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WORKING ON ALGEBRAIC THINKING

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A STARTING POINT

At the first meeting of the CERME Algebra Group in Larnaca, Cyrpus all participants were asked to write down a sentence starting with "At the beginning of the work of the Algebra Group I think that algebraic thinking *must* include…", and to keep it safe till the last session. Then we jumped to work in several problems proposed by John Mason and Janet Ainley, to begin our work by experiencing the kind of thinking we have to reflect on, and by interacting among ourselves.

Before meeting in Larnaca we had gone through the process of peer reviewing the papers, ending with 14 papers accepted for presentation.

These papers were presented in four themed groups, which organised the type of research or theoretical questions dealt with, under the labels:

- expressing generality,
- teachers,
- expressions and equations in the interface between arithmetic and algebra, and
- mathematical objects and representations.

After four sessions discussing the presented papers, we moved to discussion around two groups of issues that emerged:

- Object and meaning / situated algebra
- Emergent algebabble / Generality as central / generalisation, abstraction, formalisation

OBJECT AND MEANING / SITUATED ALGEBRA

The papers more relevant to this topic were those by Bagni; Barbosa, Palhares and Vale; Dörfler; Fischer; Lagrange and Chiappini; Mason; Wilson and Ainley; Rahman. They raised issues concerned in various ways with algebraic activities situated in the use of artefacts and representations used by teachers and learners.

Our discussion went through a set of questions on the emergence and extension of meaning, and on the characteristics of the situations in which meaning emerge.

- How can teachers extend situated meanings?
- Can meaning emerge from the practice of manipulation (of symbols or objects) with rules?
- Can meaning emerge without practice?

- What features of a situation do we anticipate can afford access beyond the situation?

The term 'situation' was used broadly to encompass task, medium, pedagogic approach by teacher, and classroom ethos, and we pointed out the central role of the teacher in going beyond it:

- Teacher stressing purpose.
- Teacher's choice of emphasis in original situation.
- Teacher referring back from new context to original situation.

EMERGENT ALGEBABBLE / GENERALITY AS CENTRAL / GENERALISATION, ABSTRACTION, FORMALISATION

The papers more relevant to this topic were those by Alexandrou-Leonidou and Philippou; Drury; Dörfler; Fisher; Gómez; Hadjidemetriou; Mason; Molina, Castro and Mason; Papaieronymou. They raised issues concerned in various ways with expressing generality or symbol manipulation or problem solving as the core principle of school algebra, and with pupils trying to make sense of the algebraic language they are learning.

Curricular issues

Discussion about the central role of generalisation in the teaching and learning of school algebra led to the question *what is school algebra about?*, and the discussion of curricular issues.

Generalisation was contrasted with symbol manipulation and with problem solving, as the core principle of the curriculum of school algebra.

One position was that algebraic expressions can be used as tools or as objects. The approach is then to use them as objects in their own right, and use manipulation of expressions to investigate expressions – what can I do with them? Algebraic expressions are diagrams in Peirce's sense, and these diagrams are the very objects of mathematical activities.

Another position stated that a curriculum organised around problem solving does not exclude, but includes expressing generality, at least in two ways. First of all expressing generality is a problem solving activity, hence if the core principle of the curriculum is problem solving, this does not have to mean that pupils are presented only with work on quantitative arithmetic-algebraic word problems, but it should also include working with problems in which the aim is to express generality.

Secondly, expressing generality is a way to endow algebraic expressions with meaning. In order to solve word problems by using the algebraic language, pupils need to learn the use of algebraic language in a meaningful way, its syntax and the special feature this language has of calculating with the expressions without resorting to its content – a possibility due to the fact that algebraic expressions are icons (a kind of diagrams), in Peirce's terms, as he explained in Peirce C.P. 2.279.

"Meaningless symbol manipulation", or symbol manipulation following a set of conventional rules, can come later on, when one can set aside the meaning of expressions to carry on the calculations. Actually, rules of symbol manipulation are also meaningful. Ways of symbolising can be social conventions, but rules of manipulation of symbolic expressions are grounded in the observation of some structure.

What does 'meaningless symbol manipulation' mean? Setting aside meaning – but can also mean without a sense of overall purpose.

Historically, the purpose of symbol manipulation, and of solving equations is the solutions of (classes) of problems. Through a process of progressive abstraction algebraic expressions are studied as objects, and so on. Mathematisation is a practice of progressive abstraction. To jump into one level, i.e. symbol manipulation, instead of climbing up levels, or to be thrown into one level instead of being given the opportunity of climbing up the staircase of levels is not a good way to organise the curriculum.

On abstraction

Generalisation was contrasted with abstraction, by pointing out that abstraction could consist of

- taking a property – forgetting the object and asking 'what else has this property?';
- forgetting some meanings;
- transforming - a means of organisation of objects into a single object.

On idiosyncratic signs

Algebabble is an expression used to capture what pupils do when they are going through the process of giving meaning to algebraic activity. The idiosyncratic signs they produce are

- windows to pupils cognitions,
- endowed with meaning by them.

A PRODUCT OF OUR DISCUSSION

At the last meeting of the CERME Algebra Group in Larnaca, Cyrpus all participants were asked to produce the paper in which they have written their ideas on what algebraic thinking *must* include, and to reflect on what new things they could see in their papers after our discussions. One of them just said: "looking at my paper what I see is if students are able to see the structure of an equality they have begun algebraic thinking".

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RESEARCH IMPACTING ON STUDENT LEARNING: HOW CONSTRUCTION TASKS INFLUENCED LEARNERS' THINKING[1]

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Abstract

Research is commonly conceived as an exocentric process where one person probes the behaviour of another. While in many cases the intention of the researcher is to gather information about the learner, the study described in this paper illustrates how research can have an impact on student learning in terms of revealing to the learner aspects of the topic that were not previously focused upon. Twenty five students studying A-level and undergraduate engineering, pure mathematics and education were invited to construct relevant mathematical objects meeting specified constraints. Having constructed these examples, learners displayed a range of awareness of the effect of the construction tasks on their understanding of the concepts involved.

INTRODUCTION

Understanding of mathematical ideas seems to be highly cognitively situated, in the sense that learners may be well-equipped to work on the standard textbook problems in a familiar context and yet become incapacitated when faced with novel situations. This may explain why learners do not seem to be able to cope with situations beyond what they are familiar with. Following Mason (2002), I see conceptual understanding as not only the ability to use situated knowledge to solve routine problems correctly but more importantly, as the ability to extend that situatedness appropriately and efficiently into unfamiliar situations. Dealing effectively with novel situations is likely to depend on which aspects of the concept/idea become the focus of learners' attention namely, what they regard as important. In this paper, I report on a study which explores learners' awareness of integration using the 'structure of a topic' framework (Mason, 2002; Mason & Johnston-Wilder, 2004). I discuss only two of the many tasks that were prepared for the study.

