**Finite element method**

Origins: structural mechanics, calculus of variations for elliptic BVPs

Boundary value problem

\[
\begin{aligned}
L u &= f \quad \text{in } \Omega \\
 u &= g_0 \quad \text{on } \Gamma_0 \\
 n \cdot \nabla u &= g_1 \quad \text{on } \Gamma_1 \\
 n \cdot \nabla u + \alpha u &= g_2 \quad \text{on } \Gamma_2 \\
\end{aligned}
\]

Minimization problem

Given a functional \( J : V \to \mathbb{R} \)

find \( u \in V \) such that

\[ J(u) \leq J(w), \quad \forall w \in V \]

subject to the imposed BC

⊕ the functional contains derivatives of lower order
⊕ solutions from a broader class of functions are admissible
⊕ boundary conditions for complex domains can be handled easily
⊕ sometimes there is no functional associated with the original BVP

Modern FEM: weighted residuals formulation (weak form of the PDE)
Theory: 1D minimization problem

Minimize \[ J(w) = \int_{x_0}^{x_1} \psi \left( x, w, \frac{dw}{dx} \right) \, dx \] over \( w \in V = C^2([x_0, x_1]) \)

subject to the boundary condition \( w(x_0) = g_0 \)

Find \( u \in V \) such that \( J(u) \leq J(w) \) for all admissible \( w \)

\[ w(x) = u(x) + \lambda v(x), \quad \lambda \in \mathbb{R}, \quad v \in V_0 = \{ v \in V : v(x_0) = 0 \} \]

\[ \Rightarrow J(u + \lambda v) = \int_{x_0}^{x_1} \psi \left( x, u + \lambda v, \frac{du}{dx} + \lambda \frac{dv}{dx} \right) \, dx = I(\lambda) \]

By construction \( w(x) = u(x) \) for \( \lambda = 0 \) so that

\[ I(0) \leq I(\lambda), \quad \forall \lambda \in \mathbb{R} \quad \Rightarrow \quad \frac{dI}{d\lambda} \bigg|_{\lambda=0} = 0, \quad \forall v \in V_0 \]

Necessary condition of an extremum

\[ \delta J(u, v) \overset{\text{def}}{=} \frac{d}{d\lambda} J(u + \lambda v) \bigg|_{\lambda=0} = 0, \quad \forall v \in V_0 \]

the first variation of the functional must vanish
Necessary condition of an extremum

Chain rule for the function \( \psi = \psi(x, w, w') \), \( w' = \frac{dw}{dx} \)

\[
\frac{d\psi}{d\lambda} = \frac{\partial \psi}{\partial x} \frac{dx}{d\lambda} + \frac{\partial \psi}{\partial w} \frac{dw}{d\lambda} + \frac{\partial \psi}{\partial w'} \frac{dw'}{d\lambda}, \quad \text{where } \frac{dx}{d\lambda} = 0, \quad \frac{dw}{d\lambda} = v, \quad \frac{dw'}{d\lambda} = \frac{dv}{dx}
\]

First variation of the functional

\[
\frac{d}{d\lambda} J(u + \lambda v) \bigg|_{\lambda=0} = \int_{x_0}^{x_1} \left[ \frac{\partial \psi}{\partial w} v + \frac{\partial \psi}{\partial w'} \frac{dv}{dx} \right] dx = 0, \quad \forall v \in V_0
\]

Integration by parts using the boundary condition \( v(x_0) = 0 \) yields

\[
\int_{x_0}^{x_1} \left[ \frac{\partial \psi}{\partial w} - \frac{d}{dx} \left( \frac{\partial \psi}{\partial w'} \right) \right] v dx + \frac{\partial \psi}{\partial w'}(x_1)v(x_1) = 0, \quad \forall v \in V_0
\]

including \( \forall v \in \hat{V} = \{ v \in V_0 : v(x_1) = 0 \} \) \( \Rightarrow \) \( \frac{\partial \psi}{\partial w} - \frac{d}{dx} \left( \frac{\partial \psi}{\partial w'} \right) = 0 \)

Substitution \( \Rightarrow \frac{\partial \psi}{\partial w'}(x_1) v(x_1) = 0, \quad \forall v \in V_0 \) \( \Rightarrow \frac{\partial \psi}{\partial w'}(x_1) \) arbitrary
Du Bois Reymond lemma

**Lemma.** Let $f \in C([a, b])$ be a continuous function and assume that

\[ \int_a^b f(x)v(x) \, dx = 0, \quad \forall v \in \hat{V} = \{v \in C^2([a, b]) : v(a) = v(b) = 0 \} \]

Then $f(x) = 0, \quad \forall x \in [a, b]$.

