Finite element method

Origins: structural mechanics, calculus of variations for elliptic BVPs

Boundary value problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = g_0 & \text{on } \Gamma_0 \\ \mathbf{n} \cdot \nabla u = g_1 & \text{on } \Gamma_1 \\ \mathbf{n} \cdot \nabla u + \alpha u = g_2 & \text{on } \Gamma_2 \end{cases}$$

$\overset{?}{\Leftrightarrow}$

Minimization problem

Given a functional $J : V \to \mathbb{R}$ *find* $u \in V$ *such that*

$$J(u) \leq J(w), \quad \forall w \in V$$

subject to the imposed BC

- ⊕ the functional contains derivatives of lower order
- ⊕ solutions from a broader class of functions are admissible
- ⊕ boundary conditions for complex domains can be handled easily
- ⊖ sometimes there is no functional associated with the original BVP

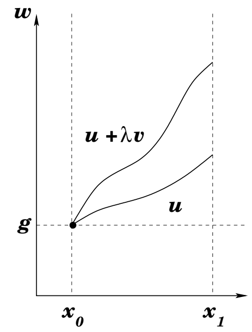
Modern FEM: weighted residuals formulation (weak form of the PDE)

Theory: 1D minimization problem

Minimize $\quad J(w) = \int_{x_0}^{x_1} \psi\left(x, w, \frac{dw}{dx}\right)\, dx \quad$ over $\quad w \in V = C^2([x_0, x_1])$

subject to the boundary condition $w(x_0) = g_0$

Find $u \in V$ such that $J(u) \leq J(w)$ for all admissible w



$$w(x) = u(x) + \lambda v(x), \qquad \lambda \in \mathbb{R}, \quad v \in V_0 = \{v \in V: \ v(x_0) = 0\}$$

$$\Rightarrow \quad J(u + \lambda v) = \int_{x_0}^{x_1} \psi\left(x, u + \lambda v, \frac{du}{dx} + \lambda \frac{dv}{dx}\right) dx = I(\lambda)$$

By construction $\quad w(x) = u(x) \quad$ for $\ \lambda = 0 \quad$ so that

$$I(0) \leq I(\lambda), \quad \forall \lambda \in \mathbb{R} \qquad \Rightarrow \qquad \left.\frac{dI}{d\lambda}\right|_{\lambda=0} = 0, \quad \forall v \in V_0$$

Necessary condition of an extremum

$$\delta J(u, v) \stackrel{\text{def}}{=} \left.\frac{d}{d\lambda} J(u + \lambda v)\right|_{\lambda=0} = 0, \quad \forall v \in V_0$$

the *first variation* of the functional must vanish

Necessary condition of an extremum

Chain rule for the function $\psi = \psi(x, w, w'), \quad w' = \frac{dw}{dx}$

$$\frac{d\psi}{d\lambda} = \frac{\partial\psi}{\partial x}\frac{dx}{d\lambda} + \frac{\partial\psi}{\partial w}\frac{dw}{d\lambda} + \frac{\partial\psi}{\partial w'}\frac{dw'}{d\lambda}, \qquad \text{where} \quad \frac{dx}{d\lambda} = 0, \quad \frac{dw}{d\lambda} = v, \quad \frac{dw'}{d\lambda} = \frac{dv}{dx}$$

First variation of the functional

$$\frac{d}{d\lambda} J(u + \lambda v)\bigg|_{\lambda=0} = \int_{x_0}^{x_1} \left[\frac{\partial\psi}{\partial w} v + \frac{\partial\psi}{\partial w'}\frac{dv}{dx}\right] dx = 0, \qquad \forall v \in V_0$$