The topic of integration is of particular interest to me because of its wide applicability in a number of areas. The topic has been considered by a number of researchers (see for example Orton (1983); Ferrini-Mundy & Graham (1994); Norman & Prichard (1994); Selden, Selden & Mason (1994)). What is reported on is learners' lack of flexibility, their inability to make necessary links/connections between concepts/ideas and their lack of understanding of underlying principles without a clear identification of the object of the research or causes for such

[1] This paper draws data presented in an earlier paper (Abdul Rahman, 2006) and a book chapter (Abdul Rahman, in preparation).

problems. The concept of integration, particularly poses problems to learners. Unlike differentiation, which is a *forward* process, the difficulties faced by learners in the reverse or *backward* process of integration are more complicated because it is essentially creative rather than algorithmic. The dual nature of integration, which is both the inverse process of differentiation and a tool for calculation of area and volume and length, can be confusing to learners.

Understanding in mathematics involves learners getting a sense of it in relation to their past experience. Tall & Vinner (1981) used the notion of *concept image* to capture what it means to have a sense of a concept. It describes “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes”. Using awareness to explain the act of coming to know mathematical ideas, Gattegno (1987) asserts that ‘knowing’ means stressing awareness of something and that this awareness is what is educable, indeed that only awareness can be educated, whereas other things can be trained. Informed by this assertion and enriching the notion of *concept image* with the three interwoven dimensions of human psyche (*cognition*, *affect*, *enaction*), Mason (2002) developed a framework referred to as ‘structure of a topic’ to describe how a mathematical topic is conceived.

The framework comprises three strands: *behaviour, emotion* and *awareness*, which are closely associated with the more familiar terms *enaction*, *affect* and *cognition*. Behaviour is trained through practice but training alone renders the individual inflexible. Flexibility arises from awareness which informs an directs behaviour. Learning then involves educating awareness which in turn directs appropriate behaviour. Energy and motivation to learn arise from the harnessing of individuals’ emotions. Placing emphasis slightly differently, Kilpatrick et. al. (2001) suggest five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. These strands can be explained in terms of awareness, behaviour and emotion. Conceptual understanding can be related to awareness as relevant actions are brought to attention when engaging in activities. Procedural fluency is attained by training of behaviour, although the training of behaviour alone may result in inflexibility of thought and a lack of adaptive reasoning. More appropriately training of behaviour must be driven by educating awareness so that students not only practice recently met ideas but also gain experience with some new concept. Productive disposition and strategic competence relate to emotion, as they deal with developing experience of identifying problem solutions and justifying conjectures. Behaviour which is to be flexible and responsive to subtle changes *must* be guided by active awareness.

A particular form of active awareness is the discernment of variation, which is what Marton (Marton & Booth, 1997) regards as learning: making distinctions, both discerning something from, and relating it to, a context. According to Marton, aspects of a phenomenon, situation or problem are discerned because they are simultaneously present in the individual's focal awareness and so define the individual's way of experiencing the object. This fits with a view of mathematics as being essentially about the study of *invariance in the midst of change*. Awareness of things that can vary and those that remain invariant in mathematical objects is essential for understanding any concept. Learners can be assisted in becoming aware of both the invariance and the possible change when constructing new mathematical objects for themselves. In the context of learning, Watson & Mason (2005) argue that learning involves extending awareness of dimensions of possible variation associated with tasks, techniques, concepts and contexts, as well as the awareness of the range of permissible change within each of those dimensions. In this regard, what learners make of mathematical examples and their awareness of what can vary and what is kept constant to maintain the exemplary nature of the examples can reveal dimensions and depth of their awareness and promote and enrich their appreciation of the mathematical topics.

Integral to effective mathematics instruction is the use of examples to illustrate and clarify mathematical concepts. While teachers may use examples to illustrate definitions and exemplify the use of a particular rule or theorem, learners may focus on the specific details of examples and may develop restricted thinking that only those kinds of examples are appropriate. So, they may overlook the generic sense of exemplification which the teacher intends for them. Constructing examples, however, involves different cognitive skills from working out given examples. Dahlberg & Housman (1997) showed how learners who generated examples and reflected on the process attained a more complete understanding of mathematical concepts by refining and expanding their evoked concept image. Hazzan & Zazkis (1997) showed how learners had difficulty managing degrees of freedom of generated examples. According to Watson & Mason (2005), encouraging learners to generate examples of mathematical objects can expand their *example space* and shift their attention away from the particularities of examples to generalizations. By prompting learners to construct examples, what they choose to change reveals dimensions, depth and scope of their awareness. Constructing examples *forces* them to attend to *form* in the example and disregard details that make up the example. Discerning generality with an awareness of particular details in mathematical examples requires learners to be sensitive to what can change and what must remain constant. Although learners do not normally encounter this type of question in their learning of concepts, it is my conjecture that example construction itself could develop their ability to discern dimensions-of-possible-variation and could reveal their awareness of the concept, which could give insight into the structure of their understanding.

METHOD

Semi-structured interviews were carried out with five pairs of students studying A-level mathematics and fifteen undergraduates around South East England in the United Kingdom. These students were in their first year, studying mathematics, engineering and education. The aim was to expose a range of responses in different learners concerning understanding of the topic. The sessions were tape-recorded and time was allotted for each task so that interviewees had enough time to answer all the questions. The interview sought to reveal the nature of students' understanding of integration. The questions in this task were intended to elicit learners' understanding in terms of what they are aware of, how their behaviour develops and the corresponding emotion that provides motivation and energy to learn.

Next, the students were invited to construct relevant mathematical objects meeting specified constraints with the aim of revealing their connections to the topic. Given the integral $\int_0^2 (1-x)dx = 0$, the students were invited to construct another integral like it where the answer is 0. Next, they were invited to construct another example and a third one. The aim was to reveal their connections to the topic by changing dimensions that could vary. In the second task, the students were given the expressions $2\int(\ln x + \frac{3}{2})dx = 2x\ln x + x = \frac{d}{dx}x^2 \ln x$ and asked to construct simpler and more complicated examples. What they chose to change in their examples could reveal the scope and nature of their awareness.

THE OUTCOME

Learners in this study displayed a range of awareness of the effect of the construction tasks on their understanding of the concept involved. To my surprise, the tasks that were intended to be research probes in fact influenced the learners' thinking of the topic concerned and changed their perception. Those who were aware of the change in their perception were able to articulate the change and express their appreciation.

Marlene, a mathematics student, constructed $\int_0^3 (1-x)dx$, $\int_0^4 (1-x)dx$ and $\int_0^5 (1-x)dx$. She later realized that they are not going to work and suggested that changing the gap might work and constructed $\int_2^4 (1-x)dx$. She constructed a general example of $\int_n^{2n} ...dx$. Her attention seemed to be focused on techniques of integration, particularly on the limits. She did not seem to reveal associations to area. Changing her perspectives slightly differently and probing her awareness, Marlene appeared to shift her attention away from the particularities of the example and revealed richer connections.

Interviewer: Why is it coming to zero?

Marlene: Because there's no area underneath it touching the graph, touching the x-axis. [After sketching] Aaahh …. they're cancelling each other out. Look at that! Nifty! Because part of the area is underneath and it's negative and it cancels out. … So we can do [change] both limits, couldn't we? … So we've simply got the situation here, we've got these little areas, when x is naught we got 1, naught and -1 and then naught to 2, if we sum both of them, we are going to get little areas that are going to cancel each other out.

She then constructed examples such as $\int_{-1}^{3}(1-x)dx$ and $\int_{-2}^{4}(1-x)dx$. Asked to express her awareness of dimensions that could vary explicitly, she noted:

> You can change the function; involve any straight line function that cuts through the origin …. It doesn't necessarily have to go through the origin, does it? You have to set the limit from either side of the point where it did go through. Any straight graph would work.

