Galerkin finite element method

Boundary value problem $\quad\longrightarrow\quad$ weighted residual formulation

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega & \text{partial differential equation} \\ u = g_0 & \text{on } \Gamma_0 & \text{Dirichlet boundary condition} \\ \mathbf{n}\cdot\nabla u = g_1 & \text{on } \Gamma_1 & \text{Neumann boundary condition} \\ \mathbf{n}\cdot\nabla u + \alpha u = g_2 & \text{on } \Gamma_2 & \text{Robin boundary condition} \end{cases}$$

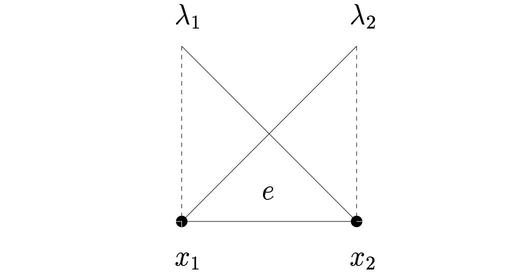
1. Multiply the residual of the PDE by a weighting function w vanishing on the Dirichlet boundary Γ_0 and set the integral over Ω equal to zero
2. Integrate by parts using the Neumann and Robin boundary conditions
3. Represent the approximate solution $u_h \approx u$ as a linear combination of polynomial basis functions φ_i defined on a given mesh (triangulation)
4. Substitute the functions u_h and φ_i for u and w in the weak formulation
5. Solve the resulting algebraic system for the vector of nodal values u_i

Construction of 1D finite elements

1. Linear finite elements

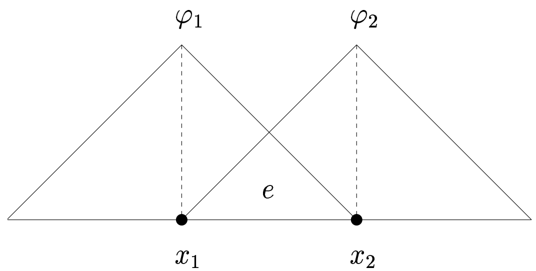
Consider the *barycentric coordinates*

$$\lambda_1(x) = \frac{x_2 - x}{x_2 - x_1}, \qquad \lambda_2(x) = \frac{x - x_1}{x_2 - x_1}$$



defined on the element $e = [x_1, x_2]$

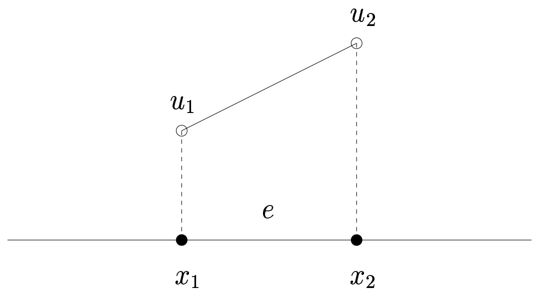
- $\lambda_i \in P_1(e), \quad i = 1, 2$
- $\lambda_i(x_j) = \delta_{ij}, \quad i, j = 1, 2$
- $\lambda_1(x) + \lambda_2(x) = 1, \quad \forall x \in e$



$\frac{d\lambda_1}{dx} = -\frac{1}{x_2 - x_1} = -\frac{d\lambda_2}{dx}$ constant derivatives

Basis functions $\quad \varphi_1|_e = \lambda_1, \quad \varphi_2|_e = \lambda_2$

$$u_h(x) = u_1\varphi_1(x) + u_2\varphi_2(x), \qquad \forall x \in e$$



Construction of 1D finite elements

2. Quadratic finite elements

$\{\lambda_1(x), \lambda_2(x)\}$ barycentric coordinates

$x_1 = \{1, 0\}, \quad x_2 = \{0, 1\}$ endpoints

$x_{12} = \frac{x_1 + x_2}{2} = \{\frac{1}{2}, \frac{1}{2}\}$ midpoint

Basis functions $\varphi_1, \varphi_2, \varphi_{12} \in P_2(e)$

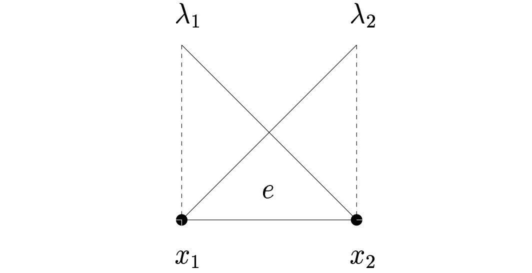
$$\varphi_1(x) = \lambda_1(x)(2\lambda_1(x) - 1)$$

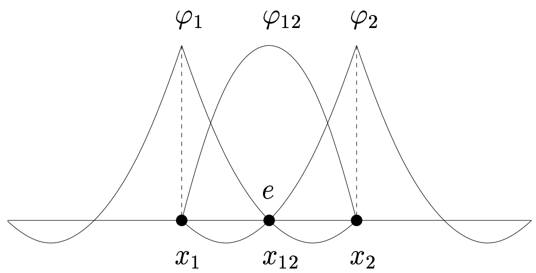
$$\varphi_2(x) = \lambda_2(x)(2\lambda_2(x) - 1)$$

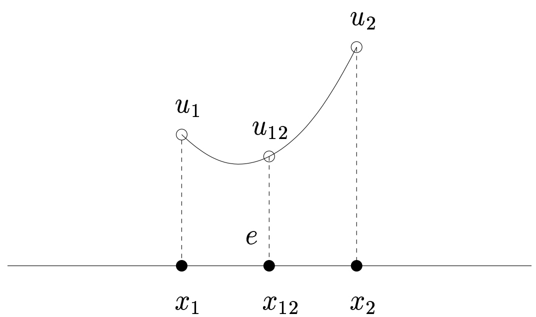
$$\varphi_{12}(x) = 4\lambda_1(x)\lambda_2(x)$$

Shape function $u_h|_e$

$$u_h(x) = u_1\varphi_1(x) + u_2\varphi_2(x) + u_{12}\varphi_{12}(x)$$







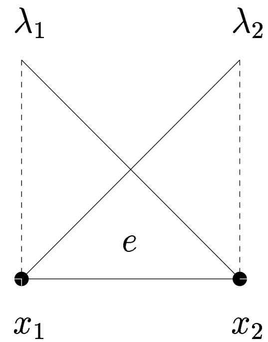
Construction of 1D finite elements

3. Cubic finite elements

$\{\lambda_1(x), \lambda_2(x)\}$ barycentric coordinates

$$x_1 = \{1, 0\}, \quad x_{12} = \frac{2x_1 + x_2}{3} = \{\tfrac{2}{3}, \tfrac{1}{3}\}$$

$$x_2 = \{0, 1\}, \quad x_{21} = \frac{x_1 + 2x_2}{3} = \{\tfrac{1}{3}, \tfrac{2}{3}\}$$