Integration by parts using the boundary condition $v(x_0) = 0$ yields

$$\int_{x_0}^{x_1} \left[\frac{\partial\psi}{\partial w} - \frac{d}{dx}\left(\frac{\partial\psi}{\partial w'}\right)\right] v\, dx + \frac{\partial\psi}{\partial w'}(x_1) v(x_1) = 0, \qquad \forall v \in V_0$$

including $\forall v \in \hat{V} = \{v \in V_0 : \; v(x_1) = 0\} \quad \Rightarrow \quad \frac{\partial\psi}{\partial w} - \frac{d}{dx}\left(\frac{\partial\psi}{\partial w'}\right) = 0$

Substitution $\Rightarrow \quad \frac{\partial\psi}{\partial w'}(x_1) \underbrace{v(x_1)}_{\text{arbitrary}} = 0, \quad \forall v \in V_0 \quad \Rightarrow \quad \frac{\partial\psi}{\partial w'}(x_1) = 0$

Du Bois Reymond lemma

LEMMA. Let $f \in C([a,b])$ be a continuous function and assume that

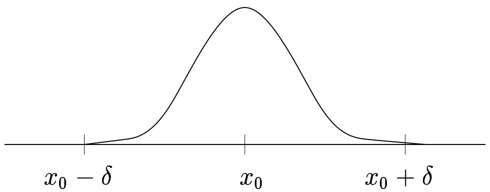
$$\int_a^b f(x)v(x)\,dx = 0, \qquad \forall v \in \hat{V} = \{v \in C^2([a,b]):\ v(a) = v(b) = 0\}$$

Then $\quad f(x) = 0, \quad \forall x \in [a,b].$

PROOF. Suppose $\exists\ x_0 \in (a,b)$ such that $f(x_0) \neq 0, \quad$ e.g. $f(x_0) > 0$

f is continuous $\quad \Rightarrow \quad f(x) > 0, \quad x \in (x_0 - \delta, x_0 + \delta) \subset (a,b)$

Let $\ v(x) = \begin{cases} \exp\left(-\frac{1}{\delta^2 - (x-x_0)^2}\right) & \text{if } |x - x_0| < \delta \\ 0 & \text{if } |x - x_0| \geq \delta \end{cases}$



$v \in \hat{V} \quad$ but $\quad \int_a^b f(x)v(x)\,dx = \int_{x_0-\delta}^{x_0+\delta} f(x)\exp\left(-\frac{1}{\delta^2-(x-x_0)^2}\right)dx > 0$

$\Rightarrow \quad f(x) = 0, \quad \forall x \in (a,b) \qquad f \in C([a,b]) \quad \Rightarrow \quad f \equiv 0 \quad$ in $[a,b] \quad \square$

Example: 1D Poisson equation

Constraints imposed on the solution $w = u$ of the minimization problem

$$\begin{cases} \frac{\partial \psi}{\partial w} - \frac{d}{dx}\left(\frac{\partial \psi}{\partial w'}\right) = 0 & \textit{Euler-Lagrange equation} \\ u(x_0) = g_0 & \textit{essential boundary condition} \\ \frac{\partial \psi}{\partial w'}(x_1) = 0 & \textit{natural boundary condition} \end{cases}$$

Poisson equation: the solution $\quad u \in V_g = \{v \in C^2([0,1]) : v(0) = g_0\}$

minimizes the functional $\quad J(w) = \int_0^1 \left[\frac{1}{2}\left(\frac{dw}{dx}\right)^2 - fw\right] dx, \quad w \in V_g$

$$\psi(x, w, w') = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - fw, \quad \frac{\partial \psi}{\partial w} = -f$$

$$\frac{\partial \psi}{\partial w'} = \frac{dw}{dx}, \quad \frac{d}{dx}\left(\frac{\partial \psi}{\partial w'}\right) = \frac{d^2 w}{dx^2}$$

$$\begin{cases} -\frac{d^2 u}{dx^2} = f & \text{in } (0,1) \\ u(0) = g_0 & \text{Dirichlet BC} \\ \frac{du}{dx}(1) = 0 & \text{Neumann BC} \end{cases}$$