Although her sense of connections seemed enhanced, she did not display enough evidence to suggest rich connection as she did not change the function. In Task 2, she constructed $2\int x^2 dx = 2\frac{x^3}{3} = \frac{d}{dx}\frac{x^4}{4}$ as the simple example and suggested that having trigonometric and logarithmic functions would make the example more complicated.

After constructing examples, Marlene was asked whether the act of constructing examples had made her aware of changes in her perception of the concept. She concluded:

> I think it helps you discern what's in front of you in the sense that you saying *what is it that makes it what it is*. Once you've isolated that, you can then identify other things which are *similar,* either more complex or less complicated, but still the same in similarities. I've never tried to make easier or more complicated ones virtually the same thing but it made me look at the object and consider the method you would need and the characteristics of the object itself to try and *discern some parameters*, *put it in a box*, deprive it in some way to make it possible to *make similar the deviant*.

The construction tasks seemed to have afforded a shift in Marlene's attention from focusing on rules of the method to use to discerning properties inherent in the examples. By tinkering with the example to construct simpler and more complicated examples, Marlene appeared to be *forced* to attend to the form present in the example. In the above extract, Marlene expresses her sensitization to and appreciation of the structure in the example once she has identified the method to use.

In Task 1, Paul, an education student, constructed $\int x^3 - 4$ and remarked:

> It is just symmetric so whatever you've got the same thing on both sides, actually opposite things on both sides, like where you stick the same value and plus and minus and that bit cancels that bit.

He then suggested one of the trigonometric function (sin or cos) as another example. Paul displayed rich connections to the topic, although he displayed emotional predisposition to the topic and hesitant in his speech.

Tina, also a mathematics student, constructed $\int_0^3 (1-\frac{x^2}{3})dx$, $\int_0^4 (1-\frac{x^3}{16})dx$ and quickly generalized to $\int_0^n (1-\frac{x^n}{n^{n-1}})dx = \left[x - \frac{x^n}{n^{n-1}}\right]_0^n = n - \frac{n^n}{n^{n-1}} = 0, n \geq 2$. In the end, Tina observed:

> It has probably made me realize that when I was saying area, actually I don't really think about it as area under the graph that much. It's more about just kind of applying some sort of transformation on the object in a way; it's sort of applying the set of rules to what you've got in different matters like that in order to come with the answer.

Tina's remarks suggest that she appears to be *chorusing* associations of the concept without necessarily having awareness of that association when she is working a problem. Realization of this contradiction between what she said and what she displayed acts to reveal her awareness of the concept involved.

She struggled to understand the expression in Task 2 because it was not in the form she is familiar with. After extensive clarification, she exclaimed 'That could be any function' and constructed $\int x dx = \frac{x^2}{2} = \frac{d}{dx}\frac{x^3}{6}$ and $\frac{d}{dx}\sin^3 x \cos x = 3\sin^2 x \cos^2 x - \sin^4 x$ $= \int 6 \sin x \cos^3 x - 10 \sin^3 x \cos x dx$ as the simpler and more complicated examples, respectively. At the end of the tasks, she noted:

> We don't normally have situations where you are told to give examples; you are just given things to do. That's normally the other way round rather than you actually thinking of the examples. I suppose as students in school, we haven't really had a chance to create many ideas like this for ourselves. We have always been given ones to derive or to evaluate ourselves.

Other learners focused on the details of the tasks themselves and did not express any appreciation of the effect of the construction task on their understanding. Robert, an engineering student, constructed $\int_{2}^{0}(x-1)dx$ as the simpler example. Being asked how he came up with the example, he remarked:

> Basically I … like … worked it out and then work it backwards … thinking along that … I don't know how to actually … [inaudible]. Basically I have some values … what I'm trying to do is like … find the expression and work it backwards. … I just basically have x^2 over something … which comes into full … whole numbers wouldn't work, so it has to be a fraction.

It seemed that Robert's attention was focused on technique and displayed limited connections to the topic. In Task 2, he displayed richer connections and constructed $\int \cosh dx = \sinh = \frac{d}{dx}\cosh$ as the simpler example. At the end of the tasks, he observed:

> Just many problems that are in fact the same in dimension although they look different.

Dan, a mathematics student, displayed rich connections to the topic and demonstrated facility with technique. He observed:

> These things were almost identical to the thought processes I go through to just solve the things. They weren't very different for me. … What I tended to do was I tended to start with the rules and then work from the rules of the method to create something that is [required].

He suggested that:

> The actual complexity has no bearing in … or the method you use to solve a certain problem is independent of how complex the thing actually is.

Dan did not display any awareness of change of perception and hence, appreciation of the effect of the construction task on his understanding. He commented on the tasks themselves and displayed little or no awareness of their effect on his understanding.

Charlotte, a mathematics student, suggested working backwards to get the answer.

> I don't know. Can we work backwards and say, "I know 25 – 25 is zero, how can I get the 25 like $5x$ and I can have x^2. I must have integrated that and that should be $x – 5$ and do that between 5 and $x^2 + 5$, $5^2 - (5 \times 5)$.

It appeared that she was caught up in the act of algebraically manipulating the integral and displayed no evidence of geometric thinking. Charlotte's attention to technique appears to overshadow her awareness of form and other associations linked to the concept. She did not construct any example in Task 2 and in the end, pointed out:

> It's really hard when you say, "Can you think of a simpler one" because you just think of the concept as the concept, don't you? You don't think of it as having that many forms.

After constructing examples that revealed restricted connections to the topic, Chris remarked:

> Chris: I'm starting to understand what classes of different problems are.

It seemed that these students' remarks suggest that they are sensitized to notice the *difference* in types of examples rather than what is the *same*.

DISCUSSION AND CONCLUSION

The different ways in which learners experience and understand mathematical concepts can reveal the nature and structure of differences in how learners experience and understand what they are supposed to learn. Particularly, the nature of learners' awareness is revealed through aspects in a mathematical example that they focus their attention to and thus, regard as important.

Becoming aware of change in one's perception and developing the ability to express appreciation such a change requires learners to become sensitized to that change. A change in perception afforded by the example construction tasks is particularly important since learners become sensitized to notice structure in mathematical example. Awareness of what can change and what must remain constant helps learners discern *form* from *details* in examples. By becoming aware of features not previously at the focus of their attention, learners who were expecting to be 'tested' about their knowledge in fact revealed to themselves aspects of the concept that were not previously salient to them. While some learners expressed appreciation of the revealed awareness, others focused on details of the tasks themselves and did not articulate any awareness of this change.

Research activities that traditionally focus on *learning about probes* from learners' responses only stress the strengths or weaknesses of probes. On the contrary, this study demonstrates the use of probes in *learning about learners*. Sensitivity of probes is important in revealing more or different things learners could be aware of because different probes elicit different responses from learners. Not only do the probes reveal to the researcher something about learners' awareness, they were also very revealing to learners themselves, about their awareness of aspects of the topic as revealed through their construction of mathematical examples. Learners who were sensitized to their own change in their perception articulated the change and expressed their appreciation while others focused on the details of the tasks themselves. Surprisingly, the tasks that were intended to be research probes had an impact on some learners' thinking of the concept and of themselves.

