle \lambda, \alpha \rangle = -2$, $\langle \lambda, \beta \rangle = \langle \lambda, \gamma \rangle = -1$ and $\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle = 0$. We shall now derive a contradiction from these assumptions. We have

$$\{\lambda + k\alpha + l\beta + m\gamma \mid k = 1, 2,\ l, m = 0, 1\} \subset \Phi_\alpha. \tag{30}$$

Thus, $\lambda + \alpha = \lambda_0 + \delta$ for some $\delta \in \Delta_0$. Since $\lambda + \alpha$ is not extremal, we have $\delta \neq 0$ and $\lambda_0 + 2\delta \in \Phi$.

Suppose that $\langle \delta, \beta \rangle, \langle \delta, \gamma \rangle \geq 0$. Then $\delta + \beta + \gamma$ is not a root, hence $\lambda + \alpha + \beta + \gamma = \lambda_1 + \varepsilon$, some $\varepsilon \in \Delta_0$. Again, since λ_1 is extremal, $\varepsilon \neq 0$. Moreover, $\lambda + \alpha + \gamma = \lambda_0 + \delta + \gamma = \lambda_1 + \varepsilon - \beta$. Since $\delta + \gamma$ is not a root, $\varepsilon - \beta$ is one, hence $\langle \varepsilon, \beta \rangle = 1$. Thus, after possibly replacing λ by $\lambda + \beta + \gamma$, replacing β, γ by their negatives and interchanging λ_0, λ_1, we may assume that

$$\langle \delta, \beta \rangle = -1, \quad \text{and thus,} \quad \langle \lambda_0, \beta \rangle = \langle \lambda + \alpha - \delta, \beta \rangle = 0.$$

This implies that $\langle \lambda_0 + 2\delta, \beta \rangle = -2$ and $\langle \lambda_0 + 2\delta, \gamma \rangle = -1 + \langle \delta, \gamma \rangle$. Thus, by Proposition 3.16, $\langle \delta, \gamma \rangle \geq 0$.

Therefore, $\delta+\gamma, \delta+\beta+\gamma \notin \Delta_0$, and hence, $\Phi_\alpha \ni \lambda+\alpha+\gamma = \lambda_1+\varepsilon$ for some $\varepsilon \in \Delta_0$ and $\Phi_\alpha \ni \lambda+\alpha+\beta+\gamma = \lambda_1+\varepsilon+\beta$ so that $\varepsilon+\beta \in \Delta_0$. Moreover, since these weights are not extremal, it follows that $\varepsilon, \varepsilon+\beta \neq 0$ and thus, $\langle \beta, \varepsilon \rangle = -1$ and $\langle \lambda_1, \beta \rangle = 0$. Therefore, $\langle \lambda_1 + 2\varepsilon, \beta \rangle = -2$ and $\langle \lambda_1 + 2\varepsilon, \gamma \rangle = 1 + \langle \varepsilon, \gamma \rangle$, whence $\langle \varepsilon, \gamma \rangle \leq 0$ by Proposition 3.16.

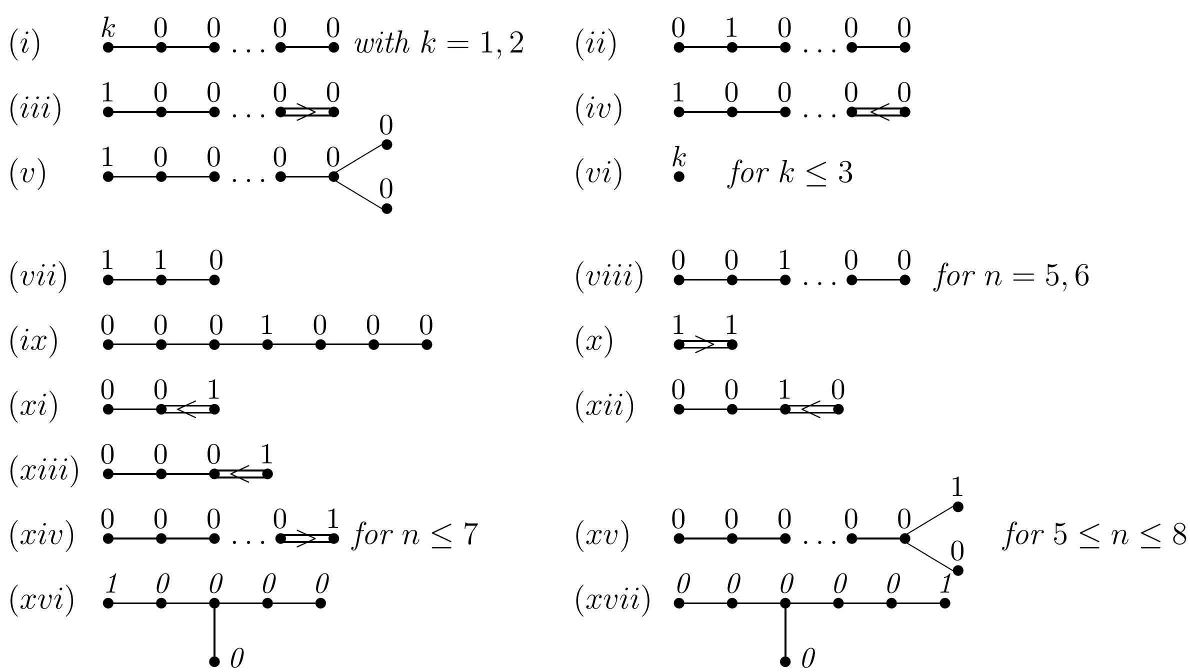
Next, $\Phi_\alpha \ni \lambda+2\alpha = \lambda_1+\alpha-\gamma+\varepsilon$, and since $\langle \alpha-\gamma+\varepsilon, \gamma \rangle \leq -2$ and $\varepsilon \neq -\alpha$, it follows that $\alpha-\gamma+\varepsilon \notin \Delta_0$, and thus, $\lambda+2\alpha = \lambda_0+\delta+\alpha$ implies that $\delta+\alpha \in \Delta_0$, i.e. $\langle \delta, \alpha \rangle = -1$. In complete analogy we also get $\langle \varepsilon, \alpha \rangle = -1$ and thus, $\langle \lambda_i, \alpha \rangle = 1$ for $i = 0, 1$.

Since $\langle \lambda_0 + 2\delta, \alpha \rangle = \langle \lambda_0 + 2(\delta+\beta), \alpha \rangle = -1$, it follows that $\Phi_\alpha \ni \lambda_0+\alpha+2\delta, \lambda_0+\alpha+2(\delta+\beta)$. On the other hand, since $\delta \neq -\alpha$, we have $\alpha+2\delta, \alpha+2(\delta+\beta) \notin \Delta_0$, so that $\lambda_0+2\delta+\alpha-\lambda_1, (\lambda_0+2\delta+\alpha-\lambda_1)+2\beta \in \Delta_0$ which happens iff $\lambda_0+2\delta+\alpha-\lambda_1 = -\beta$ by (20), so that

$$\varepsilon = -(\alpha+\beta-\gamma+\delta).$$

But now, $\langle \lambda_1 + 2\varepsilon, \alpha \rangle = -1$, so that $\Phi_\alpha \ni \lambda_1+2\varepsilon+\alpha = \lambda_0+2\gamma-\beta$, and $2\varepsilon+\alpha, 2\gamma-\beta \notin \Delta_0$ which yields a contradiction. ∎

Proposition 3.18 *Let $\mathfrak{g} \subset End(V)$ be irreducible with $\mathfrak{g}_s$ simple, Δ and Φ be as before, and suppose that there exists an extremal spanning triple $(\lambda_0, \lambda_1, \alpha)$. Then either the dominant weight is a root, i.e. $\Phi \subset \Delta_0$, or the representation of $\mathfrak{g}_s$ on V is congruent to one of the following.*



Proof. We give the proof for each type of root system.
1. Type A_n: In this case, the root system is $\Delta = \{\alpha_{i,j} := \theta_i - \theta_j \mid i \neq j \in \{1, \ldots, n+1\}\}$, and the positive roots are $\Delta^+ = \{\alpha_{i,j} \mid i < j\}$. The dominant weight of Φ can be represented in an unique way as $\lambda_0 = c_1\theta_1 + \ldots + c_n\theta_n$ with integers $c_1 \geq \ldots \geq c_n \geq 0$. For convenience,

we set $c_{n+1} = 0$. Note that due to the symmetry of the root system A_n we may assume w.l.o.g. that $c_1 - c_2 \geq c_n$.

If $n = 1$ then it is easy to see that there are extremal spanning triples iff the dominant weight is $\lambda_0 = k\alpha_{1,2}$ with $k \leq 3$, and this corresponds to (vi).

If $\mathrm{rk}(\mathfrak{g}_s) \geq 2$, then the only possibile representations (up to congruence) which satisfy the conclusions of Propositions 3.15, 3.16 and 3.17 are those with the following dominant weights:

$$\begin{array}{ll} \lambda_0 = 2\theta_1, & \\ \lambda_0 = 2\theta_1 + \theta_2 + \ldots + \theta_k, & k = 2, n-1, n, \\ \lambda_0 = \theta_1 + \ldots + \theta_k, & 1 \leq k \leq \frac{n+1}{2}. \end{array}$$

The Weyl group of A_n is the permutation group S_{n+1} which acts by permutation of the indices of $\theta_1, \ldots, \theta_{n+1}$.

From here, it is now straightforward to investigate each of these representations separately. The result is that the representations in the second row admit an extremal spanning triple iff $k = 2$ and $n = 3$, or if $k = n$; the latter correspond to the adjoint representation. In the third row, there are extremal spanning triples iff $k = 4$ and $n = 7$, or $k = 3$ and $n = 5, 6$, or $k = 1, 2$.

This yields precisely the representations $(i), (ii), (vii), (viii)$ and (ix).

2. Type B_n: The root system is $\Delta = \{\pm\theta_i,\ \pm\theta_i \pm \theta_j \mid i < j, i = 1, \ldots, n\}$, and the positive roots are $\Delta^+ = \{\theta_i,\ \theta_i \pm \theta_j \mid i < j, i = 1, \ldots, n\}$. The dominant weight is given by $\lambda_0 = c_1\theta_1 + \ldots + c_n\theta_n$ with $c_1 \geq \ldots \geq c_n \geq 0$, where either all c_k are integers, or all c_k are half-integers.

If all c_k are integers, then $0 \in \Phi$, hence, by Proposition 3.13, $\Phi \subset \Delta_0$.

Thus, let us assume that the c_k are not integers. Then the only representations satisfying the conlusions of Propositions 3.15 and 3.16 are those whose dominant weights are of the following forms:

$$\begin{array}{l} \lambda_0 = \frac{3}{2}\theta_1 + \frac{1}{2}\theta_2 + \ldots + \frac{1}{2}\theta_n, \\ \lambda_0 = \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2 + \ldots + \frac{1}{2}\theta_n. \end{array}$$

From here, one sees easily that in the first case, there is no extremal spanning triple if $n \geq 3$. The case $n = 2$ is listed in (x). In the second case, one sees that there is no extremal spanning triple if $n \geq 8$. The remaining cases are listed in (xiv).

3. Type C_n: The root system is $\Delta = \{\pm 2\theta_i,\ \pm\theta_i \pm \theta_j \mid i < j, i = 1, \ldots, n\}$, and the positive roots are $\Delta^+ = \{2\theta_i,\ \theta_i \pm \theta_j \mid i < j, i = 1, \ldots, n\}$. The dominant weight is given by $\lambda_0 = c_1\theta_1 + \ldots + c_n\theta_n$ with integers $c_1 \geq \ldots \geq c_n \geq 0$.

The only representations satisfying the conlusions of Propositions 3.15 and 3.16 are those whose dominant weights are of the following forms:

$$\begin{array}{ll} \lambda_0 = 2\theta_1, & \\ \lambda_0 = 2\theta_1 + \theta_2 + \ldots + \theta_k, & \text{with } 2 \leq k \leq n \\ \lambda_0 = \theta_1 + \ldots + \theta_k, & \text{with } 1 \leq k \leq n. \end{array}$$

The first case corresponds to the adjoint representation. In the second case, a direct investigation yields that extremal spanning triples exist iff $k = n = 2$ which is listed in (x).

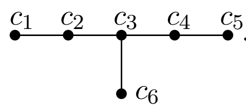
In the third case, one verifies that there are extremal spanning triples iff $k = n = 4$, or $k = 3$ and $n \leq 4$, or if $k \leq 2$. If $k = 2$ then $\Phi \subset \Delta_0$. The remaining cases are listed in (iv), (xi), (xii) and $(xiii)$.

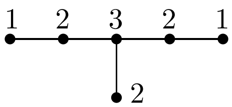
4. Type D_n: The root system is $\Delta = \{\pm\theta_i \pm \theta_j \mid i < j, i = 1, \ldots, n\}$, and the positive roots are $\Delta^+ = \{\theta_i \pm \theta_j \mid i < j, i = 1, \ldots, n\}$. The dominant weight is given by $\lambda_0 = c_1\theta_1 + \ldots + c_n\theta_n$ with $c_1 \geq \ldots \geq |c_n| \geq 0$, where either all c_k are integers, or all c_k are half-integers. Using the symmetry of the Dynkin diagram, we may assume that $c_n \geq 0$.

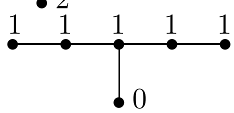
Then the only possibile representations (up to congruence) which satisfy the conclusions of Propositions 3.15, 3.16 and 3.17 are those with the following dominant weights:

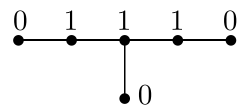
$$\begin{aligned} \lambda_0 &= \theta_1, \\ \lambda_0 &= \theta_1 + \theta_2 \\ \lambda_0 &= \tfrac{1}{2}(\theta_1 + \ldots + \theta_n). \end{aligned}$$

The first case is listed in (v), the second is the adjoint representation, and a direct investigation yields that in the last case, there is no extremal spanning triple if $n \geq 9$. The remaining cases are listed in (xv).

6. Type E_6: Let λ be given by 

. The maximal root of E_6 is given by 

, and the root system orthogonal to the maximal root is A_5 with maximal root 

. The root system orthogonal to this is 

.

It follows from Propositions 3.15, 3.16 and 3.17 that $c_2 = c_3 = c_4 = 0$, and either $c_6 = 1, c_1 = c_5 = 0$, in which case $\Phi = \Delta_0$, or $c_6 = 0$, $c_1 + c_5 \leq 1$. Using the symmetry of E_6, this yields the case (xvi).

7. Type E_7: This case is dealt with in complete analogy to E_6.

8. Types G_2, F_4, E_8: For these, it was already noted in Proposition 3.13 that they admit extremal spanning weights only if $\Phi \subset \Delta_0$. ∎

In the light of Theorem 3.12 and Proposition 3.18 we shall now investigate the representations given in Propositions 3.18 in order to classify all irreducible Berger algebras.

From the representations in Proposition 3.18, $(i), (ii), (vi)$ for $k = 1$, (xv) for $n = 5$ and (xvi) have been discussed in section 3.2.4, $(iii), (v)$ and (vi) for $k = 2$ in section 3.2.1, and $(iv), (vi)$ for $k = 3$, $(viii)$ for $n = 5$, (xi), (xv) for $n = 6$ and $(xvii)$ in section 3.2.3. We shall therefore now investigate the remaining entries from Proposition 3.18.

To simplify this investigation, we shall use the following observation.

Lemma 3.19 *Let $\mathfrak{g} \subset End(V)$ be an irreducible Berger algebra, and let $R \in K(\mathfrak{g})$ be a weight element. Suppose that for all extremal weight vectors x_0, x_1 we have $R(x_0, x_1) \in \mathfrak{t}_0$.*

Then either there is an affine hyperplane $K \subset \mathfrak{t}^*$ which contains all but two extremal weights, or $R(x_0, x_1) = 0$ for all extremal weight vectors.

An affine hyperplane $K \subset \mathfrak{t}^*$ containing all but two extremal weights will be called a *spanning plane*. Note that by Proposition 3.10 the hypothesis of Lemma 3.19 is satisfied if $R \in K(\mathfrak{g})$ is of weight ρ and there are no extremal spanning triples $(\lambda_0, \lambda_1, \alpha)$ with $\alpha - \lambda_0 - \lambda_1 = \rho$.

Proof. Suppose the hypotheses of the Lemma are satisfied, and $0 \neq R(x_0, x_1) = T + cId_V$ with $T \in \mathfrak{t}$ and $c \in \mathbb{C}$. We let $K := \{\mu \in \mathfrak{t}^* \mid \mu(T) + c = 0\}$. Then for all extremal weights $\lambda \neq \lambda_0, \lambda_1$ and $x_\lambda \in V_\lambda$ we have $R(x_i, x_\lambda) = 0$ for weight reasons and by hypothesis. Thus, the Bianchi identity implies that $0 = R(x_0, x_1)x_\lambda = (\lambda(T) + c)x_\lambda$ which implies that $\lambda \in K$. This implies, in particular, that $T \neq 0$ and hence, $K \subset \mathfrak{t}^*$ is a spanning plane. ∎

The representation $\overset{1}{\bullet}\!-\!\!\!-\!\overset{1}{\bullet}\!-\!\!\!-\!\overset{0}{\bullet}$.