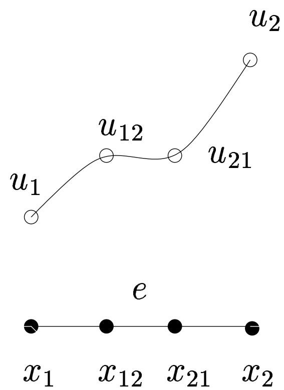
Basis functions $\varphi_1, \varphi_2, \varphi_{12}, \varphi_{21} \in P_3(e)$

$$\varphi_1(x) = \tfrac{1}{2}\lambda_1(x)(3\lambda_1(x) - 2)(3\lambda_1(x) - 1)$$

$$\varphi_2(x) = \tfrac{1}{2}\lambda_2(x)(3\lambda_2(x) - 2)(3\lambda_2(x) - 1)$$

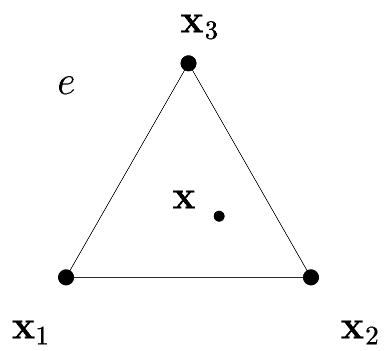
$$\varphi_{12}(x) = \tfrac{9}{2}\lambda_1(x)\lambda_2(x)(3\lambda_1(x) - 1)$$

$$\varphi_{21}(x) = \tfrac{9}{2}\lambda_1(x)\lambda_2(x)(3\lambda_2(x) - 1)$$



Shape function $u_h(x) = u_1\varphi_1(x) + u_2\varphi_2(x) + u_{12}\varphi_{12}(x) + u_{21}\varphi_{21}(x)$

Construction of triangular finite elements



Let $\mathbf{x} = (x, y) = \{\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x})\}$

Construct 2D barycentric coordinates

$$\lambda_i \in P_1(e), \quad \lambda_i(\mathbf{x}_j) = \delta_{ij}, \quad i, j = 1, 2, 3$$

Polynomial fitting: $\lambda_i(\mathbf{x}) = c_{i1} + c_{i2}x + c_{i3}y$

$$\begin{cases} \lambda_i(\mathbf{x}_1) = c_{i1} + c_{i2}x_1 + c_{i3}y_1 = \delta_{i1} \\ \lambda_i(\mathbf{x}_2) = c_{i1} + c_{i2}x_2 + c_{i3}y_2 = \delta_{i2} \\ \lambda_i(\mathbf{x}_3) = c_{i1} + c_{i2}x_3 + c_{i3}y_3 = \delta_{i3} \end{cases} \qquad \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_{A} \begin{bmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{bmatrix} = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \end{bmatrix}$$

We have 3 systems of 3 equations for 9 unknowns. They can be solved for the unknown coefficients c_{ij} by resorting to Cramer's rule.

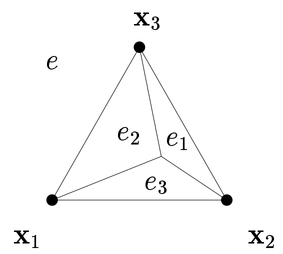
$$\det A = x_2y_3 + x_1y_2 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3$$

Area of the triangle $|e| = \frac{1}{2}|\det A|$ (also needed for quadrature rules)

Construction of triangular finite elements

Connect the point $\mathbf{x}$ to the vertices $\mathbf{x}_i, \quad i = 1, 2, 3$

to construct the *barycentric splitting* $e = \bigcup_{i=1}^{3} e_i$



Areas of the triangles $|e_i(\mathbf{x})| = \frac{1}{2}|\det A_i(\mathbf{x})|$

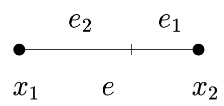
$$A_1 = \begin{bmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix}$$

Solution of the linear systems: $\lambda_i(\mathbf{x}) = \frac{|e_i(\mathbf{x})|}{|e|} = \left|\frac{\det A_i(\mathbf{x})}{\det A}\right|, \quad i = 1, 2, 3$

It is obvious that the barycentric coordinates satisfy $\lambda_i(\mathbf{x}_j) = \delta_{ij}$

$$|e_1(\mathbf{x})| + |e_2(\mathbf{x})| + |e_3(\mathbf{x})| = |e|, \quad \forall \mathbf{x} \in e \qquad \Rightarrow \quad \lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x}) + \lambda_3(\mathbf{x}) \equiv 1$$

A similar interpretation is possible in one dimension:

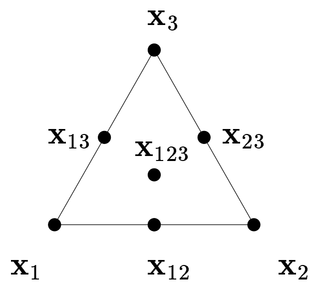


Construction of triangular finite elements

Nodal barycentric coordinates $\quad \mathbf{x}_{123} = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$

$$\mathbf{x}_1 = \{1,0,0\}, \quad \mathbf{x}_2 = \{0,1,0\}, \quad \mathbf{x}_3 = \{0,0,1\}$$

$$\mathbf{x}_{12} = \{\tfrac{1}{2}, \tfrac{1}{2}, 0\}, \quad \mathbf{x}_{13} = \{\tfrac{1}{2}, 0, \tfrac{1}{2}\}, \quad \mathbf{x}_{23} = \{0, \tfrac{1}{2}, \tfrac{1}{2}\}$$



1. Linear elements $\quad u_h(\mathbf{x}) = c_1 + c_2 x + c_3 y \in P_1(e)$

vertex-oriented $\quad \varphi_1 = \lambda_1, \quad \varphi_2 = \lambda_2, \quad \varphi_3 = \lambda_3 \quad$ (standard)

midpoint-oriented $\quad \varphi_{12} = 1 - 2\lambda_3, \quad \varphi_{13} = 1 - 2\lambda_2, \quad \varphi_{23} = 1 - 2\lambda_1$

2. Quadratic elements $\quad u_h(\mathbf{x}) = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 \in P_2(e)$

Standard P_2 basis (6 nodes)