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ELEMENTARY SCHOOL STUDENTS' UNDERSTANDING AND USE OF THE EQUAL SIGN

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Understanding the notion of equivalence and the sign that represents it are important prerequisites to the development of important algebraic concepts and procedures. Understanding the structure of an equation is critical for the development of early algebraic thinking. The present study focuses on the types of understanding that students have on the equal sign, the ways they interpret it when they fill in and construct equations and their ability to solve equations, presented in several formats. Analysing questionnaire data from 296 3rd to 6th grade Cypriot students, we found that they have serious misconceptions on the equal sign (=). This is possibly due to the long and one-dimensional use of this sign in arithmetic. Given that these misconceptions adversely affect students' efforts to proceed to more advanced algebraic concepts, such as solving equations, implications are drawn as to how far relational understanding of the equal sign could be developed in the primary school.

THEORETICAL FRAMEWORK AND RESEARCH GOALS

Developing algebraic thinking to young students is one of the main goals of modern mathematics curricula. This idea is based upon previous research findings concerning the learning difficulties that secondary school students face when studying algebra. In the same direction, mathematics education specialists as well as institutions (Kaput, 1995; NCTM, 1997; RAND, 2001; ICMI, 2004) have suggested that a greater proportion of students study algebra and that the development of algebraic concepts need to begin early as soon as the students begin the study of numbers. Consequently, the teaching of algebra needs to be according to the intellectual maturity of students, especially the younger ones. Additionally, there is a necessity for algebraic ideas to have meaning for the students' everyday life (Stacey and Chick, 2004).

The notion of equivalence and the equal sign may be of those mathematical concepts and symbols which present the mathematical thinker with the duality between process and concept. Gray and Tall (1994) introduced the idea of an amalgam of process and concept by calling this mental structure as a "procept". Consequently, the notion of equivalence and the sign that represents it may be a procept with dual meaning both in arithmetic and algebra.

Nevertheless, equivalence is one of the most important notions for the development of algebraic thinking; It is critical for equation solving, since it is the major underlying concept of the structure of any algebraic equation. Kieran (1989) refers to two types of structure for equations: (a) *surface structure* which comprises the given terms and operations of the left- and right-hand expressions and the equal sign denoting the equality of the two expressions, (b) *systemic structure* which includes the equivalent forms of the two given expressions. Much of elementary school arithmetic is oriented towards “finding the answer” (Kieran, 1989, p. 33). In order to introduce elementary school students to early algebraic thinking, it is necessary for teaching to shift emphasis to the structure of equations.

Research has shown that many students have limited understanding of equivalence and of the sign that represents it (=). The insufficient understanding of the equal sign is not limited to elementary school students; it extends to secondary and higher education (Behr, Erlwanger & Nichols, 1980) and it influences the learning of mathematics at these levels. As Cooper and Baturo (1992) have suggested, limited understanding of the equal sign may be due to the superficial and insufficient understanding of the dual meaning of the sign: (a) statically-relationally as a “scale” sign, i.e. 2 + 3 weighs 5 and (b) dynamically-operationally as a change sign, i.e. 2 changes by adding 3 and becomes 5. Previous work by Knuth, Stephens, McNeil and Alibali (2006) with students in grades 6-8 has shown that more than half of the grade 6 and grade 7 students provide an operational definition of the equal sign (gr6-53%, gr7-36% and gr8-52%). It was also found that only a small proportion of middle school students provided a relational definition (gr6-32%, gr7-43%, gr8-31%). This dual meaning may be the source of difficulties and misconceptions about the equal sign and other connected algebraic symbols, even for students at secondary and higher education.

Moreover, research has shown that many students understand the sign of equivalence as simply an “execution sign” ordering to perform the operations preceding it (McNeil & Alibali, 2000). This may be due to the frequent one-dimensional use of the equal sign in arithmetic in elementary school. Knuth et al., (2006) have found a strong positive relationship between middle school students’ understanding of the equal sign and their performance in equation-solving. They claim that relation holds irrespective of mathematical ability making the point that “… even students having no experience with formal algebra (sixth, and seventh-grade students in particular) have a better understanding of how to solve equations when they have a relational understanding of the equal sign” (p. 309).

Even though the idea of developing algebraic thinking in the elementary school has been around for some years now, the relevant research focusing on the abilities of these students for algebraic thinking leaves much to be required (Falkner, Levi & Carpenter, 1999; Saenz-Ludlow & Walgamuth, 1998; Witherspoon, 1999; Carpenter & Levi, 2000; Carraher & Earnest, 2003; Kieran & Chalouh, 1993). Research has nowadays focused on areas where arithmetic and algebra have common ground. The objective is to take advantage of procepts that occupy the intersection area in the material to be taught, in order to facilitate students’ make the transition from arithmetic to algebra, developing basic algebraic thinking from the elementary school.

One of the issues that need additional study is the teaching of equation solving (arithmetic and algebraic) in order to develop the dual meaning for the equal sign. Although a number of studies have been reported on this issue, most of them concerned middle and secondary school students (Knuth et al., 2006). Additionally, equation-solving abilities have been measured with tasks represented in symbolic formats, while we are not aware of studies that have measured elementary school students' abilities to solve equations represented in other formats (i.e. pictures, words, diagrams).

The present study focuses on the abilities that elementary school students have for algebraic thinking. The main goal of this research was to study the types of understanding that primary school students (age 8-12) have for the equal sign. Another objective was to explore the ways in which these students use the equal sign when they construct and complete equalities and how the type of understanding they have about it influences their ability to solve arithmetical and algebraic equations that are represented in different formats. The present study is part of an extended work on these issues and it includes a teaching experiment which aims to develop elementary school students' algebraic thinking.

METHODOLOGY

Participants in the study were 296 students (161 male and 135 female) from an urban and a rural primary school, 74 were 3rd graders, 68 4th graders, 81 5th graders, and 73 6th graders.

Data were collected through two tests. Test 1 (T1) comprised of three parts and aimed to grasp the type of the students' understanding of the equal sign. The first part of T1 required students to write an informal definition of the equal sign in three contexts (the sign on its own, the sign at the end in a mathematical sentence and the sign between two equivalent mathematical sentences). These tasks were used in Knuth et al. (2006) study in a similar way. The second part of T1 required students to complete equalities of different structure, that is, different number of operations, with different position of the equal sign and different position of the unknown quantity, i.e. a + b = __ + d, a + b + c = a +__. We used only single digit numbers in the first and the second part of T1, to avoid students' difficulties with the operations' algorithms. The first task of the third part required that students construct an equality with 4 numbers, i.e. __ + __ = __ + __ (This task was originally used by Witherspoon (1999). The other four tasks in this part asked students to use the four operations and their own numbers to create a given result (These tasks were originally used by Saenz-Ludlow & Walgamuth (1998).

Test 2 was designed to measure students' ability to solve equations of similar structure, which were represented in different formats, as shown in Appendix. The two types of structure that were used (the unknown quantity before the equal sign – called "start unknown" - and the unknown quantity after the equal sign – called "result unknown") were each represented in seven formats of representation (word equation, word problem, picture, diagram, equation with an unknown quantity shown in empty line, geometrical sign and algebraic symbol).