Example: 2D Poisson equation

Find $u \in V_g = \{v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\Gamma_0} = g_0\}$ that minimizes the functional

$$J(w) = \int_\Omega \left[\frac{1}{2}|\nabla w|^2 - fw\right] d\mathbf{x} - \int_{\Gamma_1} g_1 w\, ds - \int_{\Gamma_2} g_2 w\, ds + \frac{\alpha}{2}\int_{\Gamma_2} w^2\, ds, \quad w \in V_g$$

where $f = f(\mathbf{x}), \quad g_0 = g_0(\mathbf{x}), \quad g_1 = g_1(\mathbf{x}), \quad g_2 = g_2(\mathbf{x}), \quad \alpha \geq 0$

Admissible functions $w = u + \lambda v, \quad v \in V_0 = \{v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\Gamma_0} = 0\}$

Necessary condition of an extremum $\frac{d}{d\lambda} J(u + \lambda v)\big|_{\lambda=0} = 0, \quad \forall v \in V_0$

$$\frac{\partial}{\partial\lambda}|\nabla w|^2 = \frac{\partial}{\partial\lambda}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 2\left[\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y}\right] = 2\nabla w \cdot \nabla v$$

$$\int_\Omega [\nabla u \cdot \nabla v - fv]\, d\mathbf{x} - \int_{\Gamma_1} g_1 v\, ds - \int_{\Gamma_2} g_2 v\, ds + \alpha \int_{\Gamma_2} uv\, ds = 0, \quad \forall v \in V_0$$

Example: 2D Poisson equation

Integration by parts using Green's formula

$$\int_\Omega [-\Delta u - f] v \, d\mathbf{x} + \int_{\Gamma_1 \cup \Gamma_2} (\mathbf{n} \cdot \nabla u) v \, ds - \int_{\Gamma_1} g_1 v \, ds - \int_{\Gamma_2} g_2 v \, ds + \alpha \int_{\Gamma_2} uv \, ds = 0$$

$\forall v \in V_0$ including $v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_\Gamma = 0 \quad \Rightarrow \quad \int_\Omega [-\Delta u - f] v \, d\mathbf{x} = 0$

Du Bois Reymond lemma: $-\Delta u = f$ in Ω Euler-Lagrange equation

$$\int_{\Gamma_1} [\mathbf{n} \cdot \nabla u - g_1] v \, ds + \int_{\Gamma_2} [\mathbf{n} \cdot \nabla u + \alpha u - g_2] v \, ds = 0, \quad \forall v \in V_0$$

Consider $v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\Gamma_0 \cup \Gamma_1} = 0 \quad \Rightarrow \quad \int_{\Gamma_2} = 0, \quad \mathbf{n} \cdot \nabla u + \alpha u = g_2$

Substitution yields $\int_{\Gamma_1} = 0, \quad \mathbf{n} \cdot \nabla u = g_1$ and the following BVP

$$\begin{cases} -\Delta u = f & \text{in } \Omega & \text{2D Poisson equation} \\ u = g_0 & \text{on } \Gamma_0 & \text{Dirichlet BC (essential)} \\ \mathbf{n} \cdot \nabla u = g_1 & \text{on } \Gamma_1 & \text{Neumann BC (natural)} \\ \mathbf{n} \cdot \nabla u + \alpha u = g_2 & \text{on } \Gamma_2 & \text{Robin BC (natural)} \end{cases}$$

Rayleigh-Ritz method

Exact solution

$$u \in V: \quad u = \varphi_0 + \sum_{j=1}^{\infty} c_j \varphi_j$$

Approximate solution

$$u_h \in V_h: \quad u_h = \varphi_0 + \sum_{j=1}^{N} c_j \varphi_j$$

φ_0 an arbitrary function satisfying $\varphi_0 = g_0$ on Γ_0

φ_j basis functions vanishing on the boundary part Γ_0

Continuous problem

Find $u \in V$ such that

$$J(u) \leq J(w), \quad \forall w \in V$$

Discrete problem

Find $u_h \in V_h$ such that

$$J(u_h) \leq J(w_h), \quad \forall w_h \in V_h$$

$$J(c_1, \ldots, c_N) = \min_{w_h \in V_h} J(w_h) \quad \Rightarrow \quad \frac{\partial J}{\partial c_i} = 0, \qquad i = 1, \ldots, N$$