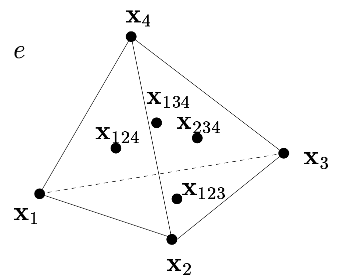
$$\varphi_1 = \lambda_1(2\lambda_1 - 1), \quad \varphi_{12} = 4\lambda_1\lambda_2$$
$$\varphi_2 = \lambda_2(2\lambda_2 - 1), \quad \varphi_{13} = 4\lambda_1\lambda_3$$
$$\varphi_3 = \lambda_3(2\lambda_3 - 1), \quad \varphi_{23} = 4\lambda_2\lambda_3$$

Extended P_2^+ basis (7 nodes)

$$\varphi_i = \lambda_i(2\lambda_i - 1) + 3\lambda_1\lambda_2\lambda_3$$
$$\varphi_{ij} = 4\lambda_i\lambda_j - 12\lambda_1\lambda_2\lambda_3$$
$$\varphi_{123} = 27\lambda_1\lambda_2\lambda_3, \quad i,j = 1,2,3$$

Construction of tetrahedral finite elements

Let $\mathbf{x} = (x, y, z) = \{\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x}), \lambda_4(\mathbf{x})\}$



Construct $\quad \lambda_i \in P_1(e) : \ \lambda_i(\mathbf{x}_j) = \delta_{ij}, \quad i, j = 1, \ldots, 4$

Polynomial fitting: $\quad \lambda_i(\mathbf{x}) = c_{i1} + c_{i2}x + c_{i3}y + c_{i4}z$

$$\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix} \begin{bmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \\ c_{i4} \end{bmatrix} = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ \delta_{i4} \end{bmatrix}$$

Barycentric splitting $\quad e = \bigcup_{i=1}^{4} e_i$

$$\lambda_i(\mathbf{x}) = \frac{|e_i(\mathbf{x})|}{|e|}, \quad i = 1, \ldots, 4$$

1. Linear elements $\quad u_h(\mathbf{x}) = c_1 + c_2x + c_3y + c_4z \in P_1(e)$

vertex-oriented $\quad \varphi_1 = \lambda_1, \quad \varphi_2 = \lambda_2, \quad \varphi_3 = \lambda_3, \quad \varphi_4 = \lambda_4 \quad$ (standard)

face-oriented $\quad \varphi_{123} = 1 - 3\lambda_4, \quad \varphi_{124} = 1 - 3\lambda_3$

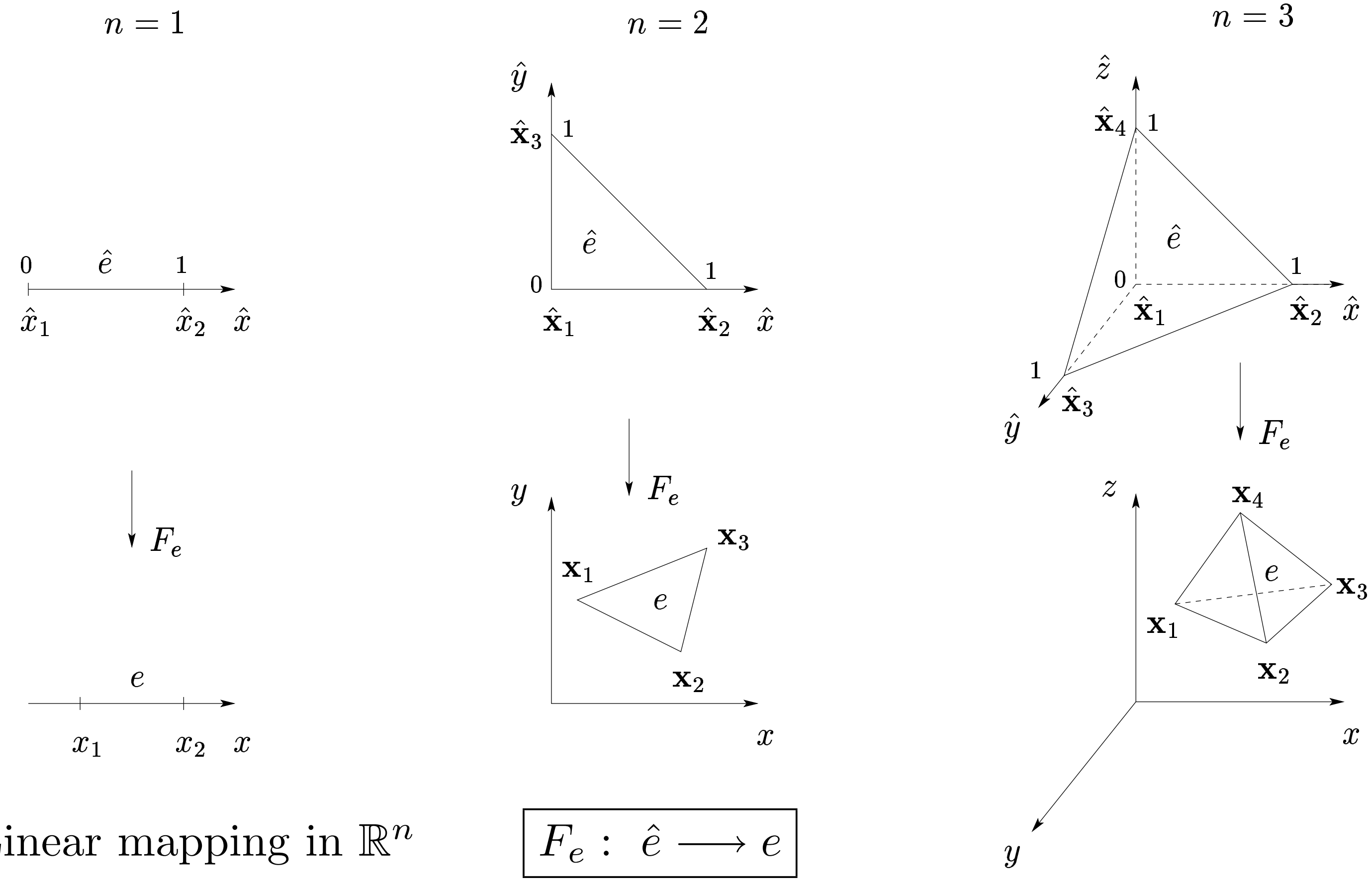
$\quad \varphi_{134} = 1 - 3\lambda_2, \quad \varphi_{234} = 1 - 3\lambda_1$