It is easy to show that there are no spanning planes and – up to the action of the Weyl group – the only extremal spanning triples are $(2\theta_1 + \theta_2, 2\theta_1 + \theta_3, \theta_1 - \theta_4)$, $(2\theta_1 + \theta_2, 2\theta_3 + \theta_1, \theta_1 - \theta_4)$ and $(2\theta_1 + \theta_2, 2\theta_3 + \theta_4, \theta_1 - \theta_4)$.

We let $\alpha := \theta_1 - \theta_4$, and let $x_0 \in V_{2\theta_1+\theta_2}$.

Suppose there is a weight element $R \in K(\mathfrak{g})$ with $0 \neq R(x_0, x_1) \in \mathfrak{g}_\alpha$ where $x_1 \in V_{2\theta_1+\theta_3}$. Then R has weight $-(3\theta_1 + \theta_2 + \theta_3 + \theta_4)$. If $x_2 \in V_{2\theta_2+\theta_4}$, then $R(x_0, x_2) \in \mathfrak{g}_{-\theta_1+2\theta_2-\theta_3} = 0$, and $R(x_1, x_2) \in \mathfrak{g}_{-\theta_1+\theta_2}$. However, $(2\theta_1+\theta_3, 2\theta_2+\theta_4, -\theta_1+\theta_2)$ is not a spanning triple, hence $R(x_1, x_2) = 0$ by Proposition 3.10. Then the first Bianchi identity for (x_0, x_1, x_2) yields that $\mathfrak{g}_\alpha V_{2\theta_2+\theta_4} = 0$, which is impossible.

Next, suppose that $0 \neq R(x_0, x_1) \in \mathfrak{g}_\alpha$ for some weight element $R \in K(\mathfrak{g})$ and $x_1 \in V_{2\theta_3+\theta_1}$. Then R has weight $-(2\theta_1 + \theta_2 + 2\theta_3 + \theta_4)$. If $x_2 \in V_{2\theta_4+\theta_1}$, then $R(x_0, x_2) \in \mathfrak{g}_{\theta_1-2\theta_3+\theta_4} = 0$, and $R(x_1, x_2) \in \mathfrak{g}_{\theta_4-\theta_2}$. However, by Proposition 3.10 and since $(2\theta_4 + \theta_1, 2\theta_3 + \theta_1, \theta_4 - \theta_2)$ is not a spanning triple, we have $R(x_1, x_2) = 0$, and from the Bianchi identity for (x_0, x_1, x_2) we get that $\mathfrak{g}_\alpha V_{2\theta_4+\theta_1} = 0$ which is a contradiction.

Finally, suppose that $0 \neq R(x_0, x_1) \in \mathfrak{g}_\alpha$ for some weight element $R \in K(\mathfrak{g})$ and $x_1 \in V_{2\theta_3+\theta_4}$. Then R has weight $-(\theta_1 + \theta_2 + 2\theta_3 + 2\theta_4)$. If $x_2 \in V_{2\theta_2+\theta_4}$, then $R(x_0, x_2) \in \mathfrak{g}_{\theta_1+2\theta_2-2\theta_3-\theta_4} = 0$, and $R(x_1, x_2) \in \mathfrak{g}_{\theta_2-\theta_1}$. However, by Proposition 3.10 and since $(2\theta_2 + \theta_4, 2\theta_3 + \theta_4, \theta_2 - \theta_1)$ is not a spanning triple, we have $R(x_1, x_2) = 0$, and from the Bianchi identity for (x_0, x_1, x_2) we get that $\mathfrak{g}_\alpha V_{2\theta_2+\theta_4} = 0$ which is a contradiction.

Thus, from Proposition 3.10, we get that $R(x_0, x_1) \in \mathfrak{t}_0$ for all extremal weight vectors x_0, x_1 and all $R \in K(\mathfrak{g})$ and since there are no spanning planes, $R(x_0, x_1) = 0$ for all such x_i and R by Lemma 3.19, and thus, $K(\mathfrak{g}) = 0$ by Lemma 3.11. Thus, $\mathfrak{g}$ is not Berger.

The representation $\overset{0}{\bullet}\!-\!\!\!-\!\overset{0}{\bullet}\!-\!\!\!-\!\overset{1}{\bullet}\!-\!\!\!-\!\overset{0}{\bullet}\!-\!\!\!-\!\overset{0}{\bullet}\!-\!\!\!-\!\overset{0}{\bullet}$.

This is the representation of $\mathfrak{gl}(7, \mathbb{C})$ on $W := \Lambda^3 V$ with $V = \mathbb{C}^7$. Every weight is extremal, and – up to the action of the Weyl group – the only spanning triples are $(\theta_1 + \theta_2 + \theta_3, \theta_4 + \theta_5 + \theta_6, \theta_1 - \theta_7)$ and $(\theta_1 + \theta_2 + \theta_3, \theta_1 + \theta_4 + \theta_5, \theta_1 - \theta_7)$. Thus, since there are no spanning planes, Proposition 3.10 and Lemma 3.19 imply that the only possible weights ρ of $K(\mathfrak{g})$ are $\rho = \theta_i$, some i.

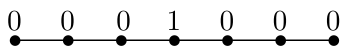
Suppose there is an $R \in K(\mathfrak{g})$ of weight $\rho = \theta_1$. We let $e_1, \ldots, e_7$ be the standard basis of V and write $e_{ijk} := e_i \wedge e_j \wedge e_k$, which spans the weight space $W_{\theta_i+\theta_j+\theta_k}$. Then, for weight reasons, we have $R(e_{123}, e_{456}) \in \mathfrak{g}_{\theta_1-\theta_7}$, and hence, there is some $c \in \mathbb{C}$ with $R(e_{123}, e_{456})y = c(e_{123} \wedge e_{456} \wedge y)e_1$, where we identify $\Lambda^7 V$ and $\mathbb{C}$.

Now $gR = R$ for all $g \in \mathrm{SL}(7, \mathbb{C})$ with $ge_1 = e_1$. Using this equivariance, we conclude that $R(e_1 \wedge \alpha_2, \alpha_3)y = c(e_1 \wedge \alpha_2 \wedge \alpha_3 \wedge y)e_1$ for all $\alpha_i \in \Lambda^i V$.

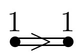
But now, applying the first Bianchi identity to $(e_{123}, e_{456}, e_{457})$ and using that for weight reasons $R(e_{456}, e_{457}) = 0$, we get that $2ce_{145} = 0$, i.e. $c = 0$, which means that $R(e_1 \wedge \alpha_2, \_) = 0$ for all $\alpha_2 \in \Lambda^2 V$. Then, from the Bianchi identity, it follows that $R(\alpha, \beta)(e_1 \wedge \alpha_2) = 0$ for all $\alpha, \beta \in W$ and $\alpha_2 \in \Lambda^2 V$. But this implies that $R(\alpha, \beta)y = cy + \tau(y)e_1$ for all $y \in V$, where $c \in \mathbb{C}$ and $\tau \in V^*$ with $\tau(e_1) = -3c$.

For weight reasons, $R(e_{234}, e_{567}) \in \mathfrak{t}_0$ and therefore, $R(e_{234}, e_{567})y = c((e_1 \wedge e_{234} \wedge e_{567})y - 3(y \wedge e_{234} \wedge e_{567})e_1)$ for some $c \in \mathbb{C}$. Using that $gR = R$ for all $g \in \mathrm{SL}(7, \mathbb{C})$ with $ge_1 = e_1$, we conclude that $R(\alpha, \beta)y = c((e_1 \wedge \alpha \wedge \beta)y - 3(y \wedge \alpha \wedge \beta)e_1)$ for all $\alpha, \beta \in W$ and some $c \in \mathbb{C}$.

But now, it is easy to show that this map R satisfies the Bianchi identity only if $c = 0$, i.e. $R = 0$ which is impossible. From here, we get that $K(\mathfrak{g}) = 0$, hence $\mathfrak{g}$ is not Berger.

The representation  .

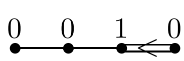
This is the representation of $\mathfrak{gl}(8, \mathbb{C})$ on $W := \Lambda^4 V$ with $V = \mathbb{C}^8$. It is easy to see that – up to the action of the Weyl group – the only spanning triple is $(\theta_1 + \theta_2 + \theta_3 + \theta_4, \theta_1 + \theta_5 + \theta_6 + \theta_7, \theta_1 - \theta_8)$. Since there are no spanning planes, it follows from Lemma 3.19 that the only possible weight of $K(\mathfrak{g})$ is $\rho = 0$, i.e. $K(\mathfrak{g})$ is a trivial $\mathfrak{g}$-module. Thus, $\dim(K(\mathfrak{g})) \leq 1$ by Proposition 3.4, and in fact, one can show that $\dim(K(\mathfrak{g})) = 1$ and it is spanned by the curvature map of the symmetric space $\mathrm{E}_7^{\mathbb{C}}/\mathrm{SL}(8, \mathbb{C})$.

The representation  .

It is easy to see that – up to the action of the Weyl group – the only extremal spanning triples are $(2\theta_1 + \theta_2, -\theta_1 - 2\theta_2, 2\theta_1)$, $(\theta_1 + 2\theta_2, \theta_1 - 2\theta_2, 2\theta_1)$, $(2\theta_1 - \theta_2, \theta_1 + 2\theta_2, 2\theta_1)$, $(2\theta_1 + \theta_2, 2\theta_1 - \theta_2, 2\theta_1)$, and $(2\theta_1 - \theta_2, -\theta_1 + 2\theta_2, \theta_1 + \theta_2)$. Moreover, there are no spanning planes.

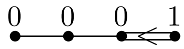
Let $x \in V_{2\theta_1 + \theta_2}, y \in V_{-\theta_1 - 2\theta_2}$ and $z \in V_{-2\theta_1 - \theta_2}$. Suppose that there is a weight element $R \in K(\mathfrak{g})$ with $0 \neq R(x, y) \in \mathfrak{g}_{2\theta_1}$. Then R has weight $-\theta_1 - \theta_2$, thus $R(x, z) \in \mathfrak{g}_{-\theta_1 - \theta_2}$ and $R(y, z) \in \mathfrak{g}_{-4\theta_1 - 2\theta_2} = 0$. However, since $(2\theta_1 + \theta_2, -\theta_1 - 2\theta_2, -\theta_1 - \theta_2)$ is not a spanning triple, we have $R(x, z) = 0$ by Proposition 3.10. Thus, from the Bianchi identity we get that $\mathfrak{g}_{2\theta_1} V_{-2\theta_1 - \theta_2} = 0$ which is impossible. Likewise, we exclude that $R(V_{2\theta_1 - \theta_2}, V_{\theta_1 + 2\theta_2}) \neq 0$ and $R(V_{2\theta_1 + \theta_2}, V_{2\theta_1 - \theta_2}) \neq 0$ for weight elements $R \in K(\mathfrak{g})$.

Thus, if $R \in K(\mathfrak{g})$ is a weight element of weight $\rho \neq 0$, then $R(x, y) \in \mathfrak{t}_0$ for all extremal weight vectors $x, y \in V$. But since there are no spanning planes, Lemma 3.19 implies that $R(x_0, x_1) = 0$ for all extremal weight vectors x_i. From there, the Bianchi identity easily implies that $R = 0$, i.e. $K(\mathfrak{g})$ has no weight elements of weight $\rho \neq 0$ and hence, $K(\mathfrak{g})$ is a trivial $\mathfrak{g}$-module. But $\mathfrak{g} \subset \mathfrak{csp}(V)$ and therefore, Proposition 3.4 implies that $K(\mathfrak{g}) = 0$ and hence, $\mathfrak{g}$ is not Berger.

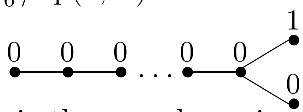
The representation  .

The only extremal spanning triples are – up to the action of the Weyl group – $(\theta_1 + \theta_3 + \theta_4, \theta_2 - \theta_3 - \theta_4, \theta_1 + \theta_2)$ and $(\theta_1 + \theta_2 + \theta_3, \theta_1 - \theta_2 - \theta_3, 2\theta_1)$, and there are no spanning planes. Thus, if $R \in K(\mathfrak{g})$ is a weight element of weight $\rho \neq 0$ then Proposition 3.10 and Lemma 3.19 imply that $R(x_0, x_1) = 0$ for all extremal weight vectors x_i, and then the Bianchi identity yields $R = 0$. Thus, the only possible weight of $K(\mathfrak{g})$ is $\rho = 0$, i.e. $K(\mathfrak{g})$ is a trivial $\mathfrak{g}$-module.

But again, $\mathfrak{g} \subset \mathfrak{csp}(V)$, and hence $K(\mathfrak{g}) = 0$ by Proposition 3.4, i.e. $\mathfrak{g}$ is not Berger.

The representation  .

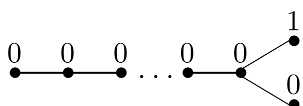
The only extremal spanning triples are, up to the action of the Weyl group, $(\theta_1 + \theta_2 + \theta_3 + \theta_4, \theta_1 - \theta_2 - \theta_3 - \theta_4, 2\theta_1)$ and there are no spanning planes. Then in complete analogy to the previous case we conclude that $K(\mathfrak{g})$ is a trivial $\mathfrak{g}$-module and hence, by Proposition 3.4, $\dim K(\mathfrak{g}) \leq 1$. But indeed, $K(\mathfrak{g})$ is one-dimensional and spanned by the curvature map of the symmetric space $\mathrm{E}_6^{\mathbb{C}}/\mathrm{Sp}(4, \mathbb{C})$.

The representation  **for** $n = 7$.

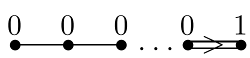
This representation is the complex spinor representation Δ_{14}^+. A calculation shows that $\Lambda^2 \Delta_{14}^+ \cong \Lambda^5 V \oplus V$ where $V = \mathbb{C}^{14}$.

Every weight is extremal, and one calculates that the only spanning triples are, up to the action of the Weyl group, $(\frac{1}{2}(\theta_1 + \ldots + \theta_7), \frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - \theta_4 - \ldots - \theta_7), \theta_1 + \theta_2)$ and $(\frac{1}{2}(\theta_1 + \ldots + \theta_7), \frac{1}{2}(\theta_1 - \theta_2 - \ldots - \theta_7), \theta_1 + \theta_2)$. Thus, for any $R \in K(\mathfrak{g})$, we have $R(x, y) = 0$ if x, y are weight vectors of weights $\frac{1}{2}(\theta_1 + \ldots + \theta_7)$ and $\frac{1}{2}(\theta_1 + \ldots + \theta_5 - \theta_6 - \theta_7)$, respectively, i.e. if $x \wedge y$ is the dominant weight vector of $\Lambda^5 V \subset \Lambda^2 \Delta_{14}^+$. Therefore, $K(\mathfrak{g}) \subset V \otimes \mathfrak{g}$.

But now, we let x, y, z be weight vectors of weights $\frac{1}{2}(\theta_1 + \ldots + \theta_7)$, $\frac{1}{2}(\theta_1 - \theta_2 - \ldots - \theta_7)$ and $\frac{1}{2}(-\theta_1 - \theta_2 + \theta_3 \ldots + \theta_7)$, respectively. Then $x \wedge z, y \wedge z \in \Lambda^5 V \subset \Lambda^2 \Delta_{14}^+$, and hence $R(x, z) = R(y, z) = 0$. Thus, by the Bianchi identity, $R(x, y)z = 0$ for all $R \in K(\mathfrak{g})$. However, if $K(\mathfrak{g}) \neq 0$ then there must be some $R \in K(\mathfrak{g})$ with $0 \neq R(x, y) \in \mathfrak{g}_{\theta_1 + \theta_2}$ by Proposition 3.10 which is a contradiction. Thus, $K(\mathfrak{g}) = 0$, and $\mathfrak{g}$ is not Berger.

The representation  **for** $n = 8$.

The only spanning triples are, up to the action of the Weyl group, $(\frac{1}{2}(\theta_1 + \ldots + \theta_8), \frac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \ldots - \theta_8), \theta_1 + \theta_2)$, and there are no spanning planes. Thus, since all weights are extremal, Proposition 3.10 and Lemma 3.19 imply that there are no weight elements $R \in K(\mathfrak{g})$ of weight $\rho \neq 0$. This implies that $K(\mathfrak{g})$ is a trivial $\mathfrak{g}$-module and so by Proposition 3.4, $\dim K(\mathfrak{g}) \leq 1$. In fact, $\dim K(\mathfrak{g}) = 1$, and is spanned by the curvature tensor of the symmetric space $\mathrm{E}_8^{\mathbb{C}}/\mathrm{Spin}(16, \mathbb{C})$.

The representations  **for** $n \leq 7$.

These are the complex spinor representations of $\mathfrak{spin}(2n+1)$ on $\Delta_{2n+1} \cong \Delta_{2n+2}^+$. Since $\mathfrak{spin}(2n+1) \subset \mathfrak{spin}(2n+2)$, it follows that $K(\mathfrak{spin}(2n+1)) \subset K(\mathfrak{spin}(2n+2))$, and hence by the above we see that $K(\mathfrak{spin}(2n+1)) = 0$ for $n = 6, 7$.