The definitions, provided by the students in the first part of T1, were coded as "relational", if the students indicated that the equal sign represents a relationship of equivalence, and as "operational", if they indicated that it announces the result or it gives the direction to do the operations. In order to classify the type of understanding students had for the equal sign, we use the term "inclusive understanding", if the students gave both operational and relational definitions and the term "restricted understanding", if they gave only operational definitions. No response or unclear responses were coded as "other".

The responses to the equalities in the second part of T1, were categorized according to how the students used the equal sign. Correct responses were coded as "Right", responses giving the sum of all the digits presented in each the equality, were coded as "Sum". Answers, which were the sum of all the operations before the equal sign, were coded as "Left Side Sum-LSS" and those, which were the sum of the operations after the equal sign, were coded as "Right Side Sum-RSS". Students also gave answers which could complete a mathematical sentence of the structure $a + b = c$ with the first three digits of the equality, regardless where the addition or the equal sign was, i.e. for an equality such as $\_\_ + 6 = 7 + 4$, they answered 1. These responses were coded as "Three First Numbers – TFN".

Students' responses to the tasks of the third part of T1 were coded as "Right", if they inserted numbers that verify the equalities. For task 1, in the case that students put the sum of the two numbers that preceded the equal sign right after it, their answers were coded as "Answer After Equal Sign–AAES", i.e. $8 + 4 = 12 + 5$. When students put the numbers in the wrong direction, in tasks involving non-commutative operations (i.e. subtraction), their answers were coded as "Wrong Direction-WD", i.e. $15 = 3 - 18$. No response or other responses in tasks of this part of the test were coded as "Other".

Correct responses to the equations of T2 were coded as 1 and wrong responses were coded as 0. Each student's score for every format of representation used to present the equations was calculated by adding the number of correct responses to the four equations of each format.

RESULTS

Table 1 presents the students' type of understanding for the equal sign. None of grade-3 students had shown inclusive understanding of the equal sign when it was presented on its own or in a mathematical sentence. Only 10% of these students had given inclusive understanding of the equal sign when it was presented in an equivalency.

Very few grade-4 students had shown that they understand that this sign represents an equivalency in any context. The data shows that about half grade-5 (46,9%) and grade-6 (54,8%) students could recognize that the equal sign represents a relationship when presented in an equivalency. In the other two contexts, most grade-5 and grade-6 students showed an inclusive understanding.

Task: How do you understand the following sign?	Type of understanding	Grade 3	Grade 4	Grade 5	Grade 6
=	Restricted	77	82,4	72,5	47,9
	Inclusive	0	7,4	15	37
	Other	23	10,3	12,5	15,1
3 + 5 + 2 + 4 =	Restricted	73	91,2	87,7	65,8
	Inclusive	0	1,5	3,7	17,8
	Other	27	7,3	8,6	16,4
5 + 6 + 4 = 5 + 10	Restricted	43,2	50	34,6	16,4
	Inclusive	9,5	15	46,9	54,8
	Other	47,3	27,9	18,5	28,8

Table 1: Percent of students' type of understanding of the equal sign in three contexts

Table 2 summarizes the students' responses to the equalities in the second part of T1; findings exemplify the misconceptions that they have about the equal sign. About half grade 3 and grade 4 students responded correctly to an equality, which had a missing number right after the equal sign. Most students in each grade (3rd – 37,7% and 4th grade – 32,8%) responded with the sum of the numbers on the left side of the equal sign, ignoring the addition of number 3 after it.

Task	Response	Grade 3	Grade 4	Grade 5	Grade 6
9 + 6 + 3 = __ + 3	Right	56,5	54,7	72,2	84,7
	Sum	0,0	4,7	2,5	1,4
	LSS	37,7	32,8	13,9	11,1
6 + 8 + 3 = 14 + __	Right	66,2	76,8	86,8	94,2
	Sum	9,2	5,4	0,0	1,4
	LSS	12,3	3,6	5,3	2,9
16 + __ = 9 + 7 + 5	Right	70,5	71,4	92,5	88,1
	Sum	0,0	0,0	1,3	1,5
	RSS	19,7	3,6	3,8	4,5
__ + 8 = 9 + 5 + 3	Right	60,7	67,3	85,9	84,1
	Sum	6,6	3,6	0,0	1,6
	RSS	13,1	0,0	2,6	4,8
	TFN	8,2	14,5	0,0	1,6

Table 2: Percent of students' responses to the completion of equalities

More students from higher grades responded correctly to the equality, although a considerable percentage of students answered with the left side sum (grade 5 - 13,9%, and grade 6 - 11.1%). Almost one out of 10 3rd graders (9,2%) responded with the sum of all numbers to the equality that had the missing number at the end of it. When the missing number was before the equal sign, almost one out of five 3rd graders (19,7%) completed the equality with the right side sum. This is an indication that they used the equal sign as an order to "find the sum", but in the opposite direction, ignoring the addition of number sixteen that preceded the gap.

Students' responses to the last equality were similar to those given to the previous one, although the missing number was at a different position. It is very interesting though to notice the responses given by grade 3 and grade 4 students for this equality; they answered 1, which completed the equality when considering only the first three numbers, that is, 1 + 8 = 9 and dismissed the following two addends. This is an indication that younger students could work out only the first three numbers of the equalities, ignoring the rest of the terms and operations.

Task	Response	Grade 3	Grade 4	Grade 5	Grade 6
___ + ___ = ___ + ___	Right	62,1	74,1	89,9	92,2
	AAES	28,8	14,8	7,6	0
	Other	9,1	11,1	2,5	7,8
15 = ___ + ___	Right	97	94,5	98,8	100
	Other	3	5,5	1,2	0
15 = ___ - ___	Right	56,1	71,9	87,5	90,6
	Wrong direction	33,3	15,8	3,8	1,6
	Other	10,6	12,3	8,8	7,8
15 = ___ X ___	Right	98,5	96,4	100	100
	Other	1,5	3,6	0	0
15 = ___ ÷ ___	Right	41,5	61,2	83,8	92,2
	Wrong direction	26,2	18,4	2,5	3,1
	Other	32,3	20,4	13,8	4,7

Table 3: Percent of students' responses to the construction of equalities

Table 3 shows that the proportion of students answering correctly the first task of Part 3 in T1 was increasing by grade level. Similarly, the proportion of students who put the sum of the numbers before the equal sign right after it has a constant decrease grade by grade. The students who gave the latter response completed the fourth missing number with one that was irrelevant to the previous three. It is important to note that the number of students of all grades who correctly completed the second and fourth task of this part of T1 was quite large. On the contrary, for tasks 3 and 5 the number of correct responses decreased significantly. The percentage of correct responses to the third and fifth task increased constantly grade by grade. A significant number of students completed the equalities by putting the missing numbers in the opposite direction, i.e. 15 = 3 – 18 or 15 = 3 ÷ 45. The proportion of students who used the equal sign in this way decreased grade by grade.

We applied analysis of variance to check for differences among students in different grades with respect to their ability to complete and construct equalities; it was found that students in elementary school have significant grade difference ($F_{3\text{-}292}$ = 12,02, p<0,01). Besides, their overall ability to solve equations was also found to vary significantly across grades ($F_{3\text{-}292}$ = 9,24, p<0,01). More specifically, it was found that student's ability to solve equations represented in different formats increases significantly with grade in each of the formats of T2, except those involving a geometrical sign. This may be due to the use of geometrical symbols to represent the missing numbers in equations as early as first grade. Table 4 shows the size of F ratio, the degrees of freedom and the significant level for each test.

