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ALGEBRAIC FLUX CORRECTION FOR FINITE ELEMENT DISCRETIZATIONS OF COUPLED SYSTEMS

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Summary. An algebraic approach to the design of high-resolution schemes for convection-dominated flow problems is revisited. A new multidimensional flux limiter of TVD type is developed in the framework of a scalar transport equation, and various extensions to systems of hyperbolic conservation laws are discussed. The implementation of an algebraic flux correction scheme based on edge-by-edge transformations of the compressible Euler equations to local characteristic variables is explained in some detail. Numerical results are presented for the shock tube problem and for a 2D supersonic flow in a scramjet inlet.

1 INTRODUCTION

It is well known that conventional Galerkin discretizations of convection-dominated flow problems tend to produce nonphysical oscillations in the vicinity of steep gradients. In particular, densities, energies, and other thermodynamic variables may become negative and trigger numerical instabilities, especially in the case of strongly coupled hyperbolic systems. Algebraic flux correction [1, 3] makes it possible to satisfy the positivity constraint *a posteriori* by adding a suitably designed perturbation to an *a priori* unstable Galerkin operator. First of all, a nonoscillatory low-order scheme is constructed using an artificial diffusion operator to enforce the M-matrix property. In the next step, a limited amount of compensating antidiffusion is applied to recover the high accuracy in regions where the solution is sufficiently smooth. In the present paper, a new multidimensional flux limiter is introduced and extended to the Euler equations of gas dynamics.

2 FLUX LIMITING FOR SCALAR EQUATIONS

The standard Galerkin discretization of a scalar conservation law gives rise to an algebraic system of the form $Au = b$. The resulting solution u remains nonoscillatory if $A = \{a_{ij}\}$ is an M-matrix. This property can be readily enforced by adding an artificial diffusion operator $D = \{d_{ij}\}$ designed to be a symmetric matrix with zero row and column sums. Importantly, the vector Du admits a decomposition into a sum of internodal fluxes

$$f_{ij} = d_{ij}(u_j - u_i) = -f_{ji} \tag{1}$$

which can be associated with edges of the sparsity graph for the global stiffness matrix. The default values of the artificial diffusion coefficients d_{ij} are given by [1]

$$d_{ij} = -\max\{a_{ij}, 0, a_{ji}\}, \tag{2}$$

whence all off-diagonal coefficients of $\tilde{A} = A + D$ are nonpositive, as required by the M-matrix criterion. Therefore, the solution of the perturbed system $\tilde{A}u = b$ is positivity-preserving. By construction, the discrete operators A and $\tilde{A}$ satisfy the relation

$$Au = b \quad \Leftrightarrow \quad \tilde{A}u = b + Du, \qquad (Du)_i = \sum_{j\neq i} f_{ij}. \tag{3}$$

Thus, the artificial diffusion built into $\tilde{A}$ can be removed by adding the sums of raw antidiffusive fluxes f_{ij} to the corresponding rows of b. Some fluxes are harmless but others may need to be limited in order to suppress spurious oscillations. To this end, every flux is multiplied by a solution-dependent correction factor α_{ij}, which yields

$$\tilde{A}u = \tilde{b}, \qquad \tilde{b}_i = b_i + \sum_{j\neq i} \alpha_{ij} f_{ij}, \qquad 0 \leq \alpha_{ij} \leq 1. \tag{4}$$

Note that the original Galerkin scheme is recovered for $\alpha_{ij} \equiv 1$. Hence, the correction factors α_{ij} should be as close to 1 as possible without violating the positivity constraint.