For $n = 5$, we consider $K(\mathfrak{h}) \subset K(\mathfrak{spin}(12))$ where $\mathfrak{h} = \mathfrak{spin}(11)$ acts on $V = \Delta_{11} \cong \Delta_{12}^+$. By Proposition 3.8, each $R \in K(\mathfrak{h})$ must be of the form $R(x, y) = \Omega(x, y)h + x \circ (hy) - y \circ (hx)$ for some $h \in \mathfrak{spin}(12)$.

Let $v \in \mathfrak{h}^\perp$. Then $0 = (R(x, y), v) = \Omega(x, y)B(h, v) + \Omega(vx, hy) - \Omega(vy, hx) = \Omega(x, y)B(h, v) - \Omega((hv + vh)x, y)$ for all $x, y \in \Delta_{11}$ and hence,

$$hv + vh = B(h, v) Id_V$$

for all $h \in K(\mathfrak{h})$ and $v \in \mathfrak{h}^\perp$.

However, a calculation then shows that this implies that $h = 0$, i.e. $K(\mathfrak{h}) = 0$ and $\mathfrak{h}$ is not Berger.

For $n = 4$, we consider $\mathfrak{h} = \mathfrak{spin}(9)$ acting on $V = \Delta_9$. It is well known that $\mathfrak{h} \subset \mathfrak{so}(V)$, and hence $\mathfrak{h} \oplus \mathbb{C}Id_V$ is not Berger by Proposition 3.2. Also, a calculation shows that $K(\mathfrak{h})$ is one-dimensional and is spanned by the curvature of the symmetric space $\mathrm{F}_4^{\mathbb{C}}/(\mathrm{Spin}(9,\mathbb{C}))$.

For $n = 3$, we have $\mathfrak{h} = \mathfrak{spin}(7)$ acting on $V = \Delta_7$. Again, $\mathfrak{h} \subset \mathfrak{so}(V)$, hence $\mathfrak{h} \oplus \mathbb{C}Id_V$ is not Berger. On the other hand, $\mathfrak{spin}(7)$ is one of the classically known examples of Riemannian holonomies, hence it is Berger.

The adjoint representations

Let $\mathfrak{g}$ be a complex simple Lie algebra with $\mathrm{rk}(\mathfrak{g}) \geq 2$, acting on $V = \mathfrak{g}$ via the adjoint representation $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Then $ad(\mathfrak{g}) \subset \mathfrak{so}(\mathfrak{g})$ with the inner product being given by the Killing form on $\mathfrak{g}$, and hence by Proposition 3.2, $ad(\mathfrak{g}) \oplus \mathbb{C}Id_{\mathfrak{g}}$ is not Berger. Fix elements $0 \neq A_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Delta$. Moreover, we denote elements of $\mathfrak{t}$ by $A_0, B_0, \ldots$.

Suppose there is an element $R \in K(ad(\mathfrak{g}))$ of weight $\rho \in \Delta$. We denote $R(x, y)$ by $\{x, y\}$, and thus, the first Bianchi identity reads

$$[\{x, y\}, z] + [\{y, z\}, x] + [\{z, x\}, y] = 0. \tag{31}$$

Moreover, since R is symmetric, we have the identity

$$B(\{x, y\}, [z, w]) = B(\{z, w\}, [x, y]). \tag{32}$$

In particular, $\{x, y\} = 0$ whenever $[x, y] = 0$, thus $\{A_0, B_0\} = 0$ for all $A_0, B_0 \in \mathfrak{t}$.

Let $-\rho \neq \alpha \in \Delta$ be a root such that $\alpha + \rho \in \Delta$. Then, for weight reasons, there is a $\sigma_\alpha \in \mathfrak{t}^*$ such that $\{A_0, A_\alpha\} = \sigma_\alpha(A_0)A_{\alpha+\rho}$ for all $A_0 \in \mathfrak{t}$. Now applying (31) to $(x, y, z) = (A_0, A_\alpha, A_{-(\alpha+\rho)})$ implies that $\sigma_\alpha(A_0)[A_{\alpha+\rho}, A_{-(\alpha+\rho)}] + \sigma_{-(\alpha+\rho)}(A_0)[A_\alpha, A_{-\alpha}] = 0$. Since $\alpha, \alpha+\rho$ are linearly independent, so are $[A_\alpha, A_{-\alpha}]$ and $[A_{\rho+\alpha}, A_{-(\rho+\alpha)}]$, and thus $\sigma_\alpha = 0$ for all $\alpha \neq -\rho$, i.e.

$$\{A_0, A_\alpha\} = 0 \ \text{ for all } \alpha \in \Delta,\ \alpha \neq -\rho.$$

If $\alpha, \beta \neq -\rho$ are roots such that $\alpha+\beta+\rho \neq 0$, then applying (31) to $(x, y, z) = (A_0, A_\alpha, A_\beta)$, we conclude that $\{A_\alpha, A_\beta\} = 0$ for all such roots.

If $\alpha, \beta, \gamma \in \Delta$ are pairwise different roots such that $\alpha + \beta + \rho = 0$, then applying (31) to $(x, y, z) = (A_\alpha, A_\beta, A_\gamma)$ and using the preceding remark, we get $\gamma(\{A_\alpha, A_\beta\}) = 0$ for all such roots γ. All this now implies that

$$\{A_\alpha, A_\beta\} = 0 \ \text{ for all roots } \alpha, \beta \neq -\rho.$$

But then, if $x, y \in \mathfrak{g}$ are arbitrary, applying (32) with $z = A_0, w = A_\alpha$, we get that $B(\{x, y\}, A_\alpha) = 0$ for all $\alpha \neq -\rho$; choosing $\alpha, \beta \in \Delta$ with $\alpha + \beta + \rho = 0$ and applying (32) with $z = A_\alpha, w = A_\beta$, we get $B(\{x, y\}, A_{-\rho}) = 0$. Finally, for $z = A_\alpha, w = A_{-\alpha}$, we get $B(\{x, y\}, [A_\alpha, A_{-\alpha}]) = 0$ for all $\alpha \neq \pm\rho$. All of this implies that $\{x, y\} = 0$ for all $x, y \in \mathfrak{g}$, i.e. $R = 0$ which is impossible.

This implies that $K(ad(\mathfrak{g}))$ does not have roots as weights, and hence is a trivial $\mathfrak{g}$-module. From here, Proposition 3.4 implies that $\dim K(ad(\mathfrak{g})) \leq 1$, and clearly, $\dim K(ad(\mathfrak{g})) = 1$, spanned by $\{x, y\} = [x, y]$. This is the curvature of the symmetric space $(\mathrm{G} \times \mathrm{G})/\Delta\mathrm{G}$.

The representations whose dominant weight is a short root

A similar argument as for the adjoint representations applies to these representations, as long as $\dim V_0 \geq 2$, where V_0 denotes the 0-weight space, and shows that all those are symmetric. This implies in particular, that the corresponding subgroups are symmetric if Δ is of type C_n with $n \geq 3$, or if Δ is of type F_4. The corresponding symmetric spaces are $\mathrm{SL}(2n,\mathbb{C})/\mathrm{Sp}(n,\mathbb{C})$ for $n \geq 3$, and $\mathrm{E}_6^{\mathbb{C}}/\mathrm{F}_4^{\mathbb{C}}$.

If Δ is of type B_n, then we obtain the standard representation of $\mathfrak{so}(2n+1)$ which was already discussed. If Δ is of type G_2, we get the 7-dimensional representation of G_2; this representation is orthogonal, thus its conformal extension is *not* Berger by Proposition 3.2. The representation of G_2 itself, however, is one of the classically known examples of a Berger group [Bes].

3.5 Complex tensor representations

In this section, we shall classify the irreducible Berger algebras whose semi-simple part is not simple. In the complex category, this implies that the representation is a tensor representation. That is, we have $V = V_1 \otimes V_2$. Moreover, there is a natural map $\mathrm{End}(V_1) \oplus \mathrm{End}(V_2) \to \mathrm{End}(V)$ which is induced by the tensor representation. We denote the image of this map by $\mathfrak{g} \subset \mathrm{End}(V)$ and its semi-simple part by $\mathfrak{g}_0$.

It is not hard to see that any irreducible Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is of the form $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with irreducible $\mathfrak{h}_i \subset \mathrm{End}(V_i)$. We denote the sets of weights and roots of $\mathfrak{h}_i$ by Φ^i and Δ^i, respectively. Then $\Delta = \Delta^1 \cup \Delta^2$ and $\Phi = \Phi^1 + \Phi^2$. Also, if $\alpha \in \Delta^1$, then $\Phi_\alpha = \Phi^1_\alpha + \Phi_2$.

We first consider the case where $\dim V_i \geq 3$ for $i = 1, 2$. We get the following classification.

Proposition 3.20 *Let V_1, V_2 be finite dimensional complex vector spaces with $n_i := \dim V_i \geq 3$, and let $V = V_1 \otimes V_2$, $\mathfrak{g}, \mathfrak{g}_0 \subset$ End(V) as above.*

If $\mathfrak{h} \subset \mathfrak{g}$ acts irreducibly on V, then $\mathfrak{h}$ is a Berger algebra iff it is congruent to an entry of the following list where in each case, $V = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$.

$\mathfrak{h}$	$K(\mathfrak{h})$	symmetric
$\mathfrak{gl}(n_1,\mathbb{C}) \oplus \mathfrak{gl}(n_2,\mathbb{C})$	$V^* \otimes V^*$	*no*
$\mathfrak{sl}(n_1,\mathbb{C}) \oplus \mathfrak{sl}(n_2,\mathbb{C})$	$\odot^2 V^*$	*no*
$\mathfrak{so}(n_1,\mathbb{C}) \oplus \mathfrak{so}(n_2,\mathbb{C})$	$\mathbb{C}$	*yes*
$\mathfrak{sp}\left(\frac{n_1}{2},\mathbb{C}\right) \oplus \mathfrak{sp}\left(\frac{n_2}{2},\mathbb{C}\right)$	$\mathbb{C}$	*yes*

For the proof, we need several Lemmas.

Lemma 3.21 *Let $V = V_1 \otimes V_2$, $\mathfrak{g}, \mathfrak{g}_0 \subset$ End(V) as above, and suppose that $\mathfrak{h} \cong \mathfrak{h}_1 \oplus \mathfrak{h}_2 \subset \mathfrak{g}$ is an irreducible Berger algebra. Then Φ^i_α consists of at most two elements for every $\alpha \in \Delta^i$.*

Proof. Suppose there is an $\alpha \in \Delta^1$ for which Φ^1_α has more than two elements. By Proposition 3.10, there is a spanning triple $(\lambda_0 + \mu_0, \lambda_1 + \mu_1, \alpha)$, $\lambda_i \in \Phi^1, \mu_i \in \Phi^2$. Since $\dim V_2 \geq 3$,

Φ^2 contains at least three elements. Thus, there are elements $\lambda \in \Phi^1_\alpha$, $\lambda \neq \lambda_0, \lambda_1$ and $\mu \in \Phi^2$, $\mu \neq \mu_1, \mu_2$. But then, $\lambda + \mu \in \Phi_\alpha$, and $(\lambda - \lambda_i) + (\mu - \mu_i) \notin \Delta$. This contradiction finishes the proof. ∎

Lemma 3.22 *Let $\mathfrak{h} \subset End(V)$ be an irreducible subalgebra, and let $\mathfrak{h}_s$ be the semi-simple part of $\mathfrak{h}$. Suppose that for some $\alpha \in \Delta$ the set Φ_α contains at most two elements. Then $\mathfrak{h}_s$ is conjugate to one of the following representations.*

1. *$\mathfrak{sl}(n, \mathbb{C})$ acting on $\mathbb{C}^n$; in this case, Φ_α is singleton for all $\alpha \in \Delta$.*
2. *$\mathfrak{so}(n, \mathbb{C})$ acting on $\mathbb{C}^n$. In this case, Φ_α contains two elements for all $\alpha \in \Delta$, and their sum equals α.*
3. *$\mathfrak{sp}(n, \mathbb{C})$ acting on $\mathbb{C}^{2n}$. In this case, Φ_α contains two elements if $\alpha \in \Delta$ is short, and their sum equals α, and $\Phi_\alpha = \{\frac{1}{2}\alpha\}$ if $\alpha \in \Delta$ is long.*
4. *$\mathfrak{g}_2$ acting on $\mathbb{C}^7$. Then Φ_α contains two elements if α is long, but three elements if α is short.*
5. *$\mathfrak{spin}(7, \mathbb{C})$ acting on $\mathbb{C}^8$. Then Φ_α contains two elements if α is long, and their sum equals α, and Φ_α contains three elements if α is short.*
6. *$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^n$; in this case, Φ_α contains two elements if α is a root of the $\mathfrak{sl}(n, \mathbb{C})$-summand, and contains n elements if α is a root of the $\mathfrak{sl}(2, \mathbb{C})$-summand.*
7. *$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sp}(n, \mathbb{C})$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^{2n}$; in this case, Φ_α contains two elements if α is a long root of the $\mathfrak{sp}(n, \mathbb{C})$-summand; Φ_α contains four elements if α is a short root of the $\mathfrak{sp}(n, \mathbb{C})$-summand, and it contains $2n$ elements if α is a root of the $\mathfrak{sl}(2, \mathbb{C})$-summand.*

Proof. Suppose that Φ_α contains at most two elements for some $\alpha \in \Delta$. Clearly, $|\langle \lambda, \alpha \rangle| \leq 2$ for all $\lambda \in \Phi$, since otherwise $\lambda + k\alpha \in \Phi_\alpha$ for $k = 1, 2, 3$.

Suppose that $\langle \lambda, \alpha \rangle = -2$ for some $\lambda \in \Phi$. Then $\Phi_\alpha = \{\lambda + \alpha, \lambda + 2\alpha\}$. If there is a $\beta \in \Delta$ with $\langle \beta, \alpha \rangle = 1$ then, after replacing β by $\alpha - \beta$ if necessary, we may assume that $\langle \lambda, \beta \rangle < 0$, and thus $\lambda + \alpha + \beta \in \Phi_\alpha$, which is a contradiction, as $\beta \neq 0, \alpha$. Thus, there is no $\beta \in \Delta$ with $\langle \beta, \alpha \rangle = 1$. This means that either rk $\mathfrak{h}_s = 1$, or Δ is of type B_n with α short. In the first case $\mathfrak{h}_s \subset \mathrm{End}(V)$ is the standard representation of $\mathfrak{so}(3, \mathbb{C})$ on $\mathbb{C}^3$, while in the second case we have the standard representation of $\mathfrak{so}(2n+1, \mathbb{C})$ on $\mathbb{C}^{2n+1}$, $n \geq 2$.

Next, suppose that $|\langle \lambda, \alpha \rangle| \leq 1$ for all $\lambda \in \Phi$, and suppose there is a $\beta \in \Delta^\perp_\alpha$ with $\langle \lambda, \beta \rangle = -1$. Then $\Phi_\alpha = \{\lambda + \alpha, \lambda + \alpha + \beta\}$. Thus, $\beta \in \Delta^\perp_\alpha$ with this property is unique, and it follows that $\{\pm\beta\}$ is a direct summand of $\Delta^\perp_\alpha$.

This implies that Δ is of type A_3, B_n (with α long), C_n (with α short), D_n, G_2 or Δ contains A_1 as a direct summand. For all these, one can show that they yield the representations listed above.

Finally, suppose that $\langle \lambda, \alpha \rangle = 1$ and $\langle \lambda, \beta \rangle = 0$ for all $\beta \in \Delta^\perp_\alpha$. If Δ is not of type A_n, then this implies that $\lambda = \frac{1}{2}\alpha$ which is possible only if Δ is of type C_n, and this yields the standard representation of $\mathfrak{sp}(n, \mathbb{C})$. If Δ is of type A_n, then all this implies that the representation is the standard representation of $\mathfrak{sl}(n, \mathbb{C})$ on $\mathbb{C}^n$. ∎

Proof of Proposition 3.20. Since by Lemma 3.21 Φ^i_α must contain at most two elements for *all* $\alpha \in \Delta^i$, it follows from Lemma 3.22 that only cases 1,2,3 and case 6 for $n = 2$ can occur. In the latter case, we have $\mathfrak{h}_s \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$, which is equivalent to the standard representation of $\mathfrak{so}(4, \mathbb{C})$ on $\mathbb{C}^4$.

Thus, if $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \subset \mathfrak{g}$ is a Berger algebra, then the semi-simple parts $(\mathfrak{h}_i)_s$ are either $\mathfrak{sl}(n_i, \mathbb{C})$, $\mathfrak{so}(n_i, \mathbb{C})$ or $\mathfrak{sp}(\frac{n_i}{2}, \mathbb{C})$ with their standard representations. But now from the explicit description of the curvature tensor in (25), it follows that $\mathfrak{h}_1 = \mathfrak{so}(n_1, \mathbb{C})$ implies $\mathfrak{h}_2 = \mathfrak{so}(n_2, \mathbb{C})$ and $\dim K(\mathfrak{h}) = 1$, and likewise, $\mathfrak{h}_1 = \mathfrak{sp}(\frac{n_1}{2}, \mathbb{C})$ implies $\mathfrak{h}_2 = \mathfrak{sp}(\frac{n_2}{2}, \mathbb{C})$ and $\dim K(\mathfrak{h}) = 1$. This proves the proposition. ∎

Now we turn to the case where $V = V_1 \otimes V_2$ with $\dim V_1 = 2$. In this case, $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{h}_2$ with an irreducible subalgebra $\mathfrak{h}_2 \subset \mathrm{End}(V_2)$. We begin with the following proposition.