The students' ability to complete and construct equalities was found to have a significant correlation with their ability to solve equations represented in all different formats in T2. The Pearson correlation indices for each format of representation were between 0,323 and 0,456 (p<0,01). According to the size of these correlation indices, the representation formats can be arranged as follows: Geometrical symbol (0,323), Numbers (0,345), Diagram (0,373), Picture (0,379), Word equation (0,429), Algebraic symbol (0,441) and Word problem (0,456). It was found that solving equations represented numerically, pictorially and with geometrical symbols affect students' ability to complete and construct equations less than solving equations represented verbally and algebraically.

Results of tests	Picture	Diagram	Word equation	Word problem	Numbers	Geometrical symbol	Algebraic symbol
Analysis of variance							
F-ratio	4,54	7,43	5,98	6,59	3,90	1,18	6,07
Df (Between-Within)	3-292	3-292	3-292	3-292	3-292	3-292	3-292
Significance.	0,00	0,00	0,00	0,00	0,00	0,31	0,00

Table 4: Results of analysis of variance by representation format of the equations

Another important finding is that students' type of understanding of the equal sign has an effect on their ability to solve equations and to complete and construct equalities. Analysis of variance for the total score for solving equations has given an $F_{3\text{-}292}$ = 17,55, p<0,01 indicating a significant difference among students who have an inclusive understanding of the equal sign, those who have a restricted understanding and those who do not have a clear understanding. Additionally, quite similar results have been found ($F_{3\text{-}292}$ = 20,27, p<0,01) between and within the same groups for their ability to complete and construct equalities.

CONCLUSIONS

The meaning that elementary school students (grade 3 – 6) give to the equal sign is found to be insufficient, particularly in the lower grades. It was found that elementary school students have not developed successfully proceptual thinking for the equal sign. This study has also shown that a large proportion of students tend to interpret the equal sign incorrectly, while completing and constructing equalities. The misconceptions, which have been observed, are influenced by two factors: the position of the equal sign and the position of the missing number in the equality. The completion of the equalities that have the missing number right after the equal sign seem to confuse the students the most.

Maturation and instruction seems to have contributed to a partial relief of the problem. Clearly, the performance of a considerable proportion of 6th graders in both equation solving and equality completion and construction is a sign of inclusive understanding of the equal sign. Exposure to a short introduction to the concept of algebraic equation seems to have positive results, at least for some students. Though the findings seem to raise a question about the high proportion of 6th graders who failed to conceptualize the relational meaning of the sign (see Table 1).

Three main questions seem to arise. First, what exactly was the cause of the improved picture that appears in 6th graders? Second, what could we do to help lower grade students develop inclusive understanding and, third, how far can we help elementary school students develop inclusive understanding? Yet the question for the lower grade students continues to be there. It is important to keep in mind that elementary school students' restricted understanding and ability to use the equal sign correctly may influence later development of important mathematical concepts and processes, such as, the notion of equivalence and algebraic equation solving.

It has been considered that the misconceptions for the equal sign that elementary school learners have, may be due to the previous use of this sign during arithmetic. Its introduction occurs along with the first student contact with numbers. Further systematic study of the notion of equivalence or of the equal sign does not happen later. Consequently, the development of this notion depends on the students' maturity level and the study of relevant concepts. We think that it will be useful to develop and implement an intervention program, which will focus on developing and mastering these two important mathematical ideas: equivalence and proceptual thinking of the equal sign. If algebraic thinking is to be developed in elementary school mathematics curriculum, it is necessary to emphasize the structural characteristics of an equation. When students understand and recognize these characteristics, they can generalize their thinking and, as a result, they can think algebraically.

It is important to treat the difficulties and misconceptions that elementary school students have about the equal sign so that they develop the correct meaning for the notion of equivalence and the sign that it represents it. Furthermore, it is essential that secondary school teachers become aware of these difficulties and misconceptions that elementary students have in order to treat them with remedial teaching before the introduction of algebraic concepts, symbolism and algorithms. Developing correct meaning for these important algebraic symbols can be crucial to the success that students will have in their future work with higher mathematics.

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APPENDIX

Examples of tasks included in T2

Numbers (Result unknown)	4 + 8 + 5 = ___
Word equation (Result unknown)	When I add 5 to 8 and subtract 3, what's the answer?
Word problem (Start unknown)	Chris gathered 9 shells from the beach. He put then in his collection and he now has 10 altogether. How many did he have at the beginning?
Geometrical symbol (Result unknown)	4 + 8 = □
Algebraic symbol (Start unknown)	$7 + \gamma + 6 = 20$

Picture (Start unknown)

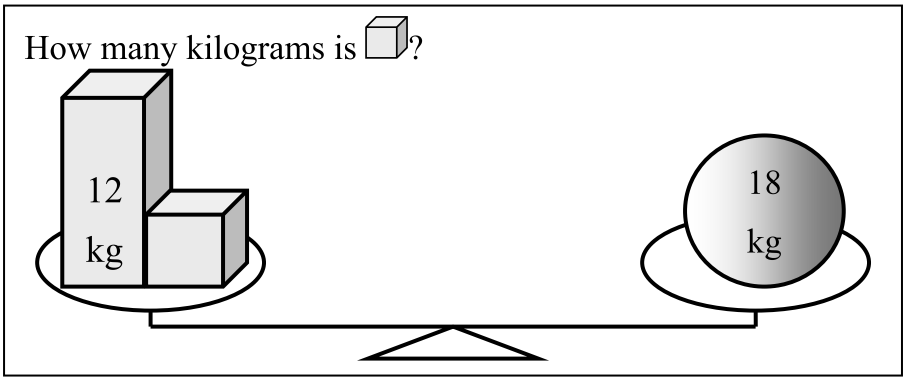
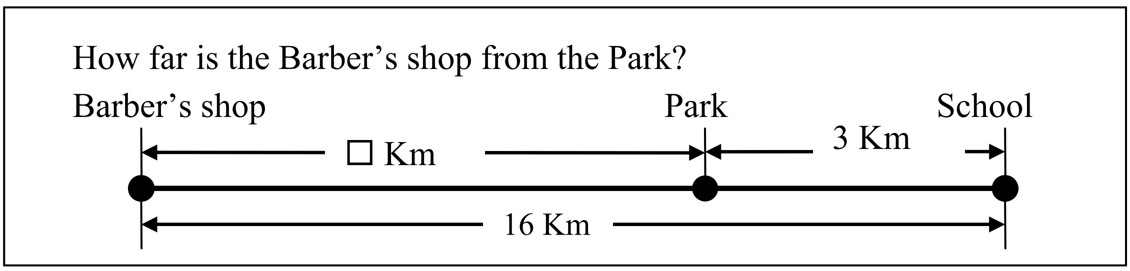


Diagram (Start unknown)



A CONTRIBUTION OF ANCIENT CHINESE ALGEBRA: SIMULTANEOUS EQUATIONS AND COUNTING RODS

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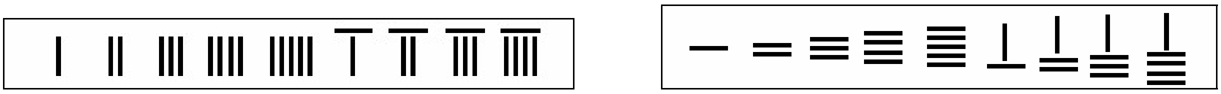
We shall discuss some educational possibilities connected to the use of an ancient Chinese artifact. Our theoretical framework is based upon works by Bruner, Vygotskij, Wartofsky and some recent papers by Bartolini Bussi. We posed to a group of 10-11 year-old pupils a problem (from the Chinese treatise entitled Jiuzhang Suanshu, 1st cent. BC) that can be solved by a couple of simultaneous linear equations. Empirical data suggest that, particularly in the context of the game, the use of the primary artifacts (counting rods, the counting board), taking into account the secondary artifacts referring to the modes of action (rules and prescriptions), can be important in order to approach some mathematical contents.