A very general approach to the computation of α_{ij}, which includes nodal limiters of FCT and TVD type, is presented in [3]. A brand new representative of the latter family is as follows: For each pair of neighboring nodes i and j such that $\tilde{a}_{ji} \leq \tilde{a}_{ij} \leq 0$

1. Compute the sums of positive/negative antidiffusive fluxes to be limited

$$P_i^+ := P_i^+ + \max\{0, f_{ij}\}, \qquad P_i^- := P_i^- + \min\{0, f_{ij}\}. \tag{5}$$

2. Compute the upper/lower bounds $Q_i^\pm$ to be imposed on the sums $P_i^\pm$

$$\begin{aligned} Q_i^+ &:= Q_i^+ + \max\{0, -f_{ij}\}, \qquad & Q_j^+ &:= Q_j^+ + \max\{0, f_{ij}\}, \\ Q_i^- &:= Q_i^- + \min\{0, -f_{ij}\}, \qquad & Q_j^- &:= Q_j^- + \min\{0, f_{ij}\}. \end{aligned} \tag{6}$$

3. Apply the nodal correction factor $R_i^\pm$ evaluated at the 'upwind' node i

$$R_i^\pm = \min\{1, Q_i^\pm / P_i^\pm\}, \qquad \alpha_{ij} = \begin{cases} R_i^+, & \text{if } f_{ij} > 0, \\ R_i^-, & \text{otherwise.} \end{cases} \tag{7}$$

This algorithm differs from its predecessors in the definition of the bounds $Q_i^\pm$ which were formerly constructed using the coefficients of $\tilde{A}$. In our experience, the new flux limiter (5)-(7) is the only one that does not inhibit convergence to a steady state limit. For an extension to time-dependent problems, the interested reader is referred to [3, 4].

3 FLUX LIMITING FOR SYSTEMS OF EQUATIONS

An extension of high-resolution schemes to coupled systems is nontrivial due to the lack of reliable physical and mathematical criteria on which to build. In many cases, the use of scalar limiting techniques produces disappointing results. In the context of algebraic flux correction, the following limiting strategies have been explored so far [2]

- naive approach: stand-alone discretization/upwinding/limiting for all variables;
- separate limiting followed by synchronization of the resulting correction factors;
- limiting in terms of arbitrary variables (conservative, primitive, characteristic).

The first approach is likely to fail if the coupling of the governing equations is very strong. Flux-limited solutions computed using the synchronization of correction factors and/or change of variables are typically much more accurate but the optimal choice of the variables to be limited is highly problem-dependent. The use of characteristic variables is appropriate for hyperbolic systems such as the compressible Euler equations

$$\frac{\partial U}{\partial t} + \sum_d \frac{\partial F^d}{\partial x_d} = 0 \qquad \Leftrightarrow \qquad \frac{\partial U}{\partial t} + \sum_d A^d \frac{\partial U}{\partial x_d} = 0, \tag{8}$$

where $U = [\rho, \rho\mathbf{v}, \rho E]^T$ is the vector of conservative variables (density, momentum, and total energy) and $A^d = \frac{\partial F^d}{\partial U}$ is the Jacobian matrix for the d^{th} coordinate direction.

The lumped-mass Galerkin discretization of (8) can be represented as follows [2]

$$\left[M_L \frac{\mathrm{d}\mathrm{U}}{\mathrm{d}t}\right]_i + \sum_d \sum_{j\neq i} \mathrm{A}^d_{ij}(\mathrm{U}_j - \mathrm{U}_i) = 0, \tag{9}$$

where A^d_{ij} denotes the unidirectional *Roe matrix* for the edge ij. The hyperbolicity of the Euler equations implies the existence of the factorization $\mathrm{A}^d_{ij} = \mathrm{R}_{ij}\Lambda_{ij}\mathrm{R}^{-1}_{ij}$, where $\Lambda_{ij} = \mathrm{diag}\{\lambda_k\}$ is the matrix of eigenvalues and R_{ij} is the matrix of right eigenvectors.