**Proof.** Suppose $\exists x_0 \in (a, b)$ such that $f(x_0) \neq 0$, e.g. $f(x_0) > 0$

If $f$ is continuous $\Rightarrow f(x) > 0, \quad x \in (x_0 - \delta, x_0 + \delta) \subset (a, b)$

Let $v(x) = \begin{cases} 
\exp \left( -\frac{1}{\delta^2 - (x-x_0)^2} \right) & \text{if } |x - x_0| < \delta \\
0 & \text{if } |x - x_0| \geq \delta 
\end{cases}$

$v \in \hat{V}$ but $\int_a^b f(x)v(x) \, dx = \int_{x_0-\delta}^{x_0+\delta} f(x) \exp \left( -\frac{1}{\delta^2 - (x-x_0)^2} \right) \, dx > 0$

$\Rightarrow f(x) = 0, \quad \forall x \in (a, b)$

$f \in C([a, b]) \Rightarrow f \equiv 0$ in $[a, b] \quad \Box$
Example: 1D Poisson equation

Constraints imposed on the solution $w = u$ of the minimization problem

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial \psi}{\partial w} - \frac{d}{dx} \left( \frac{\partial \psi}{\partial w'} \right) = 0 \\
u(x_0) = g_0 \\
\frac{\partial \psi}{\partial w'}(x_1) = 0
\end{array} \right. \\
Euler-Lagrange
equation
\\essential
boundary
condition
\\natural
boundary
condition
\end{align*}
$$

Poisson equation: the solution $u \in V_g = \{ v \in C^2([0, 1]) : v(0) = g_0 \}$

minimizes the functional $J(w) = \int_0^1 \left[ \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - fw \right] dx, \ w \in V_g$

$$
\psi(x, w, w') = \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - fw, \ \frac{\partial \psi}{\partial w} = -f \quad \begin{cases}
-d^2 u/dx^2 = f & \text{in } (0, 1) \\
u(0) = g_0 & \text{Dirichlet BC} \\
\frac{du}{dx}(1) = 0 & \text{Neumann BC}
\end{cases}
$$
Example: 2D Poisson equation

Find \( u \in V_g = \{ v \in C^2(\Omega) \cap C(\tilde{\Omega}) : v|_{\Gamma_0} = g_0 \} \) that minimizes the functional

\[
J(w) = \int_{\Omega} \left( \frac{1}{2} |\nabla w|^2 - fw \right) d\mathbf{x} - \int_{\Gamma_1} g_1 w \, ds - \int_{\Gamma_2} g_2 w \, ds + \frac{\alpha}{2} \int_{\Gamma_2} w^2 \, ds, \quad w \in V_g
\]

where \( f = f(x), \quad g_0 = g_0(x), \quad g_1 = g_1(x), \quad g_2 = g_2(x), \quad \alpha \geq 0 \)

Admissible functions \( w = u + \lambda v, \quad v \in V_0 = \{ v \in C^2(\Omega) \cap C(\tilde{\Omega}) : v|_{\Gamma_0} = 0 \} \)

Necessary condition of an extremum \( \frac{d}{d\lambda} J(u + \lambda v) \big|_{\lambda=0} = 0, \quad \forall v \in V_0 \)

\[
\frac{\partial}{\partial \lambda} |\nabla w|^2 = \frac{\partial}{\partial \lambda} \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] = 2 \left[ \frac{\partial w}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial v}{\partial y} \right] = 2 \nabla w \cdot \nabla v
\]

\[
\int_{\Omega} [\nabla u \cdot \nabla v - fv] \, d\mathbf{x} - \int_{\Gamma_1} g_1 v \, ds - \int_{\Gamma_2} g_2 v \, ds + \alpha \int_{\Gamma_2} uv \, ds = 0, \quad \forall v \in V_0
\]
Example: 2D Poisson equation