$$\mathbf{x}_{ijk} = \frac{\mathbf{x}_i + \mathbf{x}_j + \mathbf{x}_k}{3}$$

2. Higher-order approximations are possible but rather expensive in 3D

Coordinate and element transformations

Idea: define the basis functions on a geometrically simple *reference element*



Linear mapping in $\mathbb{R}^n$

$$F_e : \hat{e} \longrightarrow e$$

Coordinate and element transformations

Mapping of the reference element $\hat{e}$ onto an element e with vertices $\mathbf{x}_i$

$$F_e : \quad e = F_e(\hat{e}) \qquad \mathbf{x} = F_e(\hat{\mathbf{x}}) = \sum_{i=1}^{n+1} \mathbf{x}_i \lambda_i(\hat{\mathbf{x}}), \qquad \forall \mathbf{x} \in e$$

where λ_i are the barycentric coordinates. This mapping is of the form

$$F_e(\hat{\mathbf{x}}) = B_e \hat{\mathbf{x}} + b_e, \qquad B_e \in \mathbb{R}^{n \times n}, \quad b_e \in \mathbb{R}^n$$

It is applicable to arbitrary simplex elements with straight sides

$$n = 1 \qquad B_e = x_2 - x_1, \qquad b_e = x_1$$

$$n = 2 \qquad B_e = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}, \qquad b_e = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$n = 3 \qquad B_e = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \\ z_2 - z_1 & z_3 - z_1 & z_4 - z_1 \end{bmatrix}, \qquad b_e = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

Coordinate and element transformations

Properties of the linear mapping $\qquad F_e(\hat{\mathbf{x}}) = B_e\hat{\mathbf{x}} + b_e$

- vertices are mapped onto vertices $\qquad \mathbf{x}_i = F_e(\hat{\mathbf{x}}_i)$
- midpoints of sides are mapped onto midpoints of sides

$$\mathbf{x}_{ij} = \tfrac{\mathbf{x}_i+\mathbf{x}_j}{2} = F_e\left(\tfrac{\hat{\mathbf{x}}_i+\hat{\mathbf{x}}_j}{2}\right) = F_e(\hat{\mathbf{x}}_{ij})$$

- barycenters are mapped onto barycenters

$$\mathbf{x}_{ijk} = \tfrac{\mathbf{x}_i+\mathbf{x}_j+\mathbf{x}_k}{3} = F_e\left(\tfrac{\hat{\mathbf{x}}_i+\hat{\mathbf{x}}_j+\hat{\mathbf{x}}_k}{3}\right) = F_e(\hat{\mathbf{x}}_{ijk})$$

The values of φ_i on the physical element e are defined by the formula

$$\boxed{\varphi_i(\mathbf{x}) = \hat{\varphi}_i(F_e^{-1}(\mathbf{x})), \quad \forall \mathbf{x} \in e} \qquad \varphi_i(\mathbf{x}) = \hat{\varphi}_i(\hat{\mathbf{x}}), \quad \mathbf{x} = F_e(\hat{\mathbf{x}})$$

Note that $\varphi_i(\mathbf{x}_j) = \hat{\varphi}_i(\hat{\mathbf{x}}_j) = \delta_{ij}$ and the degree of basis functions (linear, quadratic, cubic etc.) is preserved since $\mathbf{x}$ depends linearly on $\hat{\mathbf{x}}$

Coordinate and element transformations

Derivative transformations $\quad \varphi_i(\mathbf{x}) = \hat{\varphi}_i(\hat{\mathbf{x}}), \quad \mathbf{x} \in e, \quad \hat{\mathbf{x}} \in \hat{e}$

Chain rule $\quad \boxed{\hat{\nabla}\hat{\varphi}_i = J\nabla\varphi_i} \quad$ where J is the Jacobian of the (inverse) mapping as introduced before in the context of the finite difference method

$$\frac{\partial\varphi_i}{\partial x} = \frac{1}{\det J}\left[\frac{\partial\hat{\varphi}_i}{\partial\hat{x}}\frac{\partial y}{\partial\hat{y}} - \frac{\partial\hat{\varphi}_i}{\partial\hat{y}}\frac{\partial y}{\partial\hat{x}}\right]$$

$$\frac{\partial\varphi_i}{\partial y} = \frac{1}{\det J}\left[\frac{\partial\hat{\varphi}_i}{\partial\hat{y}}\frac{\partial x}{\partial\hat{x}} - \frac{\partial\hat{\varphi}_i}{\partial\hat{x}}\frac{\partial x}{\partial\hat{y}}\right]$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial\hat{x}} & \frac{\partial y}{\partial\hat{x}} \\ \frac{\partial x}{\partial\hat{y}} & \frac{\partial y}{\partial\hat{y}} \end{bmatrix}$$

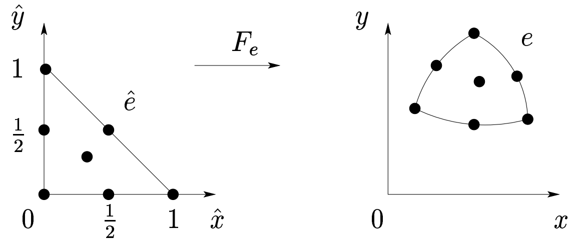
must be nonsingular for F_e to be invertible

Isoparametric mappings: it is possible to define curved elements e using a mapping F_e of the same degree as the basis functions on the reference element $\hat{e}$

Example. Extended quadratic element P_2^+

$$e = F_e(\hat{e}) \qquad \hat{\mathbf{x}}_i \to \mathbf{x}_i, \quad \hat{\mathbf{x}}_{ij} \to \mathbf{x}_{ij}, \quad \hat{\mathbf{x}}_{123} \to \mathbf{x}_{123}$$

$$\mathbf{x} = F_e(\hat{\mathbf{x}}) = \sum_{i=1}^{3} \mathbf{x}_i\hat{\varphi}_i(\hat{\mathbf{x}}) + \sum_{ij} \mathbf{x}_{ij}\hat{\varphi}_{ij}(\hat{\mathbf{x}}) + \mathbf{x}_{123}\hat{\varphi}_{123}(\hat{\mathbf{x}})$$



Construction of quadrilateral finite elements

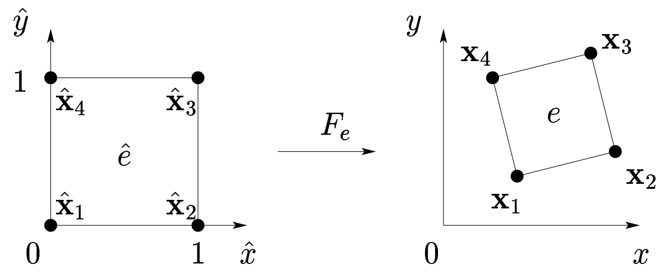
Idea: construct 2D basis functions as a tensor product of 1D ones defined on $\hat{e}$