INTRODUCTION

The main focus of this paper is on the educational possibilities connected to the use of an ancient Chinese artifact. Recent research (Bagni, forthcoming) considers an experience based upon the counting-rods and the counting board. In this paper we shall make reference to a procedure described in the chapter 8 (*Fangcheng*) of the *Jiuzhang Suanshu* (*Nine Chapters on the Mathematical Art*, 1st cent. BC), an anonymous handbook consisting of 246 problems. Of course this method (i.e. Gauss elimination) can be performed with any kind of numerals; the focus or our work is not just calculating with Chinese rods or learning *fangcheng*. We shall analyse pupils' behaviour in order to point out the role of the considered artifacts.

The idea of using ancient Chinese methods for solving a system of equations can be considered as a suggestion for doing pre-algebra. Let us notice that in a positional number system a group of rods represents a set whose size is determined by its position on the table; in a linear equation, a group of rods represents an unknown set whose size, as we shall see, is determined by the box in which it is arranged. Thus, the Chinese notion of a system of linear equations is basically formed by their way of representing numbers.

Clearly, the traditional Chinese representation of numbers by counting-rods on a board can be referred to the fingers of a hand. The counting-rods are arranged in columns placed side by side, with the rightmost column representing the units, the next column representing the tens, and so on. Before the 7th-8th cent. AD, there are no known written Chinese symbols which may be interpreted as zero (Martzloff, 1997), so in order to avoid misunderstandings Chinese mathematicians used two different dispositions: the aforementioned *Tsung* disposition for units, thousands and so on, and the *Heng* for tens, hundreds and so on (Figure 1; let us remember that from 200 BC Chinese mathematicians were used to representing positive and negative numbers by red and black counting-rods).



(Tsung) 1 2 3 4 5 6 7 8 9 (Heng) 10 20 30 40 50 60 70 80 90

Figure 1: Tsung and Heng counting-rods dispositions

In this paper, our goal is mainly of a cognitive nature: to study how pupils reproduce the rules of number manipulation and, in particular, to study the relationship between artifacts and cognition in a particular context (as we shall see, in a context related to the game). Our aim is therefore to discuss a *case study* whose importance is based upon the following elements. The subject is not included in the usual cultural system, so the influence of previous activities is rather small; moreover, it is relevant to the connections between "seeing", "doing" and "saying". We shall explain this point in the following section.

THEORETICAL FRAMEWORK

In a Vygotskijan perspective, the function of semiotic mediation can be connected to technical and psychological tools. Wartofsky identifies technical gadgets as *primary artifacts*; *secondary artifacts* are used in order to preserve and to transmit the acquired skills or "modes of action" (Wartofsky, 1985). So counting-rods can be considered as primary artifacts; prescriptions and representative rules (expressed in original books and commentaries, e.g. in the *Nine Chapters on the Mathematical Art*) are secondary artifacts. Finally, a mathematical theory is a *tertiary artifact* which organizes the secondary artifacts and hence the models constructed in order to represent the modes of action by which primary artifacts are used (see for instance: Bartolini Bussi, Mariotti, & Ferri, 2005).

We must consider the distinction between artifact and *tool* (according to Rabardel, 1995), i.e. the artifact associated with a personal or social *schema* of action. If we make reference to an object as artifact, in order to be able to consider it as a tool, we need a constructive mediated activity on the part of the subject (Radford, 2002), so the considered artifact must be framed in a wide social and cultural context. When a pupil uses a primary artifact with reference to a secondary artifact, he or she *follows the rules*, so uses the primary artifact in a *rational* way. There is an important socio-cultural element in this point, taking into account that rule-following must be framed in an essentially collective practice.

An aspect to be carefully considered is the context in which we are going to propose an activity with counting-rods: as we shall see, pupils will not be asked to approach an explicit *mathematical* (in particular: algebraic) activity. Let us remember the following remark by Lakoff and Núñez (2000, p. 30):

> "Mathematical abilities are not independent of the cognitive apparatus used outside mathematics. Rather, it appears that the cognitive structure of advanced mathematics makes use of the kind of conceptual apparatus that is the stuff of ordinary everyday thought".

With particular regard to algebraic procedures, Steinbring emphasises that in order to record algebraic relations, one does not necessarily require the typical algebraic signs (Steinbring, 2006). So we are going to investigate whether the (non-typically algebraic) context of the game can be useful in order to allow the internalisation of some mathematical contents.

Let's finally underline another aspect. According to Bruner's constructivist approach, we create our own realities through interaction with the social world and with symbols (Bruner, 1987): learning itself must be considered within a cultural context, which involves the shared symbols of a community, its artifacts, and its traditions. A key concept is that of *context embeddedness*, where the term refers to the institutional and cultural context (Godino & Batanero, 1997). As we shall see, the method children used can be approached as a *new game*, which is not included in the usual cultural system, not as a traditional mathematical task.

METHODOLOGY

Let us now summarize the *fangcheng* method. In order to solve a system of linear equations, ancient Chinese mathematicians placed coefficients and numbers on the counting board. Nowadays, their arrangement can be changed according to the following rules:

(1) the expression *biancheng* ("multiplication throughout", Martzloff, 1997, p. 253) is an instruction to multiply all the terms of a row by a given number;

(2) the expression *zhichu* ("direct reduction") carries out a series of term-by-term subtractions of a row from another row.

We are going to summarize an experience involving a group of 11 year-old pupils. The experience took place in a 6th grade classroom (in Treviso, Italy). Pupils were in the very first period of the Italian middle school; they came from several different primary schools. They had not been taught about equations (only some of them, in the primary school, had dealt with simple exercises like "what is the unknown sum, if you know that three times half of this sum is…") and they had not been taught negative numbers. The experience took place during a lesson in the classroom; all the pupils (i.e. 18 pupils), the teacher and the researcher were present (the researcher was not the teacher of the pupils in question).

Let us consider the three stages of our experience: firstly, the teacher presented to the pupils how to express numbers with counting-rods. He demonstrated several examples and the pupils were invited to represent some numbers. Secondly, a counting board with labels was created and simple problems were represented by counting-rods. Finally, the pupils were divided into six groups of three pupils. The teacher posed the following problem (*problem 1*) and represented it with counting-rods on a counting board; then the teacher emphasised that "the rods arrangement represents problem data" and presented (with several examples, too) rules (1) and (2) by which rods arrangements can be changed:

(*Problem 1*) Suppose we have 5 bundles of type A cereals and 3 bundles of type B cereal, amounting to 19 *dou* of grain. Suppose we also have 3 bundles of type A cereals and 2 bundles of type B cereals amounting to 12 *dou* of grain. Question: how many *dou* of grain in 1 bundle of type A and type B cereals respectively?