Linear system: $\quad Ac = b, \quad A \in \mathbb{R}^{N \times N}, \quad b \in \mathbb{R}^N, \quad c = [c_1, \ldots, c_N]^T$

Example: 1D Poisson equation

Find the coefficients $c_1, \dots, c_N$ that minimize the functional

$$J(w_h) = \int_0^1 \left[\frac{1}{2} \left(\frac{dw_h}{dx} \right)^2 - f w_h \right] dx, \qquad w_h = \sum_{j=1}^{N} c_j \varphi_j$$

Necessary condition of an extremum

$$\frac{\partial J}{\partial c_i} = \frac{\partial}{\partial c_i} \left[\frac{1}{2} \int_0^1 \left(\sum_{j=1}^{N} c_j \frac{d\varphi_j}{dx} \right)^2 dx - \int_0^1 f \left(\sum_{j=1}^{N} c_j \varphi_j \right) dx \right] = 0$$

$$\int_0^1 \frac{d\varphi_i}{dx} \left(\sum_{j=1}^{N} c_j \frac{d\varphi_j}{dx} \right) dx = \int_0^1 f \varphi_i \, dx, \qquad i = 1, \dots, N$$

This is a linear system of the form $Ac = b$ with coefficients

$$a_{ij} = \int_0^1 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \, dx, \qquad b_i = \int_0^1 f \varphi_i \, dx, \qquad c = [c_1, \dots, c_N]^T$$

Example: poor choice of the basis functions

Consider the polynomial basis $\varphi_0 = 0, \quad \varphi_i = x^i, \quad i = 1, \dots, N$

$$u_h(x) = \sum_{j=1}^{N} c_j x^j, \qquad a_{ij} = \int_0^1 ij\, x^{i+j-2}\, dx = \frac{ij}{i+j-1}, \qquad b_i = \int_0^1 f x^i\, dx$$

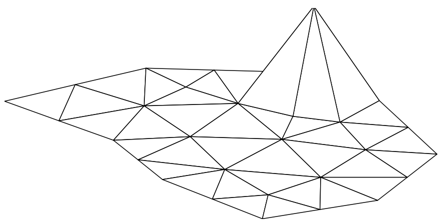
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdot & 1 \\ 1 & 4/3 & 6/4 & 8/5 & \cdot & 2N/(N+1) \\ 1 & 6/4 & 9/5 & 12/6 & \cdot & 3N/(N+2) \\ 1 & 8/5 & 12/6 & 16/7 & \cdot & 4N/(N+3) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & N^2/(2N-1) \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \cdot \\ b_N \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \cdot \\ c_N \end{bmatrix}$$

- A is known as the Hilbert matrix which is SPD but full and ill-conditioned so that the solution is expensive and corrupted by round-off errors.
- for A to be sparse, the basis functions should have a compact support

Fundamentals of the FEM

The *Finite Element Method* is a systematic approach to generating piecewise-polynomial basis functions with favorable properties

- The computational domain Ω is subdivided into many small subdomains K called *elements*: $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$.



- The *triangulation* $\mathcal{T}_h$ is admissible if the intersection of any two elements is either an empty set or a common vertex / edge / face of the mesh.
- The finite element subspace V_h consists of piecewise-polynomial functions. Typically, $V_h = \{v \in C^m(\Omega) : v|_K \in P_k,\ \forall K \in \mathcal{T}_h\}$.
- Any function $v \in V_h$ is uniquely determined by a finite number of *degrees of freedom* (function values or derivatives at certain points called *nodes*).
- Each basis function φ_i accommodates exactly one degree of freedom and has a small support so that the resulting matrices are sparse.