1. Bilinear finite elements

Let $\quad \lambda_1(t) = 1 - t, \quad \lambda_2(t) = t, \quad t \in [0, 1]$

$$\hat{\varphi}_1(\hat{\mathbf{x}}) = \lambda_1(\hat{x})\lambda_1(\hat{y}), \quad \hat{\varphi}_3(\hat{\mathbf{x}}) = \lambda_2(\hat{x})\lambda_2(\hat{y})$$

$$\hat{\varphi}_2(\hat{\mathbf{x}}) = \lambda_2(\hat{x})\lambda_1(\hat{y}), \quad \hat{\varphi}_4(\hat{\mathbf{x}}) = \lambda_1(\hat{x})\lambda_2(\hat{y})$$



The space $Q_1(\hat{e})$ spanned by $\hat{\varphi}_i$ consists of functions which are P_1 for each variable

In general $\quad Q_k(\hat{e}) = \text{span}\{x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}\}, \qquad 0 \le k_i \le k, \quad i = 1, \dots, n$

Isoparametric mapping $\quad \mathbf{x} = F_e(\hat{\mathbf{x}}) = \sum_{i=1}^{4} \mathbf{x}_i \hat{\varphi}_i(\hat{\mathbf{x}}), \qquad \varphi_i(\mathbf{x}) = \hat{\varphi}_i(F_e^{-1}(\mathbf{x}))$

The physical element $e = F_e(\hat{e})$ is a quadrilateral with straight sides which must be convex for F_e to be invertible. It is easy to verify that $F_e(\hat{\mathbf{x}}_i) = \mathbf{x}_i, \quad i = 1, \dots, 4$

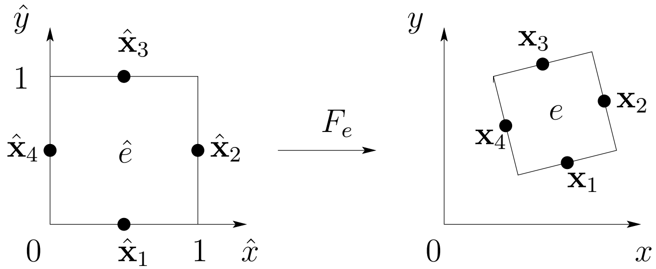
Construction of quadrilateral finite elements

2. Nonconforming rotated bilinear elements *(Rannacher and Turek, 1992)*

Let $\tilde{Q}_1(\hat{e}) = \text{span}\{1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2\}$

$$u_h(\mathbf{x}) = c_1 + c_2\hat{x} + c_3\hat{y} + c_4(\hat{x}^2 - \hat{y}^2)$$

$$\mathbf{x} = F_e(\hat{\mathbf{x}}) = \sum_{i=1}^{4} \alpha_i \mathbf{x}_i \quad \text{bilinear mapping}$$



Degrees of freedom: $u_i = \frac{1}{|S_i|}\int_{S_i} u_h(\mathbf{x}(s))\,ds \approx u_h(\mathbf{x}_i)$ edge mean values

Edge-oriented basis functions: $u_h(\mathbf{x}) = \sum_{j=1}^{4} u_j \hat{\varphi}_j(\hat{\mathbf{x}}) = \sum_{j=1}^{4} c_j \hat{\psi}_j(\hat{\mathbf{x}}), \quad \forall \mathbf{x} \in e$

$$u = [u_1, u_2, u_3, u_4]^T, \qquad \hat{\varphi} = [\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3, \hat{\varphi}_4]^T, \qquad u_i = \sum_j a_{ij} c_j$$

$$c = [c_1, c_2, c_3, c_4]^T, \qquad \hat{\psi} = [1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2]^T, \qquad a_{ij} = \frac{1}{|S_i|}\int_{S_i} \hat{\psi}_j(\hat{\mathbf{x}})ds$$

Coefficients: $Ac = u \quad \Rightarrow \quad \hat{\psi}^T c = \hat{\psi}^T A^{-1} u = \hat{\varphi}^T u \quad \Rightarrow \quad \hat{\varphi}^T = \hat{\psi}^T A^{-1}$

Construction of quadrilateral finite elements

Midpoint rule: $a_{ij} \approx \hat{\psi}_j(\hat{\mathbf{x}}_i), \quad u_i \approx u_h(\mathbf{x}_i)$ exact for linear functions

$\tilde{Q}_1^a$ {edge mean values}

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{3} \\ 1 & 1 & \frac{1}{2} & \frac{2}{3} \\ 1 & \frac{1}{2} & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{2} & -\frac{1}{3} \end{bmatrix}$$

$$\hat{\varphi}_1(\hat{\mathbf{x}}) = \frac{3}{4} + \frac{3}{2}\hat{x} - \frac{5}{2}\hat{y} - \frac{3}{2}(\hat{x}^2 - \hat{y}^2)$$
$$\hat{\varphi}_2(\hat{\mathbf{x}}) = -\frac{1}{4} - \frac{1}{2}\hat{x} + \frac{3}{2}\hat{y} + \frac{3}{2}(\hat{x}^2 - \hat{y}^2)$$
$$\hat{\varphi}_3(\hat{\mathbf{x}}) = -\frac{1}{4} + \frac{3}{2}\hat{x} - \frac{1}{2}\hat{y} - \frac{3}{2}(\hat{x}^2 - \hat{y}^2)$$
$$\hat{\varphi}_4(\hat{\mathbf{x}}) = \frac{3}{4} - \frac{5}{2}\hat{x} + \frac{3}{2}\hat{y} + \frac{3}{2}(\hat{x}^2 - \hat{y}^2)$$

$\tilde{Q}_1^b$ {edge midpoint values}

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{4} \\ 1 & 1 & \frac{1}{2} & \frac{3}{4} \\ 1 & \frac{1}{2} & 1 & -\frac{3}{4} \\ 1 & 0 & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

$$\hat{\varphi}_1(\hat{\mathbf{x}}) = \frac{3}{4} + \hat{x} - 2\hat{y} - (\hat{x}^2 - \hat{y}^2)$$
$$\hat{\varphi}_2(\hat{\mathbf{x}}) = -\frac{1}{4} + \hat{y} + (\hat{x}^2 - \hat{y}^2)$$
$$\hat{\varphi}_3(\hat{\mathbf{x}}) = -\frac{1}{4} + \hat{x} - (\hat{x}^2 - \hat{y}^2)$$
$$\hat{\varphi}_4(\hat{\mathbf{x}}) = \frac{3}{4} - 2\hat{x} + \hat{y} + (\hat{x}^2 - \hat{y}^2)$$