Integration by parts using Green’s formula

\[ \int_{\Omega} [-\Delta u - f]v \, dx + \int_{\Gamma_1 \cup \Gamma_2} (\mathbf{n} \cdot \nabla u)v \, ds - \int_{\Gamma_1} g_1 v \, ds - \int_{\Gamma_2} g_2 v \, ds + \alpha \int_{\Gamma_2} uv \, ds = 0 \]

\( \forall v \in V_0 \) including \( v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\Gamma} = 0 \Rightarrow \int_{\Omega} [-\Delta u - f]v \, dx = 0 \)

Du Bois Reymond lemma: \(-\Delta u = f\) in \(\Omega\) Euler-Lagrange equation

\[ \int_{\Gamma_1} [\mathbf{n} \cdot \nabla u - g_1]v \, ds + \int_{\Gamma_2} [\mathbf{n} \cdot \nabla u + \alpha u - g_2]v \, ds = 0, \quad \forall v \in V_0 \]

Consider \( v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\Gamma_0 \cup \Gamma_1} = 0 \Rightarrow \int_{\Gamma_2} = 0, \quad \mathbf{n} \cdot \nabla u + \alpha u = g_2 \)

Substitution yields \( \int_{\Gamma_1} = 0, \quad \mathbf{n} \cdot \nabla u = g_1 \) and the following BVP

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = g_0 & \text{on } \Gamma_0 \\
\mathbf{n} \cdot \nabla u = g_1 & \text{on } \Gamma_1 \\
\mathbf{n} \cdot \nabla u + \alpha u = g_2 & \text{on } \Gamma_2
\end{cases}
\]

2D Poisson equation

Dirichlet BC (essential)

Neumann BC (natural)

Robin BC (natural)
Rayleigh-Ritz method

Exact solution
\[ u \in V : \quad u = \varphi_0 + \sum_{j=1}^{\infty} c_j \varphi_j \]

Approximate solution
\[ u_h \in V_h : \quad u_h = \varphi_0 + \sum_{j=1}^{N} c_j \varphi_j \]

\( \varphi_0 \) an arbitrary function satisfying \( \varphi_0 = g_0 \) on \( \Gamma_0 \)

\( \varphi_j \) basis functions vanishing on the boundary part \( \Gamma_0 \)

Continuous problem

Find \( u \in V \) such that
\[ J(u) \leq J(w), \quad \forall w \in V \]

Discrete problem

Find \( u_h \in V_h \) such that
\[ J(u_h) \leq J(w_h), \quad \forall w_h \in V_h \]

\[ J(c_1, \ldots, c_N) = \min_{w_h \in V_h} J(w_h) \quad \Rightarrow \quad \frac{\partial J}{\partial c_i} = 0, \quad i = 1, \ldots, N \]

Linear system:
\[ Ac = b, \quad A \in \mathbb{R}^{N \times N}, \quad b \in \mathbb{R}^N, \quad c = [c_1, \ldots, c_N]^T \]
Example: 1D Poisson equation

Find the coefficients $c_1, \ldots, c_N$ that minimize the functional

$$ J(w_h) = \int_0^1 \left[ \frac{1}{2} \left( \frac{dw_h}{dx} \right)^2 - f w_h \right] \, dx, \quad w_h = \sum_{j=1}^N c_j \varphi_j $$

Necessary condition of an extremum

$$ \frac{\partial J}{\partial c_i} = \frac{\partial}{\partial c_i} \left[ \int_0^1 \left( \sum_{j=1}^N c_j \frac{d\varphi_j}{dx} \right)^2 \, dx - \int_0^1 f \left( \sum_{j=1}^N c_j \varphi_j \right) \, dx \right] = 0 $$

$$ \int_0^1 \frac{d\varphi_i}{dx} \left( \sum_{j=1}^N c_j \frac{d\varphi_j}{dx} \right) \, dx = \int_0^1 f \varphi_i \, dx, \quad i = 1, \ldots, N $$