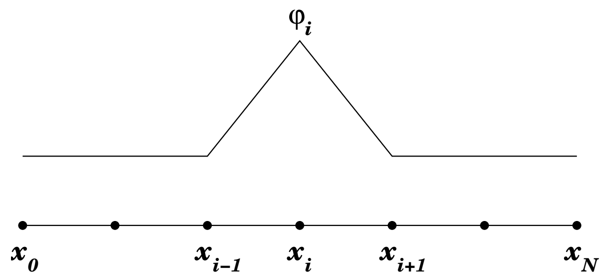
Finite element approximation

The *finite element* is a triple (K, P, Σ), where

- K is a closed subset of $\bar{\Omega}$
- P is the polynomial space for the shape functions
- Σ is the set of local degrees of freedom

Basis functions possess the property

$$\varphi_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$



$$u_h(x) = \sum_{j=1}^{N} u_j \varphi_j(x) \quad \Rightarrow \quad u_h(x_i) = \sum_{j=1}^{N} u_j \varphi_j(x_i) = \sum_{j=1}^{N} u_j \delta_{ij} = u_i$$

Approximate solution: the nodal values $u_1, \ldots, u_N$ can be computed by the Ritz method provided that there exists an equivalent minimization problem

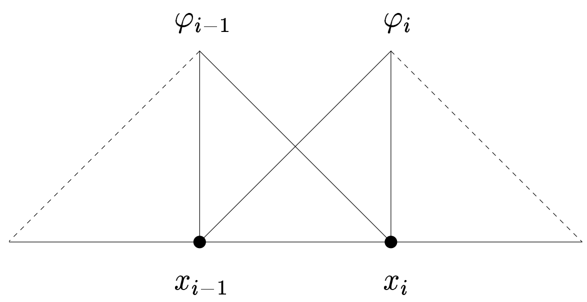
Example: 1D Poisson equation

Find the nodal values $u_1, \ldots, u_N$ that minimize the functional

$$J(w_h) = \int_0^1 \left[\frac{1}{2} \left(\frac{dw_h}{dx} \right)^2 - f w_h \right] dx, \qquad w_h = \sum_{j=1}^{N} u_j \varphi_j$$

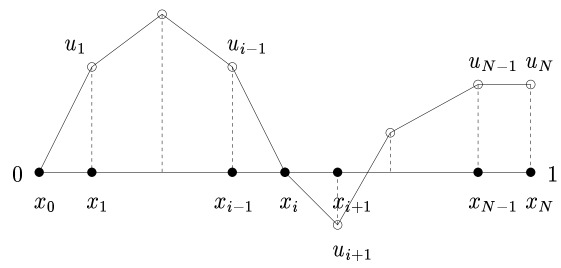
Local basis functions for $e_i = [x_{i-1}, x_i]$

$$\varphi_{i-1}(x) = \frac{x_i - x}{x_i - x_{i-1}}, \qquad \varphi_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$



Approximate solution for $x \in e_i$

$$\begin{aligned} u_h(x) &= \sum_{j=1}^{N} u_j \varphi_j = u_{i-1}\varphi_{i-1} + u_i \varphi_i \\ &= u_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}} (u_i - u_{i-1}) \end{aligned}$$



continuous, piecewise-linear

Example: 1D Poisson equation

The Ritz method yields a linear system of the form $Au = F$, where

$$a_{ij} = \sum_{k=1}^{N} \int_{e_k} \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx}\, dx, \qquad F_i = \sum_{k=1}^{N} \int_{e_k} f\varphi_i\, dx$$

These integrals can be evaluated exactly or numerically (using a quadrature rule)

Stiffness matrix and *load vector* for a uniform mesh with $\Delta x = \frac{1}{N}$ and $f \equiv 1$

$$A = \frac{1}{(\Delta x)^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ 1 \\ 1/2 \end{bmatrix}$$

This is the same linear system as the one obtained for the finite difference method!