The multiplication of the solution difference $\mathrm{U}_j - \mathrm{U}_i$ by the matrix $\mathrm{L}_{ij} = \mathrm{R}^{-1}_{ij}$ converts it to a set of local characteristic variables which are essentially decoupled and can be handled separately, as in the case of scalar equations. After the elimination of positive eigenvalues and flux limiting, the remaining artificial viscosity (if any) is transformed back to the conservative variables and inserted into the defect vector [2]. This strategy leads to the following generalization of the algorithm presented in the previous section

1. Define the raw (anti-)diffusive fluxes in terms of local characteristic variables

$$\mathrm{F}_{ij} = -|\Lambda_{ij}|\Delta\mathrm{W}_{ij} = -\mathrm{F}_{ji}, \qquad \Delta\mathrm{W}_{ij} = \mathrm{R}^{-1}_{ij}(\mathrm{U}_j - \mathrm{U}_i). \tag{10}$$

2. If the eigenvalue λ_k is positive, exchange node numbers i and j in what follows.

3. In a loop over k, update the sums $P_{i,k}^{\pm}$ and $Q_{i,k}^{\pm}$ for the corresponding field

$$\begin{cases} \mathrm{F}_{ij}^k > 0 \quad \Rightarrow \quad P_{i,k}^{+} := P_{i,k}^{+} + \mathrm{F}_{ij}^k, \quad Q_{i,k}^{-} := Q_{i,k}^{-} - \mathrm{F}_{ij}^k, \quad Q_{j,k}^{+} := Q_{j,k}^{+} + \mathrm{F}_{ij}^k, \\ \mathrm{F}_{ij}^k \leq 0 \quad \Rightarrow \quad P_{i,k}^{-} := P_{i,k}^{-} + \mathrm{F}_{ij}^k, \quad Q_{i,k}^{+} := Q_{i,k}^{+} - \mathrm{F}_{ij}^k, \quad Q_{j,k}^{-} := Q_{j,k}^{-} + \mathrm{F}_{ij}^k. \end{cases} \tag{11}$$

4. In a loop over nodes, compute the correction factors $R_{i,k}^{\pm} = \min\{1, Q_{i,k}^{\pm}/P_{i,k}^{\pm}\}$.

5. In a loop over edges, limit $\Delta \mathrm{w}_{ij}$ using the orientation convention of Step 2

$$\Delta \widehat{\mathrm{w}}_{ij}^k = \begin{cases} R_{i,k}^{+} \Delta \mathrm{w}_{ij}^k & \text{if } \mathrm{F}_{ij}^k \geq 0, \\ R_{i,k}^{-} \Delta \mathrm{w}_{ij}^k & \text{otherwise.} \end{cases} \tag{12}$$

6. Transform back to the conservative variables and insert the edge contributions

$$\mathrm{F}_{ij}^{*} = \mathrm{R}_{ij} |\Lambda_{ij}| (\Delta \mathrm{w}_{ij} - \Delta \widehat{\mathrm{w}}_{ij}), \qquad \mathrm{F}_{ji}^{*} := -\mathrm{F}_{ij}^{*} \tag{13}$$

   into the right-hand side of (9) and/or into the associated defect vector.

This sort of algebraic flux correction should be performed separately for each component of the sum over d. Further details regarding the implementation of characteristic flux limiters and boundary conditions for the Euler equations can be found in [2, 4].

4 NUMERICAL EXAMPLES

Fig. 3 illustrates the performance of the new algorithm as applied to the classical shock tube problem. The exact solution and the low-order approximation are depicted by the dashed and dotted lines, respectively. The numerical solution displayed in the right