This is a linear system of the form $Ac = b$ with coefficients

$$ a_{ij} = \int_0^1 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \, dx, \quad b_i = \int_0^1 f \varphi_i \, dx, \quad c = [c_1, \ldots, c_N]^T $$
Example: poor choice of the basis functions

Consider the polynomial basis \( \varphi_0 = 0, \quad \varphi_i = x^i, \quad i = 1, \ldots, N \)

\[
u_h(x) = \sum_{j=1}^{N} c_j x^j, \quad a_{ij} = \int_0^1 ij x^{i+j-2} \, dx = \frac{ij}{i+j-1}, \quad b_i = \int_0^1 f x^i \, dx
\]

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & \frac{4}{3} & \frac{6}{4} & \frac{8}{5} & \cdots & \frac{2N}{N+1} \\
1 & \frac{6}{4} & \frac{9}{5} & \frac{12}{6} & \cdots & \frac{3N}{N+2} \\
1 & \frac{8}{5} & \frac{12}{6} & \frac{16}{7} & \cdots & \frac{4N}{N+3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \vdots & \vdots & \vdots & \vdots & \frac{N^2}{2N-1} \\
\end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_N \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_N \end{bmatrix}
\]

- \( A \) is known as the Hilbert matrix which is SPD but full and ill-conditioned so that the solution is expensive and corrupted by round-off errors.
- for \( A \) to be sparse, the basis functions should have a compact support
Fundamentals of the FEM

The *Finite Element Method* is a systematic approach to generating piecewise-polynomial basis functions with favorable properties.

- The computational domain $\Omega$ is subdivided into many small subdomains $K$ called *elements*: $\bar{\Omega} = \bigcup_{K \in T_h} K$.
- The *triangulation* $T_h$ is admissible if the intersection of any two elements is either an empty set or a common vertex/edge/face of the mesh.
- The finite element subspace $V_h$ consists of piecewise-polynomial functions. Typically, $V_h = \{ v \in C^m(\Omega) : v|_K \in P_k, \forall K \in T_h \}$.
- Any function $v \in V_h$ is uniquely determined by a finite number of *degrees of freedom* (function values or derivatives at certain points called *nodes*).
- Each basis function $\varphi_i$ accommodates exactly one degree of freedom and has a small support so that the resulting matrices are sparse.
Finite element approximation

The finite element is a triple \((K, P, \Sigma)\), where

- \(K\) is a closed subset of \(\overline{\Omega}\)
- \(P\) is the polynomial space for the shape functions
- \(\Sigma\) is the set of local degrees of freedom

Basis functions possess the property

\[
\varphi_j(x_i) = \delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

\[
u_h(x) = \sum_{j=1}^{N} u_j \varphi_j(x) \quad \Rightarrow \quad u_h(x_i) = \sum_{j=1}^{N} u_j \varphi_j(x_i) = \sum_{j=1}^{N} u_j \delta_{ij} = u_i
\]

Approximate solution: the nodal values \(u_1, \ldots, u_N\) can be computed by the Ritz method provided that there exists an equivalent minimization problem
Example: 1D Poisson equation

Find the nodal values $u_1, \ldots, u_N$ that minimize the functional

$$J(w_h) = \int_0^1 \left[ \frac{1}{2} \left( \frac{dw_h}{dx} \right)^2 - fw_h \right] \, dx, \quad w_h = \sum_{j=1}^N u_j \varphi_j$$

Local basis functions for $e_i = [x_{i-1}, x_i]$

$$\varphi_{i-1}(x) = \frac{x - x}{x_i - x_{i-1}}, \quad \varphi_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

Approximate solution for $x \in e_i$

$$u_h(x) = \sum_{j=1}^N u_j \varphi_j = u_{i-1} \varphi_{i-1} + u_i \varphi_i$$

$$= u_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}} (u_i - u_{i-1})$$

continuous, piecewise-linear
Example: 1D Poisson equation

The Ritz method yields a linear system of the form $Au = F$, where

$$a_{ij} = \sum_{k=1}^{N} \int_{e_k} d\varphi_i \frac{d\varphi_j}{dx} dx, \quad F_i = \sum_{k=1}^{N} \int_{e_k} f \varphi_i dx$$