Existence of a minimization problem

Sufficient conditions for an elliptic PDE $\quad \mathcal{L}u = f \quad \text{in } \Omega, \quad u|_\Gamma = 0$

to be the Euler-Lagrange equation of a variational problem read

- the operator $\mathcal{L}$ is linear
- the operator $\mathcal{L}$ is self-adjoint (symmetric)

$$\int_\Omega v\mathcal{L}u\,dx = \int_\Omega u\mathcal{L}v\,dx \qquad \text{for all admissible } u, v$$

- the operator $\mathcal{L}$ is positive definite

$$\int_\Omega u\mathcal{L}u\,dx \geq 0 \quad \text{for all admissible } u; \qquad u \equiv 0 \quad \text{if } \int_\Omega u\mathcal{L}u\,dx = 0$$

In this case, the unique solution u minimizes the functional

$$J(w) = \frac{1}{2}\int_\Omega w\mathcal{L}w\,dx - \int_\Omega fw\,dx$$

over the set of admissible functions. Non-homogeneous BC modify this set and/or give rise to additional terms in the functional to be minimized

Example: 1D Poisson equation

Laplace operator $\mathcal{L} = -\frac{d^2}{dx^2}$ is linear and self-adjoint

$$\int_0^1 v\mathcal{L}u\,dx = -\int_0^1 \frac{d^2u}{dx^2}v\,dx = \int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx - \left[\frac{du}{dx}v\right]_0^1$$

$$= -\int_0^1 u\frac{d^2v}{dx^2}\,dx + \left[u\frac{dv}{dx}\right]_0^1 = \int_0^1 u\mathcal{L}v\,dx$$

Positive-definiteness: $\int_0^1 u\mathcal{L}u\,dx = -\int_0^1 u\frac{d^2u}{dx^2}\,dx = \int_0^1 \left(\frac{du}{dx}\right)^2\,dx \geq 0$

If $\int_0^1 \left(\frac{du}{dx}\right)^2\,dx = 0$ then $\frac{du}{dx} \equiv 0 \quad \Rightarrow \quad u \equiv 0$ since $u(0) = 0$

Functional for the minimization problem

$$J(w) = \int_0^1 \left[\frac{1}{2}\left(\frac{dw}{dx}\right)^2 - fw\right]dx, \qquad w(0) = 0$$

Non-homogeneous BC $\quad u(0) = g_0 \longrightarrow w(0) = g_0 \quad$ (essential)

$\frac{du}{dx}(1) = g_1 \longrightarrow J(w) = \int_0^1 \left[\frac{1}{2}\left(\frac{dw}{dx}\right)^2 - fw\right]dx - g_1w(1) \quad$ (natural)

Least-squares method

Idea: minimize the *residual* of the PDE

$$R(w) = \mathcal{L}w - f \quad \text{such that} \quad R(u) = 0 \quad \Rightarrow \quad \mathcal{L}u = f$$

Least-squares functional $\quad J(w) = \int_\Omega (\mathcal{L}w - f)^2 \, dx \quad$ always exists

Necessary condition of an extremum

$$\frac{d}{d\lambda} J(u + \lambda v)\Big|_{\lambda=0} = \frac{d}{d\lambda}\left[\int_\Omega (\mathcal{L}(u + \lambda v) - f)^2 \, dx\right]_{\lambda=0} = 0$$

Integration by parts: $\quad \int_\Omega (\mathcal{L}u - f)\mathcal{L}v \, dx = \int_\Omega \mathcal{L}^*(\mathcal{L}u - f)v \, dx - \int_\Gamma [\ldots] \, ds = 0$

Euler-Lagrange equation $\quad \boxed{\mathcal{L}^*\mathcal{L}u = \mathcal{L}^* f} \quad$ where $\mathcal{L}^*$ is the *adjoint operator*

- corresponds to a derivative of the original PDE
- requires additional boundary conditions and extra smoothness
- it makes sense to rewrite a high-order PDE as a first-order system