Proposition 3.23 *Let $V = V_1 \otimes V_2$ and $\mathfrak{h}$, $\mathfrak{h}_2$ as in the preceding paragraph, and suppose that $\mathfrak{h}$ is an irreducible Berger algebra. If $\mathfrak{sl}(2, \mathbb{C})$ acts trivially on $K(\mathfrak{h})$ then $\mathfrak{h}$ is symmetric.*

Proof. We fix a basis e_1, e_2 of V_1, and let $\langle\ ,\ \rangle$ denote the determinant of V_1. Elements of V_1 and V_2 will be denoted by $e, f, \ldots$ and $x, y, \ldots$ respectively.

Since $K^1(\mathfrak{h}) \subset V^* \otimes K(\mathfrak{h}) \cong V_1^* \otimes (V_2^* \otimes K(\mathfrak{h}))$ and since, by hypothesis, $V_2^* \otimes K(\mathfrak{h})$ is a trivial $\mathfrak{sl}(2, \mathbb{C})$-module, it follows that $K^1(\mathfrak{h}) = V_1^* \otimes W$ for some subspace $W \subset V_2^* \otimes K(\mathfrak{h})$. Pick $\phi_1 \in W$, and define an element $\phi \in K^1(\mathfrak{h})$ by

$$\phi(e_1 \otimes x) := 0, \quad \phi(e_2 \otimes x) := \phi_1(x).$$

Then the second Bianchi identity for the triple $(e_1 \otimes x, e_1 \otimes y, e_2 \otimes z)$ yields $\phi_1(z)(e_1 \otimes x, e_1 \otimes y) = 0$, and hence, by polarization, $\phi_1(z)(e \otimes x, f \otimes y) = \langle e, f \rangle\, \psi(z)(x, y)$, where $\psi(z) \in \odot^2 V_2^* \otimes \mathfrak{h}$. Since $\phi_1(z) \in W$ is $\mathfrak{sl}(2, \mathbb{C})$-invariant, so is $\psi(z)$, and hence $\psi(z) \in \odot^2 V_2^* \otimes \mathfrak{h}_2$.

Next, consider the first Bianchi identity for $\phi_1(z) \in K(\mathfrak{h})$ for the triple $(e_1 \otimes x, e_1 \otimes y, e_2 \otimes w)$. It follows that $\psi(z)(w, x) \cdot y = \psi(z)(w, y) \cdot x$, and hence, $\psi(z) \in \mathfrak{h}_2^{(2)}$.

But there are only four irreducible Lie algebras $\mathfrak{h}_2$ for which $\mathfrak{h}_2^{(2)} \neq 0$ (cf. Table 4), and for these it is easy to show that $K(\mathfrak{h})$ is *not* a trivial $\mathfrak{sl}(2, \mathbb{C})$-module. Thus, we have that $\psi = 0$, i.e. $W = 0$, and hence, $K^1(\mathfrak{h}) = 0$. ∎

Now we obtain the following classification.

Proposition 3.24 *Let $V = V_1 \otimes V_2$ and $\mathfrak{h}$, $\mathfrak{h}_2$ as above, i.e. $\dim V_1 = 2$ and $\mathfrak{h}_2 \subset End(V_2)$ is irreducible, and suppose that $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{h}_2 \subset End(V)$ is a non-symmetric irreducible Berger algebra. Then $\mathfrak{h}_2$ is congruent to the standard representation of one of the Lie algebras $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$ or $\mathfrak{gl}(n, \mathbb{C})$.*

We begin with the following Lemma.

Lemma 3.25 *Let $V = V_1 \otimes V_2$ and $\mathfrak{h}$, $\mathfrak{h}_2$ as above. If $\mathfrak{h}$ is a non-symmetric irreducible Berger algebra then Φ^2_α contains at most two elements for some $\alpha \in \Delta^2$. Moreover, if the semi-simple part of $\mathfrak{h}_2$ is simple, then this holds true for all $\alpha \in \Delta^2$.*

Proof. The claim is obvious if $\dim V_2 = 2$. Thus, we assume from now on the $\dim V_2 \geq 3$.

Let $W \subset K(\mathfrak{h})$ be the subspace spanned by weight elements $R \in K(\mathfrak{h})$ of weight $-2\psi_0+\mu$, where μ is in the weight lattice of $\mathfrak{h}_2$ and ψ_0 is the generator of the weight lattice of $\mathfrak{sl}(2,\mathbb{C})$. Since $\mathfrak{h}$ is a non-symmetric Berger algebra, Proposition 3.23 implies that $W \neq 0$. Evidently, W is $\mathfrak{h}_2$-invariant. Thus, $\mathfrak{s} := \{R(u,v) \mid u, v \in V, R \in W\} \subset \mathfrak{h}$ is also $\mathfrak{h}_2$-invariant.

Suppose $\mathfrak{s} \subset \mathfrak{h}_1$. Then the first Bianchi identity for $(e_1 \otimes x, e_1 \otimes y, e_2 \otimes z)$ for independent $x, y, z \in V_2$ yields $R = 0$, i.e. $W = 0$ which is a contradiction.

Thus, $0 \neq \mathfrak{s} \cap \mathfrak{h}_2 \lhd \mathfrak{h}_2$. If α is a root of $\mathfrak{s} \cap \mathfrak{h}_2$, then evidently, there is a weight element $R \in W$ of weight $-2\psi_0 + \mu$, and weight vectors $u, v \in V$ such that $R(u,v) = A_\alpha$. Then u, v have weights $\psi_0 + \lambda_i$, $i = 0, 1$, for some $\lambda_i \in \Phi^2$, and by Proposition 3.10, $(\psi_0+\lambda_0, \psi_0+\lambda_1, \alpha)$ is a spanning triple. Note that $\Phi_\alpha = \{\pm\psi_0 + \lambda \mid \lambda \in \Phi^2_\alpha\}$.

If there was an element $\lambda \in \Phi^2_\alpha$ with $\lambda \neq \lambda_0, \lambda_1$, then $-\psi_0 + \lambda \in \Phi^2_\alpha$, but $(-\psi_0 + \lambda) - (\psi_0 + \lambda_i) \notin \Delta$, which is a contradiction. Therefore, Φ^2_α contains at most the two elements λ_0 and λ_1.

Finally, if $\mathfrak{h}_2$ is simple, then $\mathfrak{s} \cap \mathfrak{h}_2 = \mathfrak{h}_2$, and hence the above argument applies to all $\alpha \in \Delta^2$. ∎

Proof of Proposition 3.24. By Lemmas 3.22 and 3.25, we only must rule out the representation with $\mathfrak{h}_2 = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{h}_3$ on $\mathbb{C}^2 \otimes \mathbb{C}^n$, where $\mathfrak{h}_3 = \mathfrak{sl}(n,\mathbb{C})$ or $\mathfrak{sp}(\frac{n}{2},\mathbb{C})$ with their standard representations. In these cases, $\mathfrak{h} \cong \mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{h}_3$ acting on $V = \mathbb{C}^4 \otimes \mathbb{C}^n$. However, these were already shown not to be Berger algebras in Proposition 3.20. ∎

4 Existence results

In the previous chapter, we have characterized those irreducible subalgebras $\mathfrak{h} \subset \mathrm{End}(V)$ which are Berger and hence satisfy a *necessary* condition to occur as the holonomy of a torsion free connection on some manifold. However, this is still far from showing the existence of such connections. In fact, even in the case of *Riemannian holonomies*, more than three decades passed between the classification of Riemannian Berger algebras [Ber1] and the proof of their existence in all cases [Br2].

The method that was used in the latter reference is based on the method of Exterior Differential Systems and will be described in the following section. It turns out that this method applies to most other cases of Berger algebras as well, thus showing the existence of torsion free connections with these holonomies. We shall give only a brief outline of this method in section 4.1, but shall refer the reader to [Br2, Br3, Br4] for a more thorough treatement.

There is another method to construct torsion free connections with prescribed holonomy which is based on deformations of linear Poisson structures [CMS1, CMS2]. As it turns out, this method is *universal* in the class of symplectic holonomies, that is, any torsion free connection whose irreducible holonomy group H is properly contained in $\mathrm{Sp}(V,\Omega)$ locally comes from this construction. We shall summarize this method and some of its applications in section 4.3.

4.1 Exterior Differential Systems

Let M be a manifold of dimension n, and let $\pi : \mathfrak{F} \to M$ be its total coframe bundle. Given a closed Lie subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ where $\dim V = n$, the H-structures $F \subset \mathfrak{F}$ on M correspond to the sections of the quotient bundle $S_H := \mathfrak{F}/\mathrm{H}$. We shall now describe an Exterior Differential System on S_H whose integral manifolds are the sections of S_H corresponding to *torsion free* H-structures [Br2, Br3, Br4].

We fix a basis $e_1, \ldots, e_n$ of V and let $\rho_1, \ldots, \rho_n$ be the dual basis. Since $\mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{F})$ is a linear subspace, there are constants c_{ij}^l for $i, j = 1, \ldots, n$ and $l = 1, \ldots, d$ such that

$$A \in \mathfrak{h} \;\text{ iff }\; \sum_{i,j} c_{ij}^l (A\rho_i) \wedge \rho_1 \wedge \ldots \wedge \hat{\rho}_j \wedge \ldots \wedge \rho_n = 0 \;\text{ for } l = 1, \ldots, d.$$

On $\mathfrak{F}$, we decompose the V-valued tautological 1-form θ as $\theta = \sum_i \theta_i e_i$, and define for $q = 1, \ldots n$

$$I^q(\mathfrak{h}) := \left\{ \phi = \sum_{i,j_1,\ldots,j_q} c_{ij_1\ldots j_q} d\theta_i \wedge \theta_{j_1} \wedge \ldots \wedge \theta_{j_q} \;\middle|\; A \lrcorner \phi = 0 \text{ for all } A \in \mathfrak{h} \right\}.$$

Moreover, we let $\Omega := \theta_1 \wedge \ldots \wedge \theta_n$.

Lemma 4.1 *$F \subset \mathfrak{F}$ is an integral submanifold of $(I^*(\mathfrak{h}), \Omega)$ iff F is (an open subset of) a torsion free H-structure.*

Proof. Suppose $F \subset \mathfrak{F}$ is an integral submanifold. Since $\Omega|_F \neq 0$, it follows that the restriction $\pi : F \to M$ is a submersion. Also, $A_* \in TF$ for some $A \in \mathfrak{gl}(n, \mathbb{F})$ iff $A \lrcorner I^*(\mathfrak{h}) = 0$, and since $\mathfrak{h}$ is uniquely characterized by this property, this happens iff $A \in \mathfrak{h}$. Therefore, F is an open subset of an H-structure $\pi : F' \to M$, and since clearly, $I^*(\mathfrak{h})$ is H-invariant, it follows that F' is also an integral submanifold, thus we may assume that $F = F'$.

Let ω be a connection on F, and let $\Theta = d\theta + \omega \wedge \theta$ be its torsion. Decomposing $\Theta = \sum_i \Theta_i e_i$, using that $\phi|_F \equiv 0$ for all $\phi \in I^*(\mathfrak{h})$ and substituting $d\theta_i = \Theta_i - (\omega \wedge \theta)_i$, we conclude that there is an $\mathfrak{h}$-valued 1-form α such that $\Theta = \alpha \wedge \theta$. Thus, if we replace the connection ω by the connection $\omega' = \omega - \alpha$, then ω' is torsion free, thus F admits a torsion free connection and is hence a torsion free H-structure.

Conversely, if $\pi : F \to M$ is an H-structure with a torsion free connection ω, then it is straightforward to verify that $I^*(\mathfrak{h})|_F \equiv 0$, and hence F is an integral submanifold of $(I^*(\mathfrak{h}), \Omega)$. ∎

Since $I^*(\mathfrak{h})$ is invariant under the right action of H on $\mathfrak{F}$, it follows that there is a differential ideal $\mathcal{I}^*(\mathfrak{h})$ on S_H such that $I^*(\mathfrak{h}) = \pi^*(\mathcal{I}^*(\mathfrak{h}))$ where $\pi : \mathfrak{F} \to S_H$ is the natural projection. The independence condition Ω is invariant under H up to multiples, hence there is an n-form Ω_H on S_H such that $\pi^*(\Omega_H) = f\Omega$ for some non-vanishing function f on $\mathfrak{F}$.

Therefore, if $S \subset S_H$ is an integral manifold of the differential system $(\mathcal{I}^*(\mathfrak{h}), \Omega_H)$, then $\pi^{-1}(S)$ is an H-structure which is integral to the system $(I^*(\mathfrak{h}), \Omega)$ and hence is torsion free by Lemma 4.1. Thus, we have the following result.

Corollary 4.2 *There is a one-to-one correspondence between torsion free H-structures on M and integral manifolds of the Exterior Differential System $(\mathcal{I}^*(\mathfrak{h}), \Omega_H)$ on S_H described above.*

For many subgroups $\mathrm{H} \subset \mathrm{Aut}(V)$, it turns out that the Exterior Differential System $(\mathcal{I}^*(\mathfrak{h}), \Omega_H)$ on S_H is *involutive* and therefore amenable to the Cartan-Kähler theorem [BCG^3]. This was the key to the original proof of local existence of the exceptional holonomies G_2 and Spin(7) in dimensions 7 and 8 [Br2]. In fact, the local generality of torsion free connections with holonomy H has been determined [Br4]. We list the results obtained for the *metric holonomies*, i.e. for the holonomies of Levi-Civita connections of (pseudo-)Riemannian manifolds, in Table 5.

Table 5: Local generality of metric holonomies (modulo diffeomorphisms)
(Notation: "d of l" means "d functions of l variables")

n	H	local generality
$p+q \geq 2$ $2p$	$\mathrm{SO}(p,q)$ $\mathrm{SO}(p,\mathbb{C})$	$\frac{1}{2}n(n-1)$ of n $\frac{1}{2}p(p-1)^{\mathbb{C}}$ of $p^{\mathbb{C}}$
$2(p+q) \geq 4$ $2(p+q) \geq 4$	$\mathrm{U}(p,q)$ $\mathrm{SU}(p,q)$	1 of n 2 of $n-1$
$4(p+q) \geq 8$	$\mathrm{Sp}(p,q)$	$2(p+q)$ of $(2p+2q+1)$
$4(p+q) \geq 8$ $4p \geq 8$ $8p \geq 16$	$\mathrm{Sp}(p,q) \cdot \mathrm{Sp}(1)$ $\mathrm{Sp}(p,\mathbb{R}) \cdot \mathrm{SL}(2,\mathbb{R})$ $\mathrm{Sp}(p,\mathbb{C}) \cdot \mathrm{SL}(2,\mathbb{C})$	$2(p+q)$ of $(2p+2q+1)$ $2p$ of $(2p+1)$ $2p^{\mathbb{C}}$ of $(2p+1)^{\mathbb{C}}$
7 7 14	G_2 G_2' $\mathrm{G}_2^{\mathbb{C}}$	6 of 6 6 of 6 $6^{\mathbb{C}}$ of $6^{\mathbb{C}}$
8 8 16	Spin(7) Spin(4, 3) $\mathrm{Spin}(7,\mathbb{C})$	12 of 7 12 of 7 $12^{\mathbb{C}}$ of $7^{\mathbb{C}}$

4.2 Poisson manifolds

Let us briefly recall the definition and basic properties of a Poisson manifold. For a more detailed exposition, see e.g. [LM] or [V].

Definition 4.3 *A* Poisson manifold *is a differentiable manifold P together with a bilinear map, called the Poisson bracket*

$$\{\ ,\ \} : \otimes^2 C^\infty(P,\mathbb{R}) \longrightarrow C^\infty(P,\mathbb{R}),$$

satisfying the following identities:

(i) *the bracket is skew-symmetric:*

$$\{f, g\} = -\{g, f\},$$

(ii) *the bracket satisfies the Jacobi identity:*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

(iii) *the bracket satisfies the Leibniz rule in each of its arguments:*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

It is well-known that on every Poisson manifold $(P, \{\ ,\ \})$, there exists a unique smooth bivector field $\Lambda \in \Gamma(P, \Lambda^2 TP)$ such that the Poisson bracket is given by

$$\{f, g\} = \Lambda(df, dg). \tag{33}$$

Contraction with Λ yields a homomorphism $\Lambda^\# : T^*P \to TP$. For any 1-form $\alpha \in \Omega^1(P)$ we get a vector field $\eta_\alpha := \Lambda^\# \alpha$, and by abuse of notation, we let $\eta_f := \eta_{df}$ for functions $f \in C^\infty(P)$. Vector fields of the form η_f are called *Hamiltonian vector fields*. Thus, η_f is determined by the equation

$$\eta_f(g) = (\Lambda^\# df)(g) = \{f, g\} \quad \text{for all } f, g \in C^\infty(P, \mathbb{R}). \tag{34}$$

Moreover, the Jacobi identity and the Leibniz rule imply that

$$[\eta_f, \eta_g] = \eta_{\{f,g\}} \quad \text{and} \quad \eta_{fg} = f\eta_g + g\eta_f. \tag{35}$$

The *half-rank at* $p \in P$ of the Poisson structure is the smallest integer r such that

$$\Lambda^{r+1}(p) = 0,$$

the *rank at* $p \in P$ is twice the half-rank. It follows that the rank at p equals the rank of $\Lambda^\#_p : T^*_pP \to T_pP$ and analogously, we define the *corank at* $p \in P$ to be the corank of $\Lambda^\#_p$. The Poisson structure is called *non-degenerate at* p if $\Lambda^\#_p$ is an isomorphism, i.e. if the rank at p equals the dimension of P. In particular, if P is non-degenerate at a point then P must be even dimensional, and the set of non-degenerate points is open in P. If P is non-degenerate *everywhere*, then there is a natural symplectic 2-form Ω on P such that $\Lambda^\#$ is precisely the index-raising map associated to Ω. In fact, it is well known that symplectic structures are in a natural one-to-one correspondence with non-degenerate Poisson structures.