These integrals can be evaluated exactly or numerically (using a quadrature rule)

*Stiffness matrix* and *load vector* for a uniform mesh with $\Delta x = \frac{1}{N}$ and $f \equiv 1$

$$A = \frac{1}{(\Delta x)^2} \begin{bmatrix} 2 & -1 \\ -1 & 2 & -1 \\ & \ddots & \ddots & \ddots \\ & -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 1 \\ \cdot \cdot \cdot \\ 1 \end{bmatrix}$$

This is the same linear system as the one obtained for the finite difference method!
Existence of a minimization problem

Sufficient conditions for an elliptic PDE \( \mathcal{L}u = f \) in \( \Omega \), \( u|_{\Gamma} = 0 \)
to be the Euler-Lagrange equation of a variational problem read

- the operator \( \mathcal{L} \) is linear
- the operator \( \mathcal{L} \) is self-adjoint (symmetric)
  \[ \int_{\Omega} v \mathcal{L} u \, dx = \int_{\Omega} u \mathcal{L} v \, dx \quad \text{for all admissible } u, v \]
- the operator \( \mathcal{L} \) is positive definite
  \[ \int_{\Omega} u \mathcal{L} u \, dx \geq 0 \quad \text{for all admissible } u; \quad u \equiv 0 \quad \text{if } \int_{\Omega} u \mathcal{L} u \, dx = 0 \]

In this case, the unique solution \( u \) minimizes the functional
\[
J(w) = \frac{1}{2} \int_{\Omega} w \mathcal{L} w \, dx - \int_{\Omega} f w \, dx
\]
over the set of admissible functions. Non-homogeneous BC modify this set
and/or give rise to additional terms in the functional to be minimized
**Example: 1D Poisson equation**

Laplace operator \( \mathcal{L} = -\frac{d^2}{dx^2} \) is linear and self-adjoint

\[
\int_0^1 v \mathcal{L} u \, dx = -\int_0^1 \frac{d^2 u}{dx^2} v \, dx = \int_0^1 \frac{du}{dx} \frac{dv}{dx} \, dx - \left[ \frac{du}{dx} \right]_0^1 v \, dx \\
= -\int_0^1 \frac{d^2 v}{dx^2} u \, dx + \left[ \frac{du}{dx} \right]_0^1 = \int_0^1 u \mathcal{L} v \, dx
\]

Positive-definiteness: \( \int_0^1 u \mathcal{L} u \, dx = -\int_0^1 u \frac{d^2 u}{dx^2} \, dx = \int_0^1 \left( \frac{du}{dx} \right)^2 \, dx \geq 0 \)

If \( \int_0^1 \left( \frac{du}{dx} \right)^2 \, dx = 0 \) then \( \frac{du}{dx} \equiv 0 \Rightarrow u \equiv 0 \) since \( u(0) = 0 \)

Functional for the minimization problem

\[
J(w) = \int_0^1 \left[ \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - f w \right] \, dx, \quad w(0) = 0
\]

Non-homogeneous BC \( u(0) = g_0 \) \( \rightarrow \) \( w(0) = g_0 \) (essential)

\( \frac{du}{dx}(1) = g_1 \) \( \rightarrow \) \( J(w) = \int_0^1 \left[ \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - f w \right] \, dx - g_1 w(1) \) (natural)
Least-squares method

Idea: minimize the residual of the PDE

\[ R(w) = \mathcal{L}w - f \quad \text{such that} \quad R(u) = 0 \implies \mathcal{L}u = f \]

Least-squares functional

\[ J(w) = \int_{\Omega} (\mathcal{L}w - f)^2 \, dx \quad \text{always exists} \]

Necessary condition of an extremum

\[
\frac{d}{d\lambda} J(u + \lambda v) \bigg|_{\lambda=0} = \frac{d}{d\lambda} \left[ \int_{\Omega} (\mathcal{L}(u + \lambda v) - f)^2 \, dx \right]_{\lambda=0} = 0
\]