Advantage: the matrices for a least-squares discretization are symmetric

Weighted residuals formulation

Idea: render the residual orthogonal to a space of test functions

Let $u = \sum_{j=1}^{\infty} \alpha_j \varphi_j \in V_0$ be the solution of $\begin{cases} \mathcal{L}u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \Gamma \end{cases}$

Residual is zero if its projection onto each basis function equals zero

$$\mathcal{L}u - f = 0 \quad \Leftrightarrow \quad \boxed{\int_\Omega (\mathcal{L}u - f)\varphi_i \, dx = 0} \quad \forall i = 1, 2, \ldots$$

Test functions $v = \sum_{j=1}^{\infty} \beta_j \varphi_j \quad \Rightarrow \quad \int_\Omega (\mathcal{L}u - f) v \, dx = 0, \quad \forall v \in V_0$

Weak formulation: find $u \in V_0$ such that $\boxed{a(u, v) = l(v)} \quad \forall v \in V_0$

where $a(u, v) = \int_\Omega \mathcal{L}u \, v \, dx$ is a bilinear form and $l(v) = \int_\Omega f v \, dx$

Integration by parts: $\mathcal{L}u = \nabla \cdot \mathbf{g}(u) \quad \Rightarrow \quad a(u, v) = -\int_\Omega \mathbf{g}(u) \cdot \nabla v \, dx$

Finite element discretization

Continuous problem

Find $u \in V_0$ such that

$$a(u,v) = l(v), \quad \forall v \in V_0$$

Discrete problem

Find $u_h \in V_h \subset V_0$ such that

$$a(u_h,v_h) = l(v_h), \quad \forall v_h \in V_h'$$

FEM approximations: $u_h = \sum_{j=1}^{N} u_j \varphi_j \in V_h, \quad v_h = \sum_{j=1}^{N} v_j \psi_j \in V_h'$

where $V_h = \mathrm{span}\{\varphi_1, \dots, \varphi_N\}$ and $V_h' = \mathrm{span}\{\psi_1, \dots, \psi_N\}$ may differ

(Bubnov-)Galerkin method $\quad V_h' = V_h \quad \rightarrow \quad a(u,\varphi_i) = l(\varphi_i), \quad \psi_i = \varphi_i$

Petrov-Galerkin method $\quad V_h' \neq V_h \quad \rightarrow \quad a(u,\psi_i) = l(\psi_i), \quad \psi_i \neq \varphi_i$

Linear algebraic system $\quad \sum_{j=1}^{N} a(\varphi_j,\psi_i)u_j = l(\psi_i), \qquad \forall i = 1, \dots, N$

Matrix form $\quad Au = F \quad$ with coefficients $\quad a_{ij} = a(\varphi_j,\psi_i), \quad F_i = l(\psi_i)$

Example: 1D Poisson equation

Boundary value problem

$$\begin{cases} -\frac{d^2u}{dx^2} = f \quad \text{in } (0,1) \\ u(0) = 0, \quad \frac{du}{dx}(1) = 0 \end{cases}$$

Weak formulation $\quad u \in V_0$

$$\int_0^1 \left(-\frac{d^2u}{dx^2} - f\right) v\,dx = 0, \quad \forall v \in V_0$$

Integration by parts yields

$$-\int_0^1 \frac{d^2u}{dx^2} v\,dx = \int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx - \left[\frac{du}{dx}v\right]_0^1$$

Approximate solution

$$u_h(x) = \sum_{j=1}^{N} u_j \varphi_j(x)$$

Continuous problem $\quad a(u,v) = l(v), \qquad a(u,v) = \int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx, \quad l(v) = \int_0^1 f v\,dx$

Discrete problem $\quad a(u_h, \varphi_i) = l(\varphi_i), \qquad i = 1, \ldots, N \quad$ (Galerkin method)

This is a (sparse) linear system of the form $Au = F$, where

$$a_{ij} = \int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx, \qquad F_i = \int_0^1 f\varphi_i\,dx, \qquad u = [u_1, \ldots, u_N]^T$$