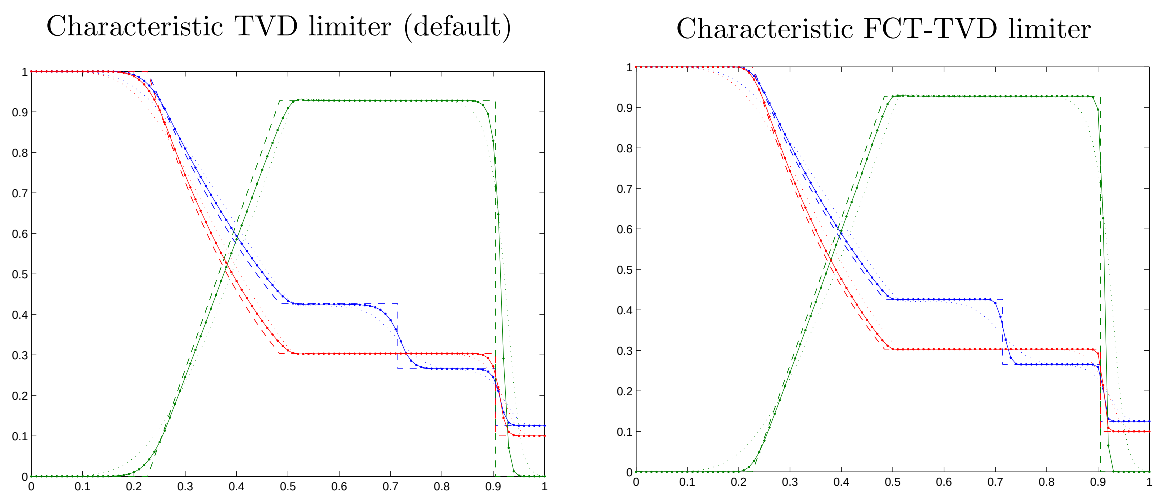

Figure 1: Shock tube: 100 linear elements, $\Delta t = 10^{-3}$, solutions at $t = 0.231$.

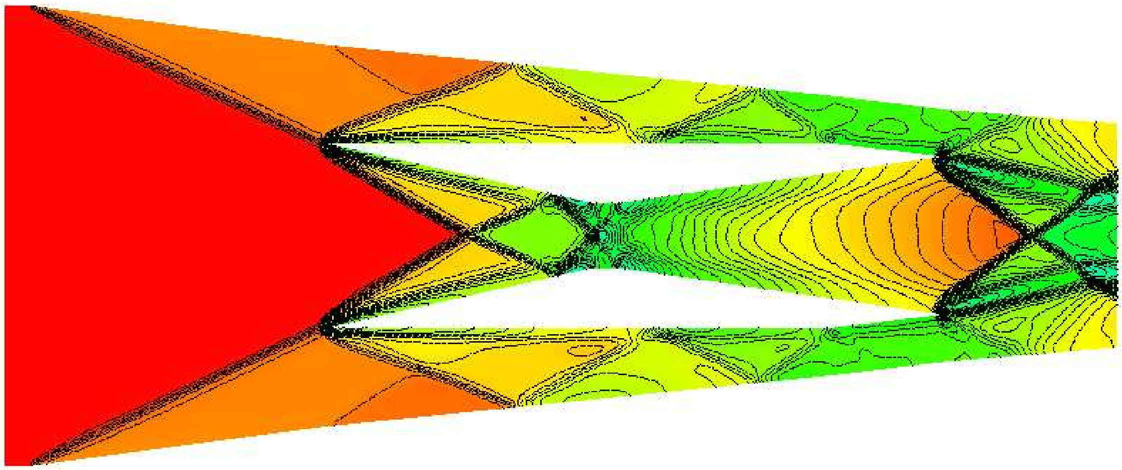

Figure 2: Scramjet inlet: $M = 3$, characteristic TVD limiter, 33,859 triangular elements.

diagram was computed using an optional FCT corrector [4] for the end-of-step solution produced by the default limiter presented in the previous section (cf. the left diagram).

The stationary Mach number distribution for a 2D supersonic flow in a scramjet inlet (Fig. 2) was computed using algorithm (10)–(13) on a relatively coarse unstructured mesh. Although the accuracy is far superior to that of the underlying low-order scheme ($R_i^\pm \equiv 0$), it is worthwhile to perform local mesh refinement in the vicinity of shocks, where a significant amount of artificial viscosity cannot be removed. An adaptive mesh refinement strategy based on limited gradient recovery/averaging is presented in [5].

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