The *characteristic field* (*characteristic cofield*, respectively) of the Poisson structure is the subset $\mathcal{C} \subset TP$ ($\mathcal{C}^* \subset T^*P$, respectively) given by

$$\mathcal{C} = \Lambda^\#(T^*P), \qquad \mathcal{C}^* = ker\Lambda^\#.$$

Thus, the dimension of $\mathcal{C}_p$ ($\mathcal{C}_p^*$, repsectively) equals the rank (corank, respectively) of the Poisson structure at p. A *characteristic leaf* $\Sigma \subset P$ is a submanifold for which $T_p\Sigma = \mathcal{C}_p$ for all $p \in \Sigma$. From (34) and (35), it follows that through each $p \in P$ there is a locally unique characteristic leaf and the set of functions which vanish on Σ form a *Poisson ideal;* hence there is a naturally induced Poisson structure on Σ. Clearly, this Poisson structure on Σ is non-degenerate. Thus it follows that *every characteristic leaf of a Poisson manifold carries a natural symplectic structure.*

On P, there is also a bracket on $\Omega^1(P)$, the space of 1-forms on P, given by

$$\{\alpha, \beta\} := \mathfrak{L}_{\eta_\alpha}\beta - \mathfrak{L}_{\eta_\beta}\alpha - d\,\langle \Lambda, \alpha \wedge \beta \rangle \tag{36}$$

which satisfies for all $f, g \in C^\infty(P)$ and $\alpha, \beta \in \Omega^1(P)$

$$\{df, dg\} = d\{f, g\}, \quad \{\alpha, f\beta\} = \eta_\alpha(f)\beta + f\{\alpha, \beta\} \quad \text{and} \quad \eta_{\{\alpha,\beta\}} = [\eta_\alpha, \eta_\beta]. \tag{37}$$

One verifies easily that $(\Omega^1(P), \{\ ,\ ,\})$ with the bracket from (36) is a Lie algebra and $\Gamma(\mathcal{C}^*) \subset \Omega^1(P)$ is an *ideal.* Moreover, by (37), for each $p \in P$ there is a unique Lie algebra structure on $\mathcal{C}_p^*$ such that the evaluation map

$$\Gamma(\mathcal{C}^*) \longrightarrow \mathcal{C}_p^*$$

is a Lie algebra epimorphism.

For each characteristic leaf $\Sigma \subset P$, the restriction $\mathcal{C}^* \to \Sigma$ is a *vector bundle.* Since conjugation with the flow of a vector field η_α for any $\alpha \in \Omega^1(P)$ induces a Lie algebra isomorphism of $\Gamma(\mathcal{C}^*)$, it follows that $\mathcal{C}_p^* \cong \mathcal{C}_q^*$ for all $p, q \in \Sigma$ so that $\mathcal{C}^* \to \Sigma$ is a Lie algebra bundle.

Definition 4.4 *Let $(P, \{\ ,\ \})$ be a Poisson manifold. A* symplectic realization *of P is a symplectic manifold (S, Ω) and a submersion*

$$\pi : S \longrightarrow P$$

which is compatible with the Poisson structures, i.e.

$$\{\pi^*(f), \pi^*(g)\}_S = \pi^*(\{f, g\}) \quad \textit{for all } f, g \in C^\infty(P, \mathbb{R}), \tag{38}$$

where the Poisson bracket $\{\ ,\ \}_S$ on S is induced by the symplectic structure.

The following fact can be proven from the local description of Poisson manifolds in suitable coordinates.

Proposition 4.5 *[V, Thm.8.2] Let $(P, \{\ ,\ \})$ be a Poisson manifold. Then for every point $p_0 \in P$, there is an open neighborhood U of p_0 and a symplectic realization $\pi : S \longrightarrow U$.*

Examples:

1. $P = \mathbb{R}^{2n+k}$ with coordinates x_i, y_i, z_α, $i = 1, \dots, n, \alpha = 1, \dots, k$. Then the following defines a Poisson bracket:

$$\{f, g\} := \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}.$$

   Let $S := \mathbb{R}^{2n+2k}$ with coordinates $x_i, y_i, z_\alpha, w_\alpha$ and the symplectic form $\Omega := \sum_i dx_i \wedge dy_i + \sum_\alpha dz_\alpha \wedge dw_\alpha$. Then the projection $\pi : S \to P$ onto the first $2n + k$ coordinates is a symplectic realization of P.

2. *The Kirillov bracket.* Let $\mathfrak{g}$ be a (finite dimensional) Lie algebra. Then its dual space $\mathfrak{g}^*$ has a Poisson structure given by $\{f, g\}(\alpha) = \langle \alpha, [df_\alpha, dg_\alpha] \rangle$. This makes sense, since $df_\alpha, dg_\alpha \in T^*_\alpha \mathfrak{g}^* \cong \mathfrak{g}^{**} \cong \mathfrak{g}$.

   Let $S := T^*\mathrm{G}$ where G is a Lie group with Lie algebra $\mathfrak{g}$. Then the right invariant dual of the Maurer-Cartan form $\omega : T^*\mathrm{G} \to \mathfrak{g}^*$ is a symplectic realization of $\mathfrak{g}$.

In general, however, we cannot expect the existence of a *global* symplectic realization of a Poisson manifold P. In fact, even if the Poisson structure on P has constant rank, the obstruction for the existence of a global symplectic realization is given by a class in $H^3_{rel}(P, \mathcal{F})$, where $\mathcal{F}$ is the foliation by symplectic leafs [V].

If $\pi : S \to P$ is a symplectic realization then, in order to avoid confusion, we denote the Hamiltonian vector fields on S by ξ_h where $h \in C^\infty(S)$, while we continue to denote the Hamiltonian vector fields on P by η_f for $f \in C^\infty(P)$. With this, we have for all $f, g \in C^\infty(P)$

$$\begin{array}{rcll} [\xi_{\pi^*(f)}, \xi_{\pi^*(g)}] & = & \xi_{\{\pi^*(f), \pi^*(g)\}_S} & \text{by (35)} \\ & = & \xi_{\pi^*(\{f,g\})} & \text{by (38).} \end{array} \tag{39}$$

This implies, on the one hand, that the space

$$\mathfrak{X} := \{\xi_{\pi^*(f)} \mid \ f \in C^\infty(P)\} \subset \mathfrak{X}(S)$$

forms an infinite dimensional subalgebra of the Lie algebra $\mathfrak{X}(S)$ of vector fields on S, and on the other hand, that the distribution Ξ on S given by

$$\Xi_s = \{(\xi_{\pi^*(f)})_s \mid \ f \in C^\infty(P)\} \quad \text{for all } s \in S$$

is integrable. Evidently, $(\xi_{\pi^*(f)})_s$ only depends on $df_{\pi(s)}$, and since π is a submersion, the map $\pi^* : T_{\pi(s)}P \to T_sS$ is injective. Thus, the canonical map

$$\begin{array}{rccl} \Theta : & \Xi_s & \longrightarrow & T^*_{\pi(s)}P \\ & (\xi_{\pi^* f})_s & \longmapsto & df_{\pi(s)} \end{array} \tag{40}$$

is a linear isomorphism and hence, Ξ has constant rank equal to the dimension of P. Also, by (34) and (38) we have

$$\pi_*(\xi_{\pi^*(f)}) = \eta_f \quad \text{for all } f \in C^\infty(P). \tag{41}$$

Thus, if $F \subset S$ is an integral leaf of Ξ then by (41), there is a symplectic leaf $\Sigma \subset P$ such that $\pi : F \to \Sigma$ is a submersion.

Definition 4.6 *Let $\pi : S \to P$ be a symplectic realization of the Poisson manifold P. Let Ξ be the distribution from before and let $F \subset S$ be an integral leaf of Ξ such that the restriction $\pi : F \to \Sigma$ is a submersion for some characteristic leaf $\Sigma \subset P$.*

We call a vector field X on F ($U \subset F$, respectively) an infinitesimal Poisson symmetry on F *(local infinitesimal Poisson symmetry, respectively) if $[\mathfrak{X}, X] = 0$, i.e. if $[\xi_{\pi^* f}, X] = 0$ for all $f \in C^\infty(P)$.*

F is called symmetrically complete *if each local infinitesimal Poisson symmetry is the restriction of a (global) infinitesimal Poisson symmetry and if all infinitesimal Poisson symmetries are complete.*

Lemma 4.7 *Let $\pi : S \to P$ be a symplectic realization of the Poisson manifold P, let $F \subset S$ be an integral leaf of Ξ and let $X \in \mathfrak{X}(F)$ a (local) infinitesimal Poisson symmetry on F. Then $\pi_*(X) = 0$.*

Proof. Let $f \in C^\infty(P)$ and note that by the Leibniz rule, $\xi_{\pi^*(f^2)} = 2\pi^*(f)\xi_{\pi^*(f)}$. Thus, by definition of infinitesimal Poisson symmetries,

$$\begin{aligned} 0 &= [X, \xi_{\pi^*(f^2)}] \\ &= 2[X, \pi^*(f)\xi_{\pi^*(f)}] \\ &= 2df(\pi_* X)\xi_{\pi^*(f)}. \end{aligned}$$

But now, if $df \neq 0$ then $\xi_{\pi^*(f)} \neq 0$ so that $df(\pi_* X) = 0$ for all $f \in C^\infty$, i.e. $\pi_* X = 0$. ∎

Fix an integral leaf $F \subset S$ of Ξ and let $F_p := \pi^{-1}(p) \subset F$ for $p \in \Sigma$. By (41), $T_s F_p = \Theta^{-1}(\mathcal{C}_p^*)$ for all $p \in \Sigma$ and $s \in F_p$, i.e. $\mu_p := \Theta^{-1}$ is a Lie algebra valued one-form on F_p. By (39), μ satisfies the *Maurer-Cartan-equation* $d\mu + \mu \wedge \mu = 0$. Thus, if F_p is simply connected, there is an immersion

$$\jmath_p : F_p \longrightarrow \mathrm{G}_p,$$

where G_p is a Lie group with Lie algebra $\mathcal{C}_p^*$, such that $\jmath_p^*(\tilde{\mu}) = \mu$ where $\tilde{\mu}$ is the Maurer-Cartan form on G_p.

A vector fields X_p on F_p satisfies $[\xi_{\pi^*(\alpha)}, X_p] = 0$ for all $\alpha \in \mathcal{C}_p^*$ iff $[d\jmath(\xi_{\pi^*(\alpha)}), d\jmath(X_p)] = 0$ for all these α. Since the vector fields $d\jmath(\xi_{\pi^*(\alpha)})$ constitute all left invariant vector fields on G_p, it follows that this happens iff $d\jmath(X_p)$ is *right invariant*. We call such a vector field X_p a *fiber symmetry on F_p*. Thus, the fiber symmetries form a Lie algebra isomorphic to $\mathcal{C}_p^*$ which we denote by $\mathcal{S}_p$ and the union $\mathcal{S} := \bigcup \mathcal{S}_p$ is a Lie algebra bundle over Σ.

The sections $X \in \Gamma(\mathcal{S})$ are those vector fields X on F with $\pi_*(X) = 0$ and $[\xi_{\pi^*(\alpha)}, X] = 0$ for all $\alpha \in \Gamma(\mathcal{C}^*)$. Since $\Gamma(\mathcal{C}^*) \subset \Omega^1(P)$ is an ideal, it follows that for an arbitrary $\alpha \in \Omega^1(P)$ we have $[\xi_{\pi^*(\alpha)}, X] \in \Gamma(\mathcal{S})$, and thus, the formula

$$\nabla_{\eta_\alpha} X := [\xi_{\pi^*(\alpha)}, X]$$

yields a well-defined connection on $\mathcal{S} \to \Sigma$. Evidently, this connection is flat, and the infinitesimal Poisson symmetries on F are the vector fields which correspond to global *flat* sections. In particular, if we assume that Σ is simply connected, then through each $X_p \in \mathcal{S}_p$ there is a unique flat section X, so that we have the following result.

Proposition 4.8 *Let $\pi : F \to \Sigma$ be the submersion from above. Then the Lie algebra of local infinitesimal Poisson symmetries is isomorphic to $\mathfrak{g} := \mathcal{C}_p^*$ for any $p \in \Sigma$. Moreover, if Σ and all fibers $F_p := \pi^{-1}(p)$ are simply connected, then every local infinitesimal Poisson symmetry is the restriction of a (global) infinitesimal Poisson symmetry.*

Since the local infinitesimal Poisson symmetries act locally transitive on F_p, it follows that F is symmetrically complete iff $\pi : F \to \Sigma$ is a principal G-bundle where G is a Lie group with Lie algebra $\mathfrak{g}$.

We end this section by the following easy

Lemma 4.9 *Let $\pi : F \to \Sigma$ be the submersion as above, and let $V \subset \mathfrak{X}$ be a linear subspace (not necessarily a subalgebra) such that the evaluation $V \to T_sF$ is surjective for each $s \in F$. Then a vector field X is a (local) infinitesimal Poisson symmetry iff $[\xi, X] = 0$ for all $\xi \in V$ and $\pi_*(X) = 0$.*

Proof. The first direction follows from the definition and Lemma 4.7. For the converse, we may work locally. Let X be a vector field with $\pi_*(X) = 0$ and $[\xi, X] = 0$ for all $\xi \in V$. Then the restriction of X to F_p yields a vector field X_p on F_p. For $\alpha_0 \in \mathcal{C}_p^*$, there is a $\xi_{\pi^*(\alpha)} \in V$ such that $\alpha_p = \alpha_0$, so that $[\xi_{\pi^*(\alpha_0)}, X_p] = 0$ for all $\alpha_0 \in \mathcal{C}_p^*$. Thus, X_p is a fiber symmetry for all p and hence X yields a section of $\mathcal{S} \to \Sigma$. Moreover, $\nabla_{\pi_*(\xi)} X = 0$ for all $\xi \in V$ by definition, and thus, X is parallel. ∎

4.3 Construction of symplectic torsion free connections

We now turn to the construction of torsion free connections via Poisson structures. First of all, let us set up some notation.

Let V be a finite dimensional vector space and let $\mathrm{H} \subset \mathrm{Aut}(V)$ be any connected closed Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathrm{End}(V)$. As before, we consider the spaces of formal curvature maps $K(\mathfrak{h})$ and of formal curvature derivatives $K^1(\mathfrak{h})$. Moreover, we define the *set of full curvature maps*

$$K_0(\mathfrak{h}) := \{R \in K(\mathfrak{h}) \mid \langle \{R(x, y) \mid x, y \in V\} \rangle = \mathfrak{h}\}. \tag{42}$$

Let $W := \mathfrak{h} \oplus V$. We shall denote elements of $\mathfrak{h}$ and V by $A, B, \ldots$ and $x, y, \ldots$, respectively, and elements of W by $w, w', \ldots$. We may regard W as the semi-direct product of Lie algebras, i.e. we define a Lie algebra structure on W by the equation

$$[A + x, B + y] := [A, B] + A \cdot y - B \cdot x.$$

This induces a Poisson structure on the dual space W^*. Now, we wish to perturb this Poisson structure. For this, we need the

Definition 4.10 *A C^∞-map $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ is called* deforming *if*

(i) *ϕ is H-equivariant,*

(ii) *for every* $p \in \mathfrak{h}^*$, *the dual map* $(d\phi_p)^* : \Lambda^2 V \to \mathfrak{h}$ *is contained in* $K(\mathfrak{h})$.

Now, the following important observation is easily proven.

Proposition 4.11 *Let* V, $\mathfrak{h} \subset End(V)$, W *and* $K(\mathfrak{h})$ *as above, and let* $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ *be a deforming map. Let* $\Phi := \phi \circ pr$, *where* $pr : W^* \to \mathfrak{h}^*$ *is the natural projection. Then the following bracket on* W^* *is Poisson:*

$$\{f, g\}(p) := p([A + x, B + y]) + \Phi(p)(x, y). \tag{43}$$

Here, $df_p = A + x$ *and* $dg_p = B + y$ *are the decompositions of* $df_p, dg_p \in T_p^* W^* \cong W$.