Integration by parts:

\[
\int_{\Omega} (\mathcal{L}u - f) \mathcal{L}v \, dx = \int_{\Omega} \mathcal{L}^*(\mathcal{L}u - f)v \, dx - \int_{\Gamma} \ldots \, ds = 0
\]

Euler-Lagrange equation

\[ \mathcal{L}^* \mathcal{L}u = \mathcal{L}^* f \quad \text{where } \mathcal{L}^* \text{ is the adjoint operator} \]

- corresponds to a derivative of the original PDE
- requires additional boundary conditions and extra smoothness
- it makes sense to rewrite a high-order PDE as a first-order system

Advantage: the matrices for a least-squares discretization are symmetric
Weighted residuals formulation

Idea: render the residual orthogonal to a space of test functions

Let  \[ u = \sum_{j=1}^{\infty} \alpha_j \varphi_j \in V_0 \] be the solution of
\[
\begin{cases}
  \mathcal{L}u = f & \text{in } \Omega \\
  u = 0 & \text{on } \Gamma
\end{cases}
\]

Residual is zero if its projection onto each basis function equals zero
\[
\mathcal{L}u - f = 0 \quad \Leftrightarrow \quad \int_{\Omega} (\mathcal{L}u - f) \varphi_i \, dx = 0 \quad \forall i = 1, 2, \ldots
\]

Test functions  \[ v = \sum_{j=1}^{\infty} \beta_j \varphi_j \]  \[ \Rightarrow \]  \[ \int_{\Omega} (\mathcal{L}u - f) v \, dx = 0, \quad \forall v \in V_0 \]

Weak formulation: find  \( u \in V_0 \) such that  \[ a(u, v) = l(v) \quad \forall v \in V_0 \]

where  \( a(u, v) = \int_{\Omega} \mathcal{L}u \, v \, dx \) is a bilinear form and  \( l(v) = \int_{\Omega} f \, v \, dx \)

Integration by parts:  \[ \mathcal{L}u = \nabla \cdot g(u) \]  \[ \Rightarrow \]  \[ a(u, v) = -\int_{\Omega} g(u) \cdot \nabla v \, dx \]
Finite element discretization

Continuous problem

Find $u \in V_0$ such that

$$a(u, v) = l(v), \quad \forall v \in V_0$$

Discrete problem

Find $u_h \in V_h \subset V_0$ such that

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h'$$

FEM approximations:

$$u_h = \sum_{j=1}^{N} u_j \varphi_j \in V_h, \quad v_h = \sum_{j=1}^{N} v_j \psi_j \in V_h'$$

where $V_h = \text{span}\{\varphi_1, \ldots, \varphi_N\}$ and $V_h' = \text{span}\{\psi_1, \ldots, \psi_N\}$ may differ

(Bubnov-)Galerkin method

$$V_h' = V_h \quad \rightarrow \quad a(u, \varphi_i) = l(\varphi_i), \quad \psi_i = \varphi_i$$

Petrov-Galerkin method

$$V_h' \neq V_h \quad \rightarrow \quad a(u, \psi_i) = l(\psi_i), \quad \psi_i \neq \varphi_i$$

Linear algebraic system

$$\sum_{j=1}^{N} a(\varphi_j, \psi_i) u_j = l(\psi_i), \quad \forall i = 1, \ldots, N$$

Matrix form

$$Au = F \quad \text{with coefficients} \quad a_{ij} = a(\varphi_j, \psi_i), \quad F_i = l(\psi_i)$$
Example: 1D Poisson equation

Boundary value problem

\[
\begin{aligned}
\begin{cases}
  -\frac{d^2 u}{dx^2} = f & \text{in } (0, 1) \\
  u(0) = 0, \quad \frac{du}{dx}(1) = 0
\end{cases}
\end{aligned}
\]

Weak formulation \( u \in V_0 \)

\[
\int_0^1 \left( -\frac{d^2 u}{dx^2} - f \right) v \, dx = 0, \quad \forall v \in V_0
\]