Nonparametric version: construct the basis functions directly using a local coordinate system rather than the transformation to a reference element

Construction of quadrilateral finite elements

3. Biquadratic finite elements

$$\theta_1(t) = (1-t)(1-2t), \quad \theta_2(t) = t(1-2t), \quad \theta_3(t) = 4t(1-t), \quad t \in [0,1]$$

Products of 1D quadratic basis functions spanning the space $Q_2(\hat{e})$

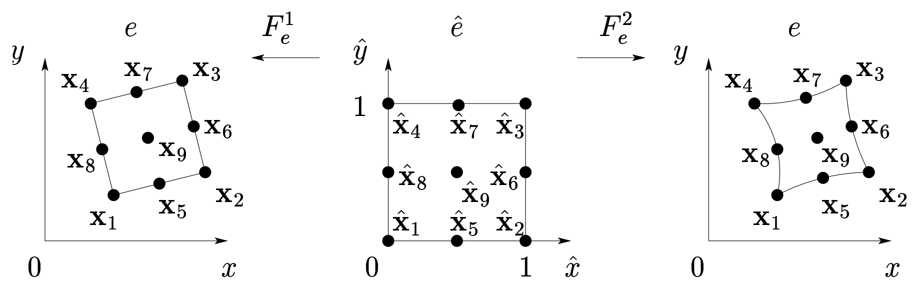
$$\hat{\varphi}_1(\hat{\mathbf{x}}) = \theta_1(\hat{x})\theta_1(\hat{y}), \quad \hat{\varphi}_4(\hat{\mathbf{x}}) = \theta_1(\hat{x})\theta_2(\hat{y}), \quad \hat{\varphi}_7(\hat{\mathbf{x}}) = \theta_3(\hat{x})\theta_2(\hat{y})$$

$$\hat{\varphi}_2(\hat{\mathbf{x}}) = \theta_2(\hat{x})\theta_1(\hat{y}), \quad \hat{\varphi}_5(\hat{\mathbf{x}}) = \theta_3(\hat{x})\theta_1(\hat{y}), \quad \hat{\varphi}_8(\hat{\mathbf{x}}) = \theta_1(\hat{x})\theta_3(\hat{y})$$

$$\hat{\varphi}_3(\hat{\mathbf{x}}) = \theta_2(\hat{x})\theta_2(\hat{y}), \quad \hat{\varphi}_6(\hat{\mathbf{x}}) = \theta_2(\hat{x})\theta_3(\hat{y}), \quad \hat{\varphi}_9(\hat{\mathbf{x}}) = \theta_3(\hat{x})\theta_3(\hat{y})$$

Basis functions on the physical element $\varphi_i(\mathbf{x}) = \hat{\varphi}_i(F_e^{-1}(\mathbf{x})), \quad \forall \mathbf{x} \in e$

Mapping: subparametric (bilinear) or isoparametric



$$\mathbf{x} = F_e(\hat{\mathbf{x}}) = \sum_{i=1}^{9} \mathbf{x}_i \hat{\varphi}_i(\hat{\mathbf{x}})$$

$e = F_e(\hat{e})$ is curved

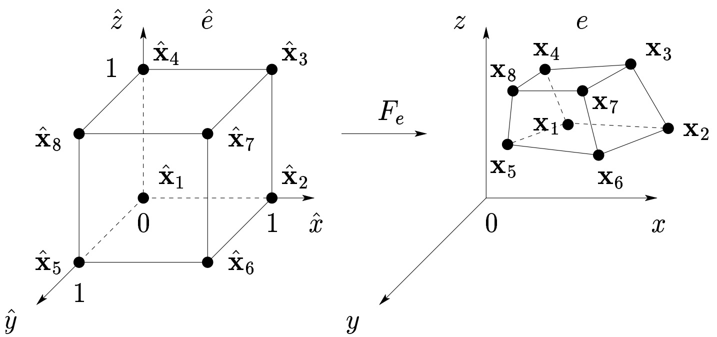
Construction of hexahedral finite elements

1. Trilinear finite elements $\lambda_1(t) = 1 - t, \quad \lambda_2(t) = t, \quad t \in [0, 1]$

Products of 1D linear basis functions spanning the space $Q_1(\hat{e})$

$$
\begin{aligned}
\hat{\varphi}_1(\hat{\mathbf{x}}) &= \lambda_1(\hat{x})\lambda_1(\hat{y})\lambda_1(\hat{z}), & \hat{\varphi}_5(\hat{\mathbf{x}}) &= \lambda_1(\hat{x})\lambda_1(\hat{y})\lambda_2(\hat{z}) \\
\hat{\varphi}_2(\hat{\mathbf{x}}) &= \lambda_2(\hat{x})\lambda_1(\hat{y})\lambda_1(\hat{z}), & \hat{\varphi}_6(\hat{\mathbf{x}}) &= \lambda_2(\hat{x})\lambda_1(\hat{y})\lambda_2(\hat{z}) \\
\hat{\varphi}_3(\hat{\mathbf{x}}) &= \lambda_2(\hat{x})\lambda_2(\hat{y})\lambda_1(\hat{z}), & \hat{\varphi}_7(\hat{\mathbf{x}}) &= \lambda_2(\hat{x})\lambda_2(\hat{y})\lambda_2(\hat{z}) \\
\hat{\varphi}_4(\hat{\mathbf{x}}) &= \lambda_1(\hat{x})\lambda_2(\hat{y})\lambda_1(\hat{z}), & \hat{\varphi}_8(\hat{\mathbf{x}}) &= \lambda_1(\hat{x})\lambda_2(\hat{y})\lambda_2(\hat{z})
\end{aligned}
$$

Basis functions on the physical element $\varphi_i(\mathbf{x}) = \hat{\varphi}_i(F_e^{-1}(\mathbf{x})), \quad \forall \mathbf{x} \in e$



Isoparametric mapping

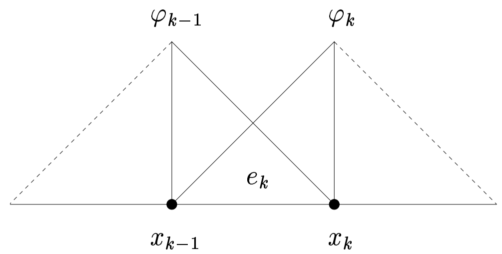
$$\mathbf{x} = F_e(\hat{\mathbf{x}}) = \sum_{i=1}^{8} \mathbf{x}_i \hat{\varphi}_i(\hat{\mathbf{x}})$$