Note that for $\phi = 0$, we simply obtain the Poisson structure induced by the Lie algebra structure on W.

Consider a Poisson structure on W^* induced by a deforming map $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ and let $\pi : S \to U$ be a symplectic realization of an open subset $U \subset W^*$. Let $F \subset S$ be an integral leaf of Ξ and $\pi : F \to \Sigma$ be the projection where $\Sigma \subset W^*$ is an integral leaf. Recall the map $\Theta : TF_s \longrightarrow T^*_{\pi(s)} W^*$ from (40). Since W^* is a vector space, we have $T^*_{\pi(s)} W^* \cong W^{**} \cong W$ canonically, so that we may regard Θ as a W-valued 1-form on F. We decompose $\Theta = \omega + \theta$ where ω and θ are 1-forms on F with values in $\mathfrak{h}$ and V, respectively.

Each $w \in W$ may be regarded as a (linear) function on $U \subset W^*$, and we shall write ξ_w instead of $\xi_{\pi^*(w)} \in \mathfrak{X}(F)$. From (39) we have $[\xi_{w_1}, \xi_{w_2}] = \xi_{\{w_1, w_2\}}$ so that by (43) we have

$$\begin{array}{rcll} [\xi_A, \xi_B] & = & \xi_{[A,B]} & \\ [\xi_A, \xi_x] & = & \xi_{A \cdot x} & \\ [\xi_x, \xi_y](s) & = & \xi_{d\Phi(p)(x,y)} & \text{where } p = \pi(s), \end{array} \tag{44}$$

where $A, B \in \mathfrak{h}$ and $x, y \in V$. In terms of $\Theta = \omega + \theta$ this is equivalent to

$$\begin{array}{rcl} d\theta & = & -\omega \wedge \theta \\ d\omega & = & -\omega \wedge \omega - \pi^*(d\Phi) \circ (\theta \wedge \theta). \end{array} \tag{45}$$

Here, $d\Phi$ is regarded as a map on U with values in $K(\mathfrak{h}) \subset \Lambda^2 V^* \otimes \mathfrak{h}$.

The first equation in (44) implies that the flow along the vector fields $\{\xi_A \mid A \in \mathfrak{h}\}$ induces a locally free H-action on $F \subset S$. After shrinking F if necessary, we may assume that $M := F/\mathrm{H}$ is a *manifold*. From (45) it follows that there is a unique torsion free connection on M and a unique immersion $\imath : F \hookrightarrow \mathfrak{F}_V$ into the V-valued coframe bundle $\mathfrak{F}_V$ of M such that $\theta = \imath^*(\underline{\theta})$ and $\omega = \imath^*(\underline{\omega})$, where $\underline{\theta}$ and $\underline{\omega}$ are the tautological and the connection 1-form on $\mathfrak{F}_V$, respectively. Clearly, the holonomy of this connection is contained in H and its curvature is represented by $\pi^*(d\Phi)$.

Definition 4.12 *Let* $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ *be a deforming map. Then a torsion free connection which is obtained from the above construction is called a* Poisson connection induced by ϕ.

We then get the following result.

Theorem 4.13 *Let V, $\mathfrak{h} \subset End(V)$ and $K(\mathfrak{h})$ be as before, and let $K_0(\mathfrak{h}) \subset K(\mathfrak{h})$ be as in (42). Consider a deforming map $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$. Furthermore, suppose that the open set $U_0 \subset \mathfrak{h}^*$ given by*

$$U_0 := (d\phi)^{-1}(K_0(\mathfrak{h}))$$

is non-empty. Then there exist Poisson connections induced by ϕ whose holonomy representations are equivalent to $\mathfrak{h}$. Moreover, if $\phi|_{U_0}$ is not affine, then not all of these connections are locally symmetric.

Proof. Let $\pi : S \to U$ be a symplectic realization where $U \subset U_0 \oplus V^* \subset W^*$ which exists by Proposition 4.5. Then the above construction produces Poisson connections induced by ϕ on some manifold $M = F/\mathrm{H}$. By (42), (45) and the *Ambrose-Singer Holonomy Theorem* [AS], the holonomy of this connection equals H.

To show the last part, let us assume that *all* connections which arise in this way are locally symmetric. Let $w := (p, q) \in U_0 \oplus V^*$. Then we may choose the symplectic realization $\pi : S \to U$ and $F \subset S$ such that $w \in \pi(F)$. It is then easy to show by (45) that the corresponding connection on $M := F/\mathrm{H}$ is locally symmetric iff $\mathfrak{L}_{\xi_x}(\pi^*(d\Phi)) = 0$ for all $x \in V$. Since π is a submersion this is equivalent to $\mathfrak{L}_{\eta_x}(d\Phi) = 0$ for all $x \in V$, or $\mathfrak{L}_{pr_*(\eta_x)}(d\phi) = 0$ for all $x \in V$. But now a calculation shows that for all $A \in \mathfrak{h}$,

$$(pr_*(\eta_x)_w)(A) = -q(A \cdot x) = -\jmath(q \otimes x)(A),$$

where $\jmath : V^* \otimes V \to \mathfrak{h}^*$ is the natural projection. Thus, by our assumption, it follows that $\mathfrak{L}_{\jmath(q\otimes x)}(d\phi)_p = 0$ for all $q \otimes x \in V^* \otimes V$ and $p \in U_0$. Since $\jmath$ is surjective this implies

$$\mathfrak{L}_\alpha(d\phi)_p = 0 \text{ for all } \alpha \in \mathfrak{h}^*,\ p \in U_0,$$

i.e. $d\phi|_{U_0}$ is constant, hence $\phi|_{U_0}$ is affine. ∎

By Theorem 4.13 it will suffice to address the question of *existence* of deforming maps ϕ in order to construct connections with prescribed holonomy.

Let $\mathcal{P}^{(k)}(\mathfrak{h})$ be the *k-th prolongation of* $K(\mathfrak{h}) \subset \Lambda^2 V^* \otimes \mathfrak{h}$ (cf. [Br4] for the definition). Then $\mathcal{P}^{(k)}(\mathfrak{h})$ is given by

$$\mathcal{P}^{(k)}(\mathfrak{h}) = \left(\odot^{k+1}(\mathfrak{h}) \otimes \Lambda^2 V^*\right) \cap \left(\odot^k(\mathfrak{h}) \otimes K(\mathfrak{h})\right),$$

where both are regarded as subspaces of $\odot^k(\mathfrak{h}) \otimes \mathfrak{h} \otimes \Lambda^2 V^*$. Suppose that there is an H-invariant element $\phi_k \in \mathcal{P}^{(k-1)}(\mathfrak{h})$. If we regard ϕ_k as a polynomial map of degree k, $\phi_k : \mathfrak{h}^* \to \Lambda^2 V^*$, then it follows that ϕ_k is deforming. Conversely, given an *analytic* map $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ with analytic expansion at $0 \in \mathfrak{h}^*$

$$\phi = \phi_0 + \phi_1 + \cdots,$$

then it is straightforward to show that ϕ is deforming iff all ϕ_k are, iff $\phi_k \in \left(\mathcal{P}^{(k-1)}(\mathfrak{h})\right)^H$.

Consider an element $\phi_2 \in \left(\mathcal{P}^{(1)}(\mathfrak{h})\right)^H$. On the one hand, we may regard ϕ_2 as an element of $\mathfrak{h} \otimes K(\mathfrak{h})$, on the other hand, it is easy to verify that also $\phi_2 \in V \otimes K^1(\mathfrak{h}) \subset V \otimes V^* \otimes K(\mathfrak{h})$. Thus, by the natural contractions, ϕ_2 induces H-equivariant linear maps

$$\begin{array}{rccc} \phi_2' : & \mathfrak{h}^* & \longrightarrow & K(\mathfrak{h}) \\ \phi_2'' : & V^* & \longrightarrow & K^1(\mathfrak{h}). \end{array} \tag{46}$$

Theorem 4.14 *Let $H \subset Aut(V)$ be a closed irreducible subgroup with Lie algebra $\mathfrak{h} \subset End(V)$, and suppose that there is an element $\phi_2 \in \left(\mathcal{P}^{(1)}(\mathfrak{h})\right)^H$ such that the corresponding H-equivariant maps ϕ_2' and ϕ_2'' from (46) are isomorphisms.*

Then every torsion free affine connection whose holonomy is contained in H is a Poisson connection induced by a polynomial map

$$\phi = \phi_2 + \tau,$$

with $\phi_2 \in \mathcal{P}^{(1)}(\mathfrak{h})$ from above and some H-invariant (possibly vanishing) 2-form τ.

For the proof, we shall need the following Lemma.

Lemma 4.15 *Let $H \subset Aut(V)$ be an irreducible representation of a connected, reductive Lie group H, and let $\mathfrak{h} \subset End(V)$ be the corresponding Lie algebra. If $\tau \in V^* \otimes V^*$ satisfies the condition*

$$\tau(x, A \cdot y) = \tau(y, A \cdot x) \quad \text{for all } x, y \in V \text{ and } A \in \mathfrak{h}, \tag{47}$$

then τ is skew-symmetric and hence an H-invariant 2-form.

Proof. Clearly, the problem is invariant under complexification, thus we assume that $\mathfrak{h}$ and V are complex. Let $P \subset V^* \otimes V^*$ be the subspace of all τ satisfying (47). It is easy to verify that P is H-invariant. If $\mathrm{rk}(\mathfrak{h}) = 1$, then by the Clebsch-Gordan formula we must have that $P = (P \cap \odot^2 V^*) \oplus (P \cap \Lambda^2 V^*)$. But it is easy to show that $P \cap \odot^2 V^* = 0$, and so the claim follows.

Let us now assume that $\mathrm{rk}(\mathfrak{h}) > 1$. Suppose there is an element $\tau \in P$ of weight $\rho \neq 0$. Let $x_\mu, x_\lambda \in V$ be elements of weights μ and λ. Then applying (47) with $A \in \mathfrak{t}$, we see that $\tau(x_\mu, x_\lambda)\lambda = \tau(x_\lambda, x_\mu)\mu$. Thus, if $\tau(x_\mu, x_\lambda) \neq 0$, we have that λ, μ are linearly dependent and $\lambda + \mu + \rho = 0$, hence

$$\text{if } \tau(x_\lambda, x_\mu) \neq 0 \text{ then } \lambda = c_1\rho,\ \mu = c_2\rho \text{ with } c_1 + c_2 + 1 = 0. \tag{48}$$

Let λ, μ be as in (48), and let $\alpha \in \Delta$ be a root independent of ρ. Then if $A_\alpha \in \mathfrak{h}_\alpha$, we have for $x_{\mu-\alpha} \in V_{\mu-\alpha}$

$$\tau(x_\lambda, A_\alpha x_{\mu-\alpha}) = \tau(x_{\mu-\alpha}, A_\alpha x_\lambda) = 0 \tag{49}$$

by (47) and (48). If $\alpha \in \Delta$ is dependent of ρ then we can write $\alpha = \beta + \gamma$ with roots β, γ independent of ρ. Thus, $\tau(x_\lambda, A_\beta A_\gamma x_{\mu-\alpha}) = \tau(x_\lambda, A_\gamma A_\beta x_{\mu-\alpha}) = 0$ by (49), and hence $\tau(x_\lambda, A_\alpha x_{\mu-\alpha}) = 0$, as $A_\alpha = [A_\beta, A_\gamma]$. Therefore $\tau(x_\lambda, A_\alpha x_{\mu-\alpha}) = 0$ for *all* $\alpha \in \Delta$, and since V_μ is spanned by $\{A_\alpha V_{\mu-\alpha} \mid \alpha \in \Delta\}$, it follows that $\tau = 0$ which is impossible.

Thus, P has only $\rho = 0$ as a weight, i.e. each $\tau \in P$ is H-invariant, and from there it is easy to show that $\tau \in \Lambda^2 V^*$. ∎

Proof of Theorem 4.14. Let $F \subset \mathfrak{F}_V$ be an H-structure on the manifold M where $\mathfrak{F}_V \to M$ is the V-valued coframe bundle of M, and denote the tautological V-valued 1-form on F by

θ. Suppose that F is equipped with a torsion free connection, i.e. an $\mathfrak{h}$-valued 1-form ω on F. Since ϕ_2' is an isomorphism, the *first and second structure equations* read

$$\begin{aligned} d\theta &= -\omega \wedge \theta \\ d\omega &= -\omega \wedge \omega - 2(\phi_2'(\mathbf{a})) \circ (\theta \wedge \theta), \end{aligned} \tag{50}$$

where $\mathbf{a} : F \to \mathfrak{h}^*$ is an H-equivariant map. Differentiating (50) and using that ϕ_2'' is an isomorphism yields the *third structure equation* for the differential of $\mathbf{a}$:

$$d\mathbf{a} = -\omega \cdot \mathbf{a} + \jmath(\mathbf{b} \otimes \theta), \tag{51}$$

for some H-equivariant map $\mathbf{b} : F \to V^*$, where $\jmath : V^* \otimes V \to \mathfrak{h}^*$ is the natural projection. The multiplication in the first term refers to the coadjoint action of $\mathfrak{h}$ on $\mathfrak{h}^*$. In other words, (51) should be read as

$$\begin{aligned} (\xi_A \mathbf{a})(B) &= \mathbf{a}([A, B]) \\ (\xi_x \mathbf{a})(B) &= \mathbf{b}(B \cdot x). \end{aligned}$$

Let us define the map $\mathbf{c} : F \to V^* \otimes V^*$ by

$$\mathbf{c}_p(x, y) := d\mathbf{b}(\xi_x)(y) - \phi_2(\mathbf{a}_p, \mathbf{a}_p, x, y). \tag{52}$$

Differentiation of (51) yields

$$\mathbf{c}_p(x, Ay) = \mathbf{c}_p(y, Ax) \quad \text{for all } x, y \in V \text{ and all } A \in \mathfrak{h}. \tag{53}$$

Then Lemma 4.15 implies that $\mathbf{c}_p \in \Lambda^2 V^*$ is H-invariant. Moreover, differentiation of (52) implies that $\xi_A(\mathbf{c}) = 0$ and $(\xi_x \mathbf{c})(y, z) = (\xi_y \mathbf{c})(x, z)$ for all $A \in \mathfrak{h}$ and $x, y, z \in V$. Since $\mathbf{c}$ is skew-symmetric, it follows that

$$d\mathbf{c} = 0,$$

i.e. $\mathbf{c}_p \equiv \tau \in \Lambda^2 V^*$ is *constant*. Thus, the H-equivariance of $\mathbf{b}$ and (52) yield

$$d\mathbf{b} = -\omega \cdot \mathbf{b} + \left(\mathbf{a}_p^2 \lrcorner \phi_2 + \tau\right) \circ \theta, \tag{54}$$

where $\lrcorner$ refers to the contraction of $\mathbf{a}_p^2 \in \odot^2 \mathfrak{h}^*$ with $\phi_2 \in \odot^2 \mathfrak{h} \otimes \Lambda^2 V^*$. In other words, (54) should be read as

$$\begin{aligned} (\xi_A \mathbf{b})(y) &= \mathbf{b}(A \cdot y) \\ (\xi_x \mathbf{b})_p(y) &= \phi_2(\mathbf{a}_p, \mathbf{a}_p, x, y) + \tau(x, y). \end{aligned}$$

Let us now define the Poisson structure on $W^* = \mathfrak{h}^* \oplus V^*$ induced by $\phi := \phi_2 + \tau$, and let $\pi := \mathbf{a} + \mathbf{b} : F \to W^*$. From (51) and (54) it follows that $\pi_*(\xi_w) = \eta_w$ for all $w \in W$, and from there it follows that, at least locally, the connection is indeed a Poisson connection induced by ϕ. ∎

Corollary 4.16 *Let $H \subset Sp(V, \Omega)$ be one of the representations listed in Corollary 3.7. Then every torsion free affine connection whose holonomy is contained in H is locally equivalent to a Poisson connection induced by the map $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ given by*

$$\langle \phi(\mathbf{a}), x \wedge y \rangle = 2\Omega(\mathbf{a}^2 x, y) + (2B(\mathbf{a}, \mathbf{a}) + \mathbf{c})\Omega(x, y)$$

for some constant $\mathbf{c}$. Here, we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the H-invariant bilinear form B on $\mathfrak{h}$ given by

$$B(A, x \circ y) = \langle Ax, y \rangle \quad \textit{for all } x, y \in V,\ A \in \mathfrak{h}.$$

Proof. We define the H-invariant element $\phi_2 \in \odot^2\mathfrak{h} \otimes \Lambda^2 V$ by

$$\phi_2(h_1, h_2, x, y) = 2\,\Omega(x, y)B(h_1, h_2) + \Omega((h_1h_2 + h_2h_1)x, y).$$

Since obviously the contraction map $\phi_2' : \mathfrak{h} \to K(\mathfrak{h})$ coincides with the isomorphism $A \mapsto R_A$ from Theorem 3.6 it follows that $\phi_2 \in (\mathcal{P}^{(1)}(\mathfrak{h}))^{\mathrm{H}}$. Moreover, it is straightforwars to verify that $\phi_2'' : V^* \to K^1(\mathfrak{h})$ is an isomorphism as well. Thus, the statement follows immediately from Theorems 4.13 and 4.14. ∎

From this complete characterization, we can now deduce the following properties.