Integration by parts yields

\[
- \int_0^1 \frac{d^2 u}{dx^2} v \, dx = \int_0^1 \frac{du}{dx} \frac{dv}{dx} \, dx - \left[ \frac{du}{dx} v \right]_0^1
\]

Approximate solution

\[
u_h(x) = \sum_{j=1}^N u_j \varphi_j(x)
\]

Continuous problem \( a(u, v) = l(v), \quad a(u, v) = \int_0^1 \frac{du}{dx} \frac{dv}{dx} \, dx, \quad l(v) = \int_0^1 f v \, dx \)

Discrete problem \( a(u_h, \varphi_i) = l(\varphi_i), \quad i = 1, \ldots, N \) (Galerkin method)

This is a (sparse) linear system of the form \( Au = F \), where

\[
a_{ij} = \int_0^1 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \, dx, \quad F_i = \int_0^1 f \varphi_i \, dx, \quad u = [u_1, \ldots, u_N]^T
\]

The Galerkin and Ritz methods are equivalent if the minimization problem exists
Example: 2D Poisson equation

Boundary value problem

\[
\begin{align*}
-\Delta u &= f \quad \text{in } \Omega \\
u &= g_0 \quad \text{on } \Gamma_0 \\
\mathbf{n} \cdot \nabla u &= g_1 \quad \text{on } \Gamma_1 \\
\mathbf{n} \cdot \nabla u + \alpha u &= g_2 \quad \text{on } \Gamma_2
\end{align*}
\]

Weak formulation

\[
\int_{\Omega} [-\Delta u - f] v \, dx = 0, \quad \forall v \in V_0
\]

\[
V_g = \{ v \in V : v|_{\Gamma_0} = g_0 \}
\]

\[
V_0 = \{ v \in V : v|_{\Gamma_0} = 0 \}
\]

Integration by parts using Green's formula

\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\Gamma} (\mathbf{n} \cdot \nabla u) v \, ds = \int_{\Omega} f v \, dx, \quad \Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2
\]

Boundary conditions

\[
\int_{\Gamma_0} (\mathbf{n} \cdot \nabla u) v \, ds = 0 \quad \text{since } v = 0 \text{ on } \Gamma_0
\]

\[
\int_{\Gamma_1} (\mathbf{n} \cdot \nabla u) v \, ds = \int_{\Gamma_1} g_1 v \, ds, \quad \int_{\Gamma_2} (\mathbf{n} \cdot \nabla u) v \, ds = \int_{\Gamma_2} g_2 v \, ds - \alpha \int_{\Gamma_2} uv \, ds
\]

Approximate solution

\[
u_h(x) = \varphi_0 + \sum_{j=1}^{N} u_j \varphi_j(x), \quad \varphi_0|_{\Gamma_0} = g_0
\]
**Example: 2D Poisson equation**

Continuous problem

\[ a(u, v) = l(v) + \int_{\Gamma_1} g_1 v ds + \int_{\Gamma_2} g_2 v ds, \quad \forall v \in V_0 \]

\[ a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \alpha \int_{\Gamma_2} u v ds, \quad l(v) = \int_{\Omega} f v \, dx \]

Discrete problem

\[ a(u_h, \varphi_i) = l(\varphi_i) + \int_{\Gamma_1} g_1 \varphi_i ds + \int_{\Gamma_2} g_2 \varphi_i ds, \quad \forall i = 1, \ldots, N \]

Piecewise-linear basis functions \( \varphi_i \in C(\bar{\Omega}), \quad \varphi_i|_K \in P_1, \quad \forall K \in T_h \)

satisfying \( \varphi_i(x_j) = \delta_{ij}, \quad \forall i, j = 1, \ldots, N \)

Linear system \( Au = F \) where

\[ a_{ij} = \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, dx + \alpha \int_{\Gamma_2} u \varphi_i ds \]

\[ F_i = \int_{\Omega} f \varphi_i \, dx + \int_{\Gamma_1} g_1 \varphi_i ds + \int_{\Gamma_2} g_2 \varphi_i ds \]

The matrix \( A \) is SPD, sparse and banded for a proper node numbering.