2. Rotated trilinear elements (6 nodes, face-oriented degrees of freedom)

Finite element matrix assembly

Example: 1D Poisson equation

$$\begin{cases} -\frac{d^2u}{dx^2} = f \quad \text{in } (0,1) \\ u(0) = 0, \quad \frac{du}{dx}(1) = 0 \end{cases}$$



Galerkin discretization: $u_h = \sum_{j=1}^{N} u_j \varphi_j$ (linear finite elements)

$$u_0 = 0, \qquad \sum_{j=1}^{N} u_j \int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx = \int_0^1 f\varphi_i\,dx, \qquad \forall i = 1,\ldots,N$$

Decomposition of integrals into element contributions $\qquad e_k = [x_{k-1}, x_k]$

$$Au = F, \qquad \sum_{j=1}^{N} u_j \underbrace{\sum_{k=1}^{N} \overbrace{\int_{e_k} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx}^{a_{ij}^k}}_{a_{ij}} = \underbrace{\sum_{k=1}^{N} \overbrace{\int_{e_k} f\varphi_i\,dx}^{F_i^k}}_{F_i}, \qquad \forall i = 1,\ldots,N$$

Example: 1D Poisson equation / linear elements

Idea: evaluate element contributions and insert them into the global matrix

$$a_{ij} = \sum_{k=1}^{N} a_{ij}^k = \int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx \qquad a_{ij}^k \neq 0 \quad \text{only for } i,j \in \{k-1,k\}$$

$$F_i = \sum_{k=1}^{N} F_i^k = \int_0^1 f\varphi_i\,dx \qquad F_i^k \neq 0 \quad \text{only for } i \in \{k-1,k\}$$

Element stiffness matrix and load vector $\qquad e_k = [x_{k-1}, x_k]$

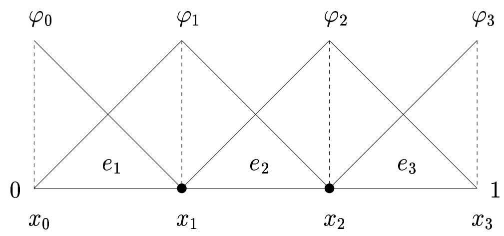
$$\mathrm{A}^k = \begin{bmatrix} \int_{e_k} \frac{d\varphi_{k-1}}{dx}\frac{d\varphi_{k-1}}{dx}\,dx & \int_{e_k} \frac{d\varphi_{k-1}}{dx}\frac{d\varphi_k}{dx}\,dx \\ \int_{e_k} \frac{d\varphi_k}{dx}\frac{d\varphi_{k-1}}{dx}\,dx & \int_{e_k} \frac{d\varphi_k}{dx}\frac{d\varphi_k}{dx}\,dx \end{bmatrix}, \qquad \mathrm{F}^k = \begin{bmatrix} \int_{e_k} f\varphi_{k-1}\,dx \\ \int_{e_k} f\varphi_k\,dx \end{bmatrix}$$

Coefficients of the global system $\quad Au = F \quad$ which are to be augmented

$$A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{k-1\,k-1} & a_{k-1\,k} & \cdot \\ \cdot & a_{k\,k-1} & a_{k\,k} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \qquad F = \begin{bmatrix} \cdot \\ F_{k-1} \\ F_k \\ \cdot \end{bmatrix}$$

Example: 1D Poisson equation / linear elements

Special case: 3 elements, $\Delta x = \frac{1}{3}$, $f \equiv 1$



$$\varphi_{k-1}(x) = \frac{x_k - x}{x_k - x_{k-1}} = \frac{k\Delta x - x}{\Delta x}, \qquad \frac{d\varphi_{k-1}}{dx} = -\frac{1}{\Delta x}$$

$$\varphi_k(x) = \frac{x - x_{k-1}}{x_k - x_{k-1}} = \frac{x - (k-1)\Delta x}{\Delta x}, \qquad \frac{d\varphi_k}{dx} = \frac{1}{\Delta x}$$

$$\forall x \in e_k = [x_{k-1}, x_k] \qquad \text{Hence,} \quad \mathrm{A}^k = \frac{1}{\Delta x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathrm{F}^k = \frac{\Delta x}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Assembly of the global stiffness matrix and load vector

$$A = \frac{1}{\Delta x}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\Delta x}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\Delta x}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow$$

$$A = \frac{1}{\Delta x}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad F = \frac{\Delta x}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\Delta x}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{\Delta x}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{\Delta x}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Example: 1D Poisson equation / linear elements

Recall that $u_0 = 0$ so the first equation drops out and the system shrinks to

$$\frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} u_1 = \frac{5}{2}(\Delta x)^2, & u_2 = 4(\Delta x)^2 \\ u_3 = \frac{9}{2}(\Delta x)^2, & \Delta x = \frac{1}{3} \end{array}$$

Implementation of Dirichlet boundary conditions

1. Row/column elimination: $u_0 = g_0 \quad \Rightarrow$ the first equation is superfluous whereas the second one turns into $a_{11}u_1 + a_{12}u_2 + a_{13}u_3 = F_1 - a_{10}g_0$
2. Row modification (replacement by a row of the identity matrix)

   $$a_{00} := 1, \quad a_{0j} := 0, \quad j = 1,2,3, \qquad F_0 := g_0$$
3. Penalty method / addition of a large number α to the diagonal

   $$a_{00} := a_{00} + \alpha, \qquad F_0 := F_0 + \alpha g_0 \qquad \textit{symmetry is preserved}$$

Implementation of Neumann boundary conditions

$$\frac{du}{dx}(1) = g_1 \quad \Rightarrow \quad F_N = \int_{e_N} f\varphi_N\,dx + g_1 \qquad \textit{a surface integral is added}$$