Corollary 4.17 *Let M be a manifold which carries a torsion free connection whose holonomy is contained in one of the groups $H \subset Sp(V, \Omega)$ from Corollary 3.7. Then the following hold.*

(1) *The connection is analytic.*

(2) *The map $\pi := \mathbf{a} + \mathbf{b} : F \to W^*$ has constant even rank $2k$ which we shall call the rank of the connection. $k = 0$ iff the connection is flat.*

(3) *$\pi(F)$ is contained in a $2k$-dimensional characteristic leaf Σ of the Poisson structure on W^* induced by ϕ. In particular, $\pi : F \to \Sigma$ is a submersion onto its image.*

(4) *Conversely, every characteristic leaf $\Sigma \subset W^*$ can be covered by open neighborhoods $\{U_\alpha\}$ such that there are Poisson connections with $\pi(F_\alpha) = U_\alpha$.*

(5) *The moduli space of torsion free connections with any of the above holonomies is finite dimensional. Indeed, the 2nd derivative of the curvature at a single point in M completely determines the connection on all of M.*

Proof. (1) – (4) follow from the construction of the Poisson connections and the analyticity of ϕ, whereas (5) follows from the structure equations in the proof of Theorem 4.14. ∎

Of course, (4) is not an optimal statement. One would like to show that there are connections such that $\pi(F)$ is an *entire characteristic leaf*. The difficulty is that, in general, one cannot expect to have a *global* symplectic realization $\pi : S \to W^*$.

Definition 4.18 *Let $p : F \to M$ be an H-structure on the manifold M and let θ and ω denote the tautological and the connection one form on F, respectively. A vector field X on F is called an* infinitesimal connection symmetry on F *if $\mathcal{L}_X\theta = \mathcal{L}_X\omega = 0$.*

A vector field X_0 on M is called an infinitesimal connection symmetry on M *if there exists an infinitesimal connection symmetry on F with $p_*(X) = X_0$.*

Note that $\mathcal{L}_X\theta = 0$ implies that $X_0 = p_*(X)$ is a well-defined infinitesimal connection symmetry on M. Conversely, if X_0 is an infinitesimal connection symmetry on M then there is a unique infinitesimal connection symmetry X on F with $p_*(X) = X_0$. Thus, the Lie algebras of infinitesimal connection symmetries on M and on F are isomorphic.

Corollary 4.19 *Let $H \subset Aut(V)$ be a subgroup satisfying the premises of Theorem 4.14, let $p : F \to M$ be an H-structure on some manifold M with a torsion free connection and let $\pi : F \to W^*$ be the symplectic realization.*

Then the Lie algebra of infinitesimal Poisson symmetries coincides with the Lie algebra $\mathfrak{s}$ of infinitesimal connection symmetries. In particular, $\dim \mathfrak{s} = \dim W - 2k$ *where k is the rank of the connection, and $\mathfrak{s} \cong \mathcal{C}_p^*$ where $p \in \pi(F)$.*

Proof. Let X be an infinitesimal connection symmetry on F. Taking the Lie derivative of the second equation in (50) yields $0 = -2\phi_2'(X(\mathbf{a})) \circ \theta \wedge \theta$ and hence, $\phi_2'(X(\mathbf{a})) = 0$. By hypothesis, ϕ_2' is an isomorphism, so that $X(\mathbf{a}) = 0$. Likewise, the Lie derivative of (51) yields $0 = \jmath(X(\mathbf{b}) \otimes \theta)$, whence $X(\mathbf{b}) = 0$. Since $\pi = \mathbf{a} + \mathbf{b}$, we have $\pi_*(X) = 0$ for every infinitesimal connection symmetry. Thus, if we let $V := \{\xi_w \mid w \in W\}$ then Lemma 4.9 implies the claim. ▮

Note that in general there may be infinitesimal connection symmetries X with $\pi_*(X) \neq 0$ which are thus not Poisson symmetries. This happens for example for Poisson connections induced by $\phi = 0$.

5 Twistor theory of torsion free connections

In this section, we shall give a brief exposition of a twistor theory which can be associated to a holomorphic torsion free connection on a complex manifold M. This twistor theory has been developed by Merkulov in [Me1, Me2, Me3, Me4]. Throughout this section, we shall work in the complex category. That is, all manifolds, functions, vector fields, forms etc. are understood to be *holomorphic*. Also, TM and T^*M stand for the *holomorphic* (co-)tangent bundle of the manifold M.

Definition 5.1 *Let Y be a manifold, let $\mathcal{D}$ be a codimension-1 distribution on Y, and define the line bundle L by the exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow TY \longrightarrow L \longrightarrow 0. \tag{55}$$

If the L-valued 2-form θ on $\mathcal{D}$ given by $\theta(x, y) := [x, y]$ mod $\mathcal{D}$ is non-degenerate, then $\mathcal{D}$ is called a contact structure *on Y, and L is called the* contact line bundle *of Y.*

A submanifold $X \subset Y$ is called a contact submanifold *if $TX \subset \mathcal{D}$. If X is a contact submanifold with* $\dim X = (\dim Y - 1)/2$ *then X is called a* Legendre submanifold.

Note that from the maximal non-integrability of $\mathcal{D}$ it follows that Legendre submanifolds are contact submanifolds of maximal dimension.

Given a contact manifold Y and a compact Legendre submanifold $X_0 \subset Y$, a natural question is when the moduli space of "close-by" Legendre submanifolds carries the structure of a manifold. To make this more precise, we need the following definition.

Definition 5.2 *Let Y be a contact manifold. An* analytic family of compact Legendre manifolds *is a submanifold $S \hookrightarrow M \times Y$ with some manifold M such that the restriction $\pi_1 : S \to M$ is a submersion, and $X_p := \pi_2(\pi_1^{-1}(p)) \subset Y$ is a compact Legendre submanifold for all $p \in M$. Here, π_i is the projection of $M \times Y$ onto the i-th factor. In this case, we call M a* moduli space of Legendre submanifolds*, and say that the submanifolds X_p, $p \in M$, are contained in the analytic family.*

S is called maximal (locally maximal, respectively) *if for every analytic family $S' \subset M' \times Y$ with $M \subset M'$ and $S \subset S'$, it follows that $S = S'$ and $M = M'$ (S open in S' and M open in M', respectively).*

Then one can show the following deformation result.

Theorem 5.3 *[Me1] Let Y be a contact manifold with contact line bundle $L \to Y$, and let $X_0 \subset Y$ be a compact Legendre submanifold. If $H^1(X_0, L_{X_0}) = 0$ then there exists a maximal analytic family $S \hookrightarrow Y \times M$ containing X_0. Moreover, there is a canonical isomorphism $T_pM \cong H^0(X_p, L_{X_p})$, and hence,* $\dim M = \dim H^0(X_0, L_{X_0})$.

Now, let Y be a contact manifold, $X \subset Y$ compact Legendre, and assume that X is homogeneous, i.e. $X = \mathrm{G}/P$ where G is a semi-simple Lie group and $P \subset \mathrm{G}$ a parabolic subgroup. W.l.o.g. we assume that $\mathrm{G} = \mathrm{Aut}(X)$ is the biholomorphism group of X. Furthermore, suppose that the restriction L_X is very ample. It is well-known that in this case $H^1(X, L_X) = 0$.

Consider the moduli space M from Theorem 5.3. Since very ample line bundles on homogeneous manifolds are stable, it follows that all (X_p, L_{X_p}) are equivalent. Let (X_0, L_0) be a *reference bundle* which is equivalent to all (X_p, L_{X_p}), and define

$$F_0 := \left\{ \imath : \begin{array}{ccc} L_{X_p} & \longrightarrow & L_0 \\ \downarrow & & \downarrow \\ X_p & \longrightarrow & X_0 \end{array} \;\middle|\; p \in M,\ \imath \text{ a bundle isomorphism} \right\}. \tag{56}$$

With the canonical projection $\pi : F_0 \to M$, this is a principal bundle with structure group $\mathrm{G}_0 := \mathrm{Aut}(X_0, L_0)$ of bundle automorphisms of (X_0, L_0), that is $\mathrm{G}_0 \cong \mathrm{G} \times \mathbb{C}^*$. Now, we define an inclusion $F_0 \hookrightarrow \mathfrak{F}$ where $\mathfrak{F}$ is the total coframe bundle of M consisting of all isomorphisms of $T_pM \to V$ where V is a fixed vector space. This is done by setting $V := H^0(X_0, L_0)$ and using the correspondence

$$\imath \in F_0 \longmapsto [T_pM \cong H^0(X_p, L_{X_p}) \xrightarrow{\imath_*} V] \in \mathfrak{F}.$$

Since (X_0, L_0) is very ample, this map yields an inclusion, and it is obviously G_0-equivariant. Thus, its image $F_0 \hookrightarrow \mathfrak{F}$ is a G_0-structure on M.

Definition 5.4 *Let $S \hookrightarrow M \times Y$ be a maximal analytic family of compact homogeneous Legendre submanifolds, and suppose that (X_p, L_{X_p}) is very ample for some (and hence for all) $p \in M$. Let $G_0 := Aut(X_p, L_{X_p})$. Then the G_0-structure $F_0 \subset \mathfrak{F}$ on M constructed above is called the* canonical G_0-structure of the moduli space.

We shall now describe how certain G-structures F on a manifold M can be regarded as reductions of the canonical G_0-structure on a Legendre moduli space. To begin with, let M be a complex manifold, and let $\pi : T^*M \to M$ be its holomorphic cotangent bundle. We let λ denote the *Liouville form* on T^*M which is given by the equation

$$\lambda(v_\theta) := \theta(\pi_*(v_\theta))$$

for all $v_\theta \in T_\theta(T^*M)$. The 2-form

$$\omega := d\lambda$$

is non-degenerate and is called the *canonical symplectic form* on T^*M. It is also easy to verify that

$$m_t^*\lambda = t\lambda \quad \text{and} \quad m_t^*\omega = t\omega,$$

where $m_t : T^*M \to T^*M$ denotes the scalar multiplication by $t \in \mathbb{C}^*$.

The following is an easy fact relating contact structures to the symplectic form.

Proposition 5.5 *Let Y be a manifold, let $\mathcal{D}$ be a codimension-1 distribution on Y, and let L be the line bundle from (55). Consider the dual embedding $\imath : L^* \hookrightarrow T^*Y$. Then $\mathcal{D}$ is a contact structure iff $\imath^*\omega$ is non-degenerate where ω denotes the canonical symplectic form on T^*Y.*

Let V be a vector space with $\dim V = \dim M =: n$, and let $\mathrm{G} \subset \mathrm{Aut}(V)$ be an irreducible Lie subgroup. We let $\tilde{\mathcal{C}} \subset V^*\backslash\{0\}$ be the G-orbit of a highest weight vector of the dual representation, and let $\mathcal{C} \subset \mathbb{P}(V^*)$ be its projectivization. $\mathcal{C}$ is called the *sky* of G.

Consider a G-structure $F \subset \mathfrak{F}$ on M. Clearly, the cotangent bundle of M and its projectivization can be expressed as $T^*M = F \times_G V^*$ and $\mathbb{P}T^*M = F \times_G \mathbb{P}(V^*)$. Let

$$\tilde{S} := F \times_G \tilde{\mathcal{C}} \subset T^*M\backslash\{0\},$$

and

$$S := F \times_G \mathcal{C} \subset \mathbb{P}T^*M.$$

Obviously, S is the quotient of $\tilde{S}$ by the natural $\mathbb{C}^*$-action. The restriction $\omega_{\tilde{S}}$ of ω is no longer non-degenerate, and we let $\mathcal{N} \subset T\tilde{S}$ be its anihilator, i.e.

$$\mathcal{N} := \{v \in T\tilde{S} \mid v \lrcorner \omega_{\tilde{S}} = 0\}.$$

If we denote the canonical projection by $\pi : \tilde{S} \to M$, then it is easy to see that for all $p \in M$,

$$\mathcal{N} \cap T\pi^{-1}(p) = 0.$$

We make the simplifying assumption that $\dim \mathcal{N}$ is constant. Since $\omega_{\tilde{S}}$ is closed, it follows that $\mathcal{N}$ is integrable. Thus, restricting to a sufficiently small open subset of M, we may assume that the set of integral leafs of $\mathcal{N}$ is a *manifold* $\tilde{Y}$, i.e. we have a submersion

$$\tilde{\mu} : \tilde{S} \longrightarrow \tilde{Y}$$

such that $\mathcal{N}$ is precisely the tangent space of the fibers of $\tilde{\mu}$.

Let v be a vector field on $\tilde{S}$ with $v_s \subset \mathcal{N}$ for all $s \in \tilde{S}$. Then $\mathfrak{L}_v \omega_{\tilde{S}} = v \lrcorner d\omega_{\tilde{S}} + d(v \lrcorner \omega_{\tilde{S}}) = 0$, and therefore $\omega_{\tilde{S}}$ can be pushed down to $\tilde{Y}$ via $\tilde{\mu}$; in other words, there is a 2-form $\tilde{\omega}$ on $\tilde{Y}$ with

$$\omega_{\tilde{S}} = \tilde{\mu}^*(\tilde{\omega}).$$

It is obvious that $\tilde{\omega}$ is nondegenerate. Moreover, $0 = d\omega_{\tilde{S}} = \tilde{\mu}^*(d\tilde{\omega})$, and since $\tilde{\mu}$ is a submersion, it follows that $d\tilde{\omega} = 0$, i.e. $(\tilde{Y}, \tilde{\omega})$ is a symplectic manifold.

Since the distribution $\mathcal{N}$ is invariant under the natural $\mathbb{C}^*$-action on $\tilde{S}$, there is an induced $\mathbb{C}^*$-action on $\tilde{Y}$ for which

$$m_t^* \tilde{\omega} = t\tilde{\omega} \text{ for all } t \in \mathbb{C}^*. \tag{57}$$

Also, $\mathcal{N}$ factors through to an integrable distribution on $S = \tilde{S}/\mathbb{C}^*$, and if we denote the leaf space of this distribution by Y then we get a submersion $\mu : S \to Y$, and Y is the quotient of $\tilde{Y}$ by the $\mathbb{C}^*$-action. We denote the canonical projection by $p : \tilde{Y} \to Y$.

Let ∂_t denote the vector field on $\tilde{Y}$ whose flow induces this $\mathbb{C}^*$-action. Then by (57), $\mathfrak{L}_{\partial_t} \tilde{\omega} = \tilde{\omega}$, and since $\tilde{\omega}$ is closed, this implies that

$$\tilde{\omega} = d\tilde{\lambda}, \quad \text{where} \quad \tilde{\lambda} = \partial_t \lrcorner \tilde{\omega}.$$

Evidently, $\tilde{\lambda}(\partial_t) = 0$, and $\tilde{\lambda}$ is nowhere vanishing. Thus, for each $\tilde{y} \in \tilde{Y}$, there is a unique 1-form $0 \neq \underline{\lambda}_{\tilde{y}} \in T_y^* Y$ where $y = p(\tilde{y})$, such that $p^*(\underline{\lambda}_{\tilde{y}}) = \tilde{\lambda}_{\tilde{y}}$. Hence, the map $\imath : \tilde{Y} \hookrightarrow T^*Y \backslash \{0\}$ with $\imath(\tilde{y}) := \underline{\lambda}_{\tilde{y}}$ is well-defined and, by (57), a $\mathbb{C}^*$-equivariant embedding whose image is a $\mathbb{C}^*$-subbundle. It is now evident that $\tilde{\lambda} = \imath^* \lambda_Y$ where λ_Y denotes the Liouville 1-form on T^*Y, and thus $\tilde{\omega} = \imath^* \omega_Y$ where ω_Y is the canonical symplectic form on T^*Y. But since $\tilde{\omega}$ is non-degenerate on $\tilde{Y}$, Proposition 5.5 implies that the distribution $\mathcal{D}$ on Y which is anihilated by $\imath(\tilde{Y})$ defines a *contact structure on* Y, and $\imath(\tilde{Y}) \subset T^*Y \backslash \{0\}$ is precisely the dual of the contact line bundle $L \to Y$. Thus, identifying $\tilde{Y}$ with its image under this inclusion, we get the following commutative diagram:

$$\begin{array}{ccccc} & & \tilde{S} & \xrightarrow{\tilde{\mu}} & L^* \backslash \{0\} \\ & \swarrow & \downarrow \mathbb{C}^* & & \downarrow \mathbb{C}^* \\ M & \leftarrow & S & \xrightarrow{\mu} & Y \end{array}$$

For $p \in M$, we let $S_p := \pi^{-1}(p) \subset S$. Since $\mathcal{N} \cap TS_p = 0$, it follows that the map $\pi \times \mu : S \to M \times Y$ is an embedding. Moreover, it follows easily from the construction that $Y_p := \mu(S_p) \subset Y$ is a *contact submanifold*, and hence, S determines a analytic family of compact contact submanifolds.