The Galerkin and Ritz methods are equivalent if the minimization problem exists

Example: 2D Poisson equation

Boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g_0 & \text{on } \Gamma_0 \\ \mathbf{n} \cdot \nabla u = g_1 & \text{on } \Gamma_1 \\ \mathbf{n} \cdot \nabla u + \alpha u = g_2 & \text{on } \Gamma_2 \end{cases}$$

Weak formulation $\quad u \in V_g$

$$\int_\Omega [-\Delta u - f] v \, d\mathbf{x} = 0, \quad \forall v \in V_0$$

$$V_g = \{v \in V : v|_{\Gamma_0} = g_0\}$$

$$V_0 = \{v \in V : v|_{\Gamma_0} = 0\}$$

Integration by parts using Green's formula

$$\int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} - \int_\Gamma (\mathbf{n} \cdot \nabla u) v \, ds = \int_\Omega f v \, d\mathbf{x}, \qquad \Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$$

Boundary conditions $\quad \int_{\Gamma_0} (\mathbf{n} \cdot \nabla u) v \, ds = 0 \quad$ since $v = 0$ on Γ_0

$$\int_{\Gamma_1} (\mathbf{n} \cdot \nabla u) v \, ds = \int_{\Gamma_1} g_1 v \, ds, \qquad \int_{\Gamma_2} (\mathbf{n} \cdot \nabla u) v \, ds = \int_{\Gamma_2} g_2 v \, ds - \alpha \int_{\Gamma_2} uv \, ds$$

Approximate solution $\quad u_h(\mathbf{x}) = \varphi_0 + \sum_{j=1}^{N} u_j \varphi_j(\mathbf{x}), \qquad \varphi_0|_{\Gamma_0} = g_0$

Example: 2D Poisson equation

Continuous problem $\quad a(u,v) = l(v) + \int_{\Gamma_1} g_1 v\, ds + \int_{\Gamma_2} g_2 v\, ds, \quad \forall v \in V_0$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, d\mathbf{x} + \alpha \int_{\Gamma_2} uv\, ds, \qquad l(v) = \int_\Omega f v\, d\mathbf{x}$$

Discrete problem $\quad a(u_h, \varphi_i) = l(\varphi_i) + \int_{\Gamma_1} g_1 \varphi_i\, ds + \int_{\Gamma_2} g_2 \varphi_i\, ds, \quad \forall i = 1, \ldots, N$

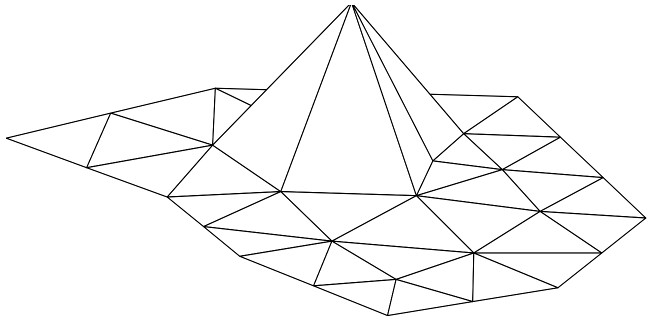
Piecewise-linear basis functions $\quad \varphi_i \in C(\bar{\Omega}), \quad \varphi_i|_K \in P_1, \ \forall K \in \mathcal{T}_h$

satisfying $\quad \varphi_i(\mathbf{x}_j) = \delta_{ij}, \quad \forall\ i, j = 1, \ldots, N$

Linear system $\quad Au = F \quad$ where

$$a_{ij} = \int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j\, d\mathbf{x} + \alpha \int_{\Gamma_2} u \varphi_i\, ds$$

$$F_i = \int_\Omega f \varphi_i\, d\mathbf{x} + \int_{\Gamma_1} g_1 \varphi_i\, ds + \int_{\Gamma_2} g_2 \varphi_i\, ds$$



The matrix A is SPD, sparse and banded for a proper node numbering