Example: 1D Poisson equation / quadratic elements

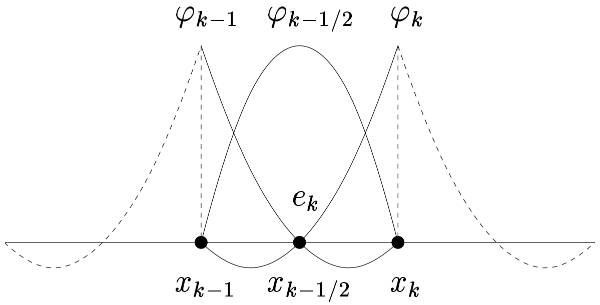
Galerkin FEM: $$\sum_{j=1/2}^{N} u_j \sum_{k=1}^{N} \int_{e_k} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx = \sum_{k=1}^{N} \int_{e_k} f\varphi_i\,dx, \qquad \forall i = 1,\dots,N$$

$$e_k = [x_{k-1}, x_k], \quad x = \{\lambda_1(x), \lambda_2(x)\}$$

$$\varphi_{k-1} = \lambda_1(2\lambda_1 - 1), \qquad \frac{d\varphi_{k-1}}{dx} = -\frac{4\lambda_1 - 1}{\Delta x}$$

$$\varphi_k = \lambda_2(2\lambda_2 - 1), \qquad \frac{d\varphi_{k-1/2}}{dx} = \frac{4\lambda_2 - 1}{\Delta x}$$

$$\varphi_{k-1/2} = 4\lambda_1\lambda_2, \qquad \frac{d\varphi_k}{dx} = 4\frac{\lambda_1 - \lambda_2}{\Delta x}$$



Element stiffness matrix and load vector

$$\mathrm{A}^k = \begin{bmatrix} \int_{e_k} \frac{d\varphi_{k-1}}{dx}\frac{d\varphi_{k-1}}{dx}\,dx & \int_{e_k} \frac{d\varphi_{k-1}}{dx}\frac{d\varphi_{k-1/2}}{dx}\,dx & \int_{e_k} \frac{d\varphi_{k-1}}{dx}\frac{d\varphi_k}{dx}\,dx \\ \int_{e_k} \frac{d\varphi_{k-1/2}}{dx}\frac{d\varphi_{k-1}}{dx}\,dx & \int_{e_k} \frac{d\varphi_{k-1/2}}{dx}\frac{d\varphi_{k-1/2}}{dx}\,dx & \int_{e_k} \frac{d\varphi_{k-1/2}}{dx}\frac{d\varphi_k}{dx}\,dx \\ \int_{e_k} \frac{d\varphi_k}{dx}\frac{d\varphi_{k-1}}{dx}\,dx & \int_{e_k} \frac{d\varphi_k}{dx}\frac{d\varphi_{k-1/2}}{dx}\,dx & \int_{e_k} \frac{d\varphi_k}{dx}\frac{d\varphi_k}{dx}\,dx \end{bmatrix} = \frac{1}{3\Delta x}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$\mathrm{F}^k = \begin{bmatrix} \int_{e_k} f\varphi_{k-1}\,dx \\ \int_{e_k} f\varphi_{k-1/2}\,dx \\ \int_{e_k} f\varphi_k\,dx \end{bmatrix} = \frac{\Delta x}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Global system: $Au = F$, where

$$u = [u_{1/2}\, u_1\, u_{3/2} \dots u_{N-1/2}\, u_N]^T$$

Numerical integration for finite elements

Change of variables theorem

$$\int_e f(\mathbf{x})\,d\mathbf{x} = \int_{\hat{e}} \hat{f}(\hat{\mathbf{x}})|\det J|\,d\hat{\mathbf{x}}$$

$$\varphi_i(\mathbf{x}) = \hat{\varphi}_i(F_e^{-1}(\mathbf{x})), \quad \forall x \in e \qquad \hat{\nabla}\hat{\varphi}_i = J\nabla\varphi_i \quad \Rightarrow \quad \nabla\varphi_i = J^{-1}\hat{\nabla}\hat{\varphi}_i$$

For instance, the entries of the element stiffness matrix are given by

$$a_{ij} = \int_e \nabla\varphi_i \cdot \nabla\varphi_j\,d\mathbf{x} = \int_{\hat{e}} (J^{-1}\hat{\nabla}\hat{\varphi}_i)\cdot(J^{-1}\hat{\nabla}\hat{\varphi}_j)|\det J|\,d\hat{\mathbf{x}}$$

Numerical integration

$$\int_{\hat{e}} \hat{g}(\hat{\mathbf{x}})\,d\hat{\mathbf{x}} \approx \sum_{i=0}^{n} \hat{w}_i\hat{g}(\hat{\mathbf{x}}_i), \quad \hat{g}(\hat{\mathbf{x}}) = \hat{f}(\hat{\mathbf{x}})|\det J|$$

Newton-Cotes formulae can be used but *Gaussian quadrature* is preferable:

$$\int_{\hat{e}} \hat{g}(\hat{x})\,d\hat{x} \approx \frac{\hat{g}(\hat{x}_1)+\hat{g}(\hat{x}_2)}{2}, \qquad \hat{x}_1 = \frac{1}{2} - \frac{1}{6}\sqrt{3}, \quad \hat{x}_2 = \frac{1}{2} + \frac{1}{6}\sqrt{3}, \qquad \hat{e} = [0,1]$$

exact for $\hat{g} \in P_3(\hat{e})$ as compared to $P_1(\hat{e})$ for the trapezoidal rule

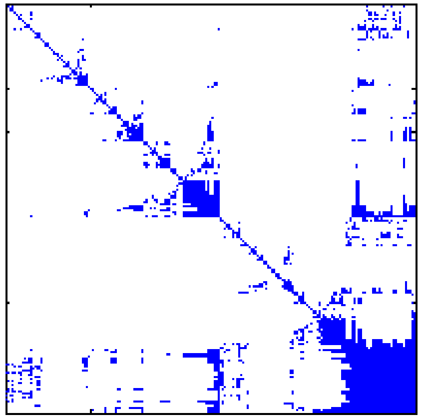
Storage of sparse matrices

Banded matrices: store the nonzero diagonals as 1D arrays

Arbitrary matrices: store the nonzero elements as a 1D array

1. Coordinate storage (inconvenient access)

A(NNZ)	nonzero elements in arbitrary order
IROW(NNZ)	auxiliary array of row numbers
ICOL(NNZ)	auxiliary array of column numbers



2. Compact storage (convenient access)

A(NNZ)	nonzero elements stored row-by-row
ILD(N+1)	pointers to the beginning of each row
ICOL(NNZ)	auxiliary array of column numbers

Example

$$A = \begin{bmatrix} 1 & 2 & 0 & 7 \\ 2 & 4 & 3 & 0 \\ 0 & 3 & 6 & 5 \\ 7 & 0 & 5 & 8 \end{bmatrix}$$

$\mathrm{A} = (1, 2, 7, 4, 2, 3, 6, 3, 5, 8, 7, 5)$

$\mathrm{ICOL} = (1, 2, 4, 2, 1, 3, 3, 2, 4, 4, 1, 3)$

$\mathrm{IROW} = (1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4)$

$\mathrm{NNZ} = 12,\ \mathrm{ILD} = (1, 4, 7, 10, 13)$