Let us now address the question under which circumstances the contact submanifolds $Y_p \subset Y$ are *Legendre*. A dimension count yields that this is the case iff $\dim \mathcal{N} = \operatorname{codim}(S \subset \mathbb{P}T^*M) = \operatorname{codim}(\tilde{S} \subset T^*M)$. Evidently, we have the inequality $\dim \mathcal{N} \leq \operatorname{codim}(\tilde{S} \subset T^*M)$, as ω is non-degenerate on T^*M. Thus, $Y_p \subset Y$ is Legendre iff the dimension of $\mathcal{N}$ is maximal. If this is the case at some point, then by semi-continuity of the rank, this holds for a neighborhood of that point as well. If $\dim \mathcal{N}$ is maximal *everywhere* then we call the G-structure F *non-degenerate.*

Proposition 5.6 *Let M be a manifold, and let $F \subset \mathfrak{F}$ be a non-degenerate G-structure with irreducible $G \subset Aut(V)$, and let $S \subset \mathbb{P}T^*M$ be as before. Then the inclusion $\pi \times \mu : S \hookrightarrow M \times Y$ is a locally maximal analytic family of Legendre submanifolds of Y.*

Moreover, if F_0 denotes the canonical G_0-structure of M, then $F \subset F_0$, and F is a reduction of F_0 with structure group $G \subset G_0$.

Proof. We have already shown that the inclusion $\pi \times \mu : S \hookrightarrow M \times Y$ yields an analytic family of Legendre submanifolds since F is non-degenerate.

Let $u \in F_p$, i.e. $u : T_pM \to V$ is a linear isomorphism. Its dual $u^* : V^* \to T_p^*M$ maps $\tilde{\mathcal{C}}$ to $\tilde{S}_p$, and hence induces a bundle equivalence $(\mathcal{C}, \mathcal{O}(-1)_{\mathcal{C}}) \to (\tilde{S}_p \to S_p)$ where $\mathcal{O}(-1)_{\mathcal{C}}$ is the restriction of the tautological line bundle on $\mathbb{P}(V^*)$ to $\mathcal{C}$. Combining this with the isomorphism $(\mu, \tilde{\mu}) : (\tilde{S}_p \to S_p) \to (Y_p, L^*_{Y_p})$ we obtain an equivalence $(\mathcal{C}, \mathcal{O}(-1)_{\mathcal{C}}) \to (Y_p, L^*_{Y_p})$, and hence the dual map yields an bundle equivalence

$$\jmath(u) : (Y_p, L_{Y_p}) \longrightarrow (\mathcal{C}, \mathcal{O}(1)_{\mathcal{C}}) \tag{58}$$

This implies, in particular, that L_{Y_p} is very ample for all $p \in M$ and $H^1(Y_p, L_{Y_p}) = 0$ and $\dim H^0(Y_p, L_{Y_p}) = \dim V = \dim M$. Thus, by Theorem 5.3 it follows that M is of the same dimension as the moduli space of the maximal analytic Legendre family, and hence, M is locally maximal.

Moreover, $\jmath(u) \in F_0$ for all $u \in F$ with the canonical G_0-structure F_0 from (56), with reference bundle $(X_0, L_0) := (\mathcal{C}, \mathcal{O}(1)_{\mathcal{C}})$. Also, $\jmath$ is clearly G-equivariant, and hence the map

$$\jmath : F \longrightarrow F_0$$

is an embedding whose image is a G-reduction of F_0. ∎

It may seem at first glance that we loose some information when passing from a non-degenerate G-structure F on M to the G_0-structure F_0. However, to see that not much information is lost, we cite the following result.

Theorem 5.7 *[St] Let $G_s \subset GL(n, \mathbb{C})$ be an irreducible simple subgroup, and let $\mathcal{C}$ be the sky of G_s. Then $G_s = Aut(\mathcal{C})$, unless G_s is one of the following subgroups.*

1. *$G_2^{\mathbb{C}} \subset GL(7, \mathbb{C})$, in which case $Aut(\mathcal{C}) = SO(7, \mathbb{C})$,*
2. *$Spin(2n+1, \mathbb{C}) \subset GL(\Delta^+_{2n+2}, \mathbb{C})$, in which case $Aut(\mathcal{C}) = Spin(2n+2, \mathbb{C})$,*
3. *$G = Sp(n, \mathbb{C}) \subset GL(2n, \mathbb{C})$, in which case $Aut(\mathcal{C}) = SL(2n, \mathbb{C})$.*

Now $\mathrm{G}_0 = \mathrm{Aut}(\mathcal{C}, \mathcal{O}(1)_{\mathcal{C}}) \cong \mathrm{Aut}(\mathcal{C}) \times \mathbb{C}^*$, and $\mathrm{Aut}(\mathcal{C}) = \mathrm{G}_s$ and hence $\mathrm{G}_0 = \mathrm{G}_s \times \mathbb{C}^*$ in almost all cases. Therefore, the only times when $F \neq F_0$ is when the semi-simple part G_s of G is one of the exceptions listed in Theorem 5.7, or if G is semi-simple in which case F_0 is the conformal extension of F.

The reason why we are particularly interested in this twistor description of non-degenerate G-structures is the following.

Theorem 5.8 *Every torsion free G structure F on M with irreducible $G \subset GL(n, \mathbb{C})$ is non-degenerate, and thus M can be realized as a locally maximal analytic family of compact homogeneous Legendre submanifolds of a contact manifold Y.*

This will follow from the next result.

Proposition 5.9 *Let M be a manifold, let $F \subset \mathfrak{F}$ be a G-structure with irreducible $G \subset Aut(V)$, and let $\tilde{S} \subset T^*M$ be as before. Then F is non-degenerate iff $\tilde{S}$ is Poisson, in the sense that $\{f, g\}|_{\tilde{S}} = 0$ for all (local) functions f, g on T^*M with $f|_{\tilde{S}} = g|_{\tilde{S}} = 0$.*

If F is torsion free then F is non-degenerate. Moreover, in this case the distribution $\mathcal{N}$ is contained in the horizontal distribution.

Proof. We choose a local coordinate system $p = (p_1, \dots, p_n)$ on M. Then we have the natural coordinates $(p, q) = (p_1, \dots, p_n, q_1, \dots, q_n)$ on T^*M where q_i corresponds to the form dp_i. In these coordinates, the canonical symplectic form is given by

$$\omega = \sum_i dp_i \wedge dq_i.$$

Since $\tilde{S}_p \subset T^*_pM$ is algebraic, we can describe $\tilde{S} \subset T^*M$ by the equations

$$f_r(p, q) = 0, \qquad r = 1, \dots, d,$$

where the f_r are homogeneous polynomials in q, i.e. $f_r(p, cq) = c^{d_r} f(p, q)$ for some integers d_r. Then for each $v \in \mathcal{N}$, we have $v \lrcorner \omega \in span\{df_r\}$, and therefore,

$$\mathcal{N} \subset \left\{ \sum_i \frac{\partial f_r}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f_r}{\partial q_i}\frac{\partial}{\partial p_i} \,\middle|\, r = 1, \dots d \right\}.$$

Thus, $\dim \mathcal{N} = \mathrm{codim}(\tilde{S} \subset T^*M) = d$ iff this inclusion is an equality, i.e. iff the right hand side above is tangent to $\tilde{S}$, i.e. iff

$$\{f_r, f_s\} = \sum_i \frac{\partial f_r}{\partial p_i}\frac{\partial f_s}{\partial q_i} - \frac{\partial f_r}{\partial q_i}\frac{\partial f_s}{\partial p_i} = 0 \quad \text{for all } r, s.$$

This means precisely that $\tilde{S} \subset T^*M$ is Poisson.

Now, suppose that F carries a torsion free connection, and let $\mathcal{H}_\nabla$ be the horizontal distribution on T^*M. Then $\mathcal{H}_\nabla$ is spanned by the vector fields

$$\mathcal{H}_\nabla = span \left\{ \frac{\partial}{\partial p_i} - \sum_{j,k} \Gamma^k_{ij} q_k \frac{\partial}{\partial q_j} \,\middle|\, i = 1, \dots n \right\},$$

where Γ^k_{ij} are the Christoffel symbols of ∇. Since $\tilde{S}$ is parallel w.r.t. any connection on F, it follows that $\mathcal{H}_\nabla$ is tangent to $\tilde{S}$, i.e.

$$\frac{\partial f_r}{\partial p_i} = \sum_{j,k} \Gamma^k_{ij} q_k \frac{\partial f_r}{\partial q_j} \quad \text{for all } i, r.$$

Therefore,

$$
\begin{aligned}
\{f_r, f_s\} &= \sum_i \frac{\partial f_r}{\partial p_i}\frac{\partial f_s}{\partial q_i} - \frac{\partial f_r}{\partial q_i}\frac{\partial f_s}{\partial p_i} \\
&= \sum_{i,j,k} \Gamma^k_{ij} q_k \frac{\partial f_r}{\partial q_j}\frac{\partial f_s}{\partial q_i} - \Gamma^k_{ij} q_k \frac{\partial f_r}{\partial q_i}\frac{\partial f_s}{\partial q_j} \\
&= \sum_{i,j,k} (\Gamma^k_{ij} - \Gamma^k_{ji}) q_k \frac{\partial f_r}{\partial q_j}\frac{\partial f_s}{\partial q_i} = 0,
\end{aligned}
$$

since ∇ is torsion free and hence $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Finally, observe that

$$
\begin{aligned}
\sum_i \frac{\partial f_r}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f_r}{\partial q_i}\frac{\partial}{\partial p_i} &= \sum_i \left(\sum_{j,k} \Gamma^k_{ij} q_k \frac{\partial f_r}{\partial q_j}\frac{\partial}{\partial q_i} - \frac{\partial f_r}{\partial q_i}\frac{\partial}{\partial p_i} \right) \\
&= \sum_i \frac{\partial f_r}{\partial q_i} \left(\sum_{j,k} \Gamma^k_{ji} q_k \frac{\partial}{\partial q_j} - \frac{\partial}{\partial p_i} \right) \in \mathcal{H}_\nabla,
\end{aligned}
$$

since $\Gamma^k_{ij} = \Gamma^k_{ji}$, and hence, $\mathcal{N} \subset \mathcal{H}_\nabla$. ∎

We conclude this summary by explaining the relation of the Spencer cohomology of irreducible representations and the cohomologies of certain homogeneous vector bundles over their sky.

Let X be a compact complex homogeneous manifold, and let $L \to X$ be a very ample line bundle. This means that for each $p \in X$, the subspace of $V := H^0(X, L)$ consisting of all sections vanishing at p is a hyperplane, and the associated map $X \hookrightarrow \mathbb{P}(V^*)$ is an embedding. Thus, $L \to X$ is the restriction of the tautological line bundle $\mathcal{O}(-1) \to \mathbb{P}(V^*)$.

Let G be the automorphism group of (X, L). Then G acts on $V = H^0(X, L)$ and the homogeneity of X implies that this action is irreducible.

Lemma 5.10 *The Lie algebra $\mathfrak{g}$ of G is isomorphic to $H^0(X, L \otimes N^*)$ where $N := J^1 L$ is the first jet bundle of L. In fact, there is an exact sequence of Lie algebras*

$$
0 \longrightarrow \mathbb{C} \longrightarrow H^0(X, L \otimes N^*) \longrightarrow H^0(X, TX) \longrightarrow 0, \tag{59}
$$

where the last term is the Lie algebra of the biholomorphism group of X.

Proof. The first jet $N := J^1 L$ is determined by the exact sequence

$$
0 \longrightarrow T^*X \otimes L \longrightarrow N \longrightarrow L \longrightarrow 0, \tag{60}
$$

Taking the tensor product of the dual of (60) with L yields

$$
0 \longrightarrow \mathbb{C} \longrightarrow L \otimes N^* \longrightarrow TX \longrightarrow 0,
$$

and (59) is the beginning of its long exact sequence. Note that the kernel of the homomorphism $\mathrm{Aut}(X, L) \to \mathrm{Aut}(X)$ consists of global non-vanishing functions on X, and since X is compact, we have a sequence

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{G} \longrightarrow \mathrm{Aut}(X) \longrightarrow 0.$$

Since $\mathrm{Aut}(X)$ is semi-simple, there are no non-trivial extensions, whence this sequence splits and therefore, $\mathfrak{g} \cong \mathbb{C} \oplus H^0(X, TX) \cong H^0(X, L \otimes N^*)$ as claimed. ∎

Theorem 5.11 *[Me1] Let $L \to X$ be a very ample line bundle over a compact homogeneous complex manifold X and let $\mathfrak{g} := H^0(X, L \otimes N^*) \subset End(V)$ where $V := H^0(X, L)$ be as in Lemma 5.10. Then the following hold.*

1. *$\mathfrak{g}^{(k)} \cong H^0(X, L \otimes \odot^{k+1} N^*)$ for all $k \geq 0$.*
2. *There is an exact sequence of $\mathfrak{g}$-modules*

$$0 \longrightarrow H^{k,2}(\mathfrak{g}) \longrightarrow H^1(X, L \otimes \odot^{k+2} N^*) \longrightarrow H^1(X, L \otimes \odot^{k+1} N^*) \otimes V^*.$$

Proof. Since L is very ample there is a exact sequence of sheaves $V \otimes \mathcal{O}_X \to N \to 0$ whose dual yields an embedding

$$0 \longrightarrow N^* \longrightarrow V^* \otimes \mathcal{O}_X.$$

Thus, symmetrization induces maps $\odot^k N^* \otimes \Lambda^n V^* \to \odot^{k-1} N^* \otimes \Lambda^{n+1} V^*$. Moreover, since all sheaves involved are locally free, these maps form exact sequences

$$0 \to \odot^k N^* \to \odot^{k-1} N^* \otimes V^* \to \odot^{k-2} N^* \otimes \Lambda^2 V^* \to \ldots \to N^* \otimes \odot \Lambda^{k-1} V^* \to \Lambda^k V^* \to 0. \quad (61)$$

Tensoring (61) with L, the beginning of the associated long exact sequence yields

$$0 \to H^0(X, L \otimes \odot^k N^*) \to H^0(X, L \otimes \odot^{k-1} N^*) \otimes V^* \to H^0(X, L \otimes \odot^{k-2} N^*) \otimes \Lambda^2 V^*,$$

and the first assertion follows from the inductive definition of the prolongation from (7).

Next, we define the sheaves E_k from (61) by requiring the following sequences to be exact:

$$0 \longrightarrow L \otimes \odot^k N^* \longrightarrow L \otimes \odot^{k-1} N^* \otimes V^* \longrightarrow E_k \longrightarrow 0,$$

$$0 \longrightarrow E_k \longrightarrow L \otimes \odot^{k-2} N^* \otimes \Lambda^2 V^* \longrightarrow L \otimes \odot^{k-3} N^* \otimes \Lambda^3 V^*.$$

The long exact sequence associated to these sequences and the first part of the theorem induces exact sequences

$$0 \to \mathfrak{g}^{(k-1)} \to \mathfrak{g}^{(k-2)} \otimes V^* \to H^0(X, E_k) \to H^1(X, L \otimes \odot^k N^*) \to H^1(X, L \otimes \odot^{k-1} N^*) \otimes V^*,$$

and

$$0 \to H^0(X, E_k) \longrightarrow \mathfrak{g}^{(k-3)} \otimes \Lambda^2 V^* \longrightarrow \mathfrak{g}^{(k-4)} \otimes \Lambda^3 V^*.$$

It is easy to verify that the maps $\mathfrak{g}^{(k-1)} \to \mathfrak{g}^{(k-2)} \otimes V^*$ and $\mathfrak{g}^{(k-3)} \otimes \Lambda^2 V^* \longrightarrow \mathfrak{g}^{(k-4)} \otimes \Lambda^3 V^*$ in these sequences are the boundary maps of the Spencer complex, whence the assertion follows. ∎

The main application of this theory to the holonomy problem is the following

Corollary 5.12 *Let $\mathfrak{h} \subset End(V)$ be an irreducible complex Lie subalgebra, let $X \subset \mathbb{P}(V^*)$ be the sky of $\mathfrak{h}$ and $L \to X$ the restriction of the tautological bundle. If $\mathfrak{h}^{(1)} \neq 0$, i.e. if $\mathfrak{h}$ is not one of the entries of Table 4, then $\mathfrak{h}$ is a Berger subalgebra iff the kernel of the map*

$$H^1(X, L \otimes \odot^3 N^*) \longrightarrow H^1(X, L \otimes \odot^2 N^*) \otimes V^* \tag{62}$$

is non-trivial. If $\mathfrak{h}$ is a non-symmetric Berger algebra then the dimension of this kernel must be at least 2.

Proof. Using the exact sequence (8), we see that $K(\mathfrak{h}) \cong H^{1,2}(\mathfrak{h})$ if $\mathfrak{h}^{(1)} = 0$. Thus, $K(\mathfrak{h})$ is the kernel of (62) by Theorem 5.11, and the assertion follows immediately from the definition and Lemma 2.3. ∎

This corollary turns out to be an important tool in the discussion of irreducible complex Berger algebras. In fact, it was this result which was responsible for the discovery of the Berger algebras corresponding to the quaternionic symmetric spaces in Theorem 1.1 [CMS1, CMS2, MeSc1]. It also provides an efficient way of computing the curvature spaces $K(\mathfrak{h})$ for a given Lie algebra $\mathfrak{h} \subset \mathrm{End}(V)$ (cf. Proposition 3.8).

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