This problem (based upon an original Chinese problem, with some variations of data) leads to the pair of simultaneous equations, in our modern notation:

$$5x + 3y = 19$$
$$3x + 2y = 12$$

Let us see (Figure 2) the rods arrangements suggested by some pupils in order to obtain the solution (we report the original counting board used; the translation of the labels is: "type A" and "type B bundles", "cereals"). In particular, a group of three pupils obtained the correct solution of the problem: one of the pupils (S.) proposed the steps and another pupil (F.) made some interesting comments; the third pupil's (R.) role was not active. During the experience, the teacher gave no suggestions; he just pointed out the mistakes.

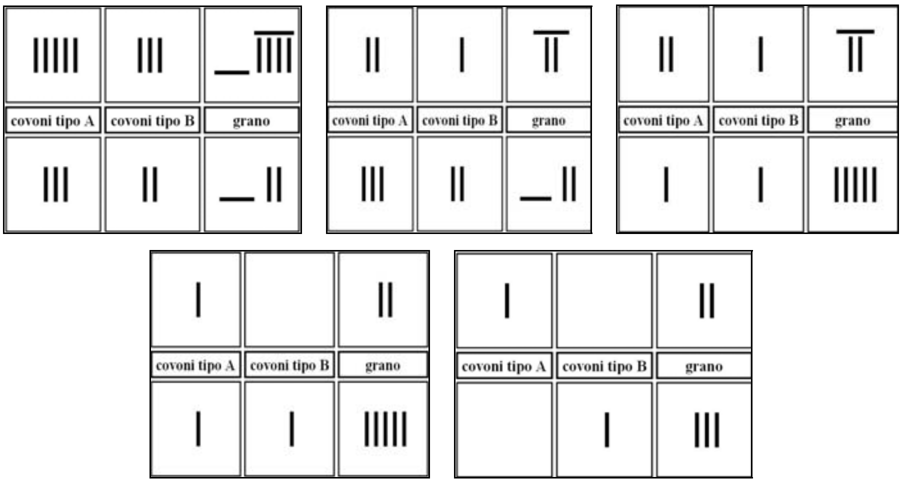


Figure 2: Rods arrangements in order to obtain the solution of the first problem

In order to deepen our comprehension of pupils' behaviour, we posed to the same group of pupils another problem:

(*Problem 2*) Suppose we have 4 bundles of type A cereals and 1 bundle of type B cereal, amounting to 6 *dou* of grain. Suppose we also have 2 bundles of type A cereals and 3 bundles of type B cereals amounting to 8 *dou* of grain. Question: how many *dou* of grain are there in 1 bundle of type A and type B cereals respectively?

This problem leads to the pair of simultaneous equations (in our modern notation):

$$4x + y = 6$$
$$2x + 3y = 8$$

Let us see (Figure 3) the rods arrangements suggested by some pupils in order to obtain the solution. Of course it is impossible for the pupils to solve *problem 2* by using only rule (2): its application would imply the use of negative numbers.

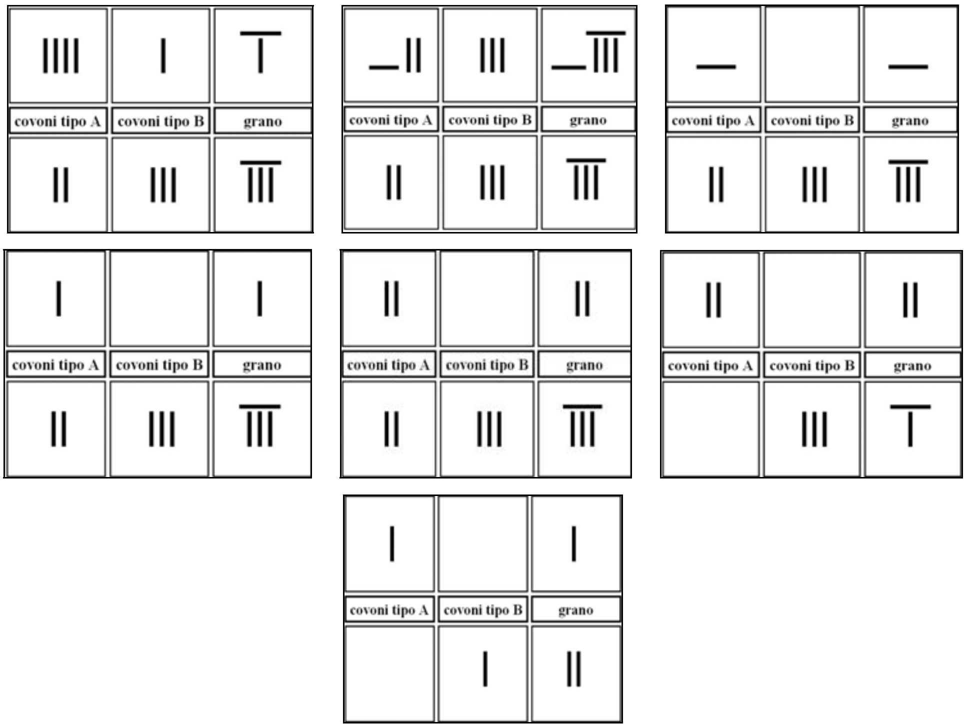


Figure 3: Rods arrangements in order to obtain the solution of the second problem

DATA

Recorded audio material and transcriptions allowed us to point out some salient short passages that were analyzed using techniques of qualitative research (see: Bagni, forthcoming). The experimental excerpts allowed us to highlight some elements that are relevant to our discussion.

Let us now examine the following transcript of the resolution of *problem 2*. It is particularly representative because, as we shall see, it gives us the opportunity to highlight some elements related to pupils' use of the artifacts (other data and results are not very interesting); the following transcript is to be considered as a *case study*:

[1] Pupil S.: Mm, no, we cannot take off these from those, there are not enough of them there. [S. indicates the second row, then the first row and touches the rods of the first square; she looks at the teacher]

[2]	Pupil F.:	The other also is impossible… Well, this time the exercise is impossible, isn't it? [S. touches the rods of the second row several times. One minute goes by]
[3]	Pupil S.:	Alright, we must take off the rods, but if we now increase them… using the other rule, we can multiply these, that is… yes… until there aren't enough of them.
[4]	Pupil F.:	Or those in the upper row.
[5]	Pupil S.:	Let's do it here … by three. [S. indicates the second square of the first row]
[6]	Pupil F.:	Oh yes, yes, we must make this equal to these! [She indicates the second squares of the first row and of the second row]
[7]	Pupil S.:	Go on, here, let's do it here by three. This one becomes three, four times three is twelve and the grain, six, gives eighteen. [She changes the arrangement of the rods]
[8]	Pupil S.:	Now we take them off… that is, here they goes away, twelve, ten, and ten here, too. [S. takes off the rods]
[9]	Pupil F.:	They are equal.
[10]	Pupil S.:	The first is done. [S. takes off the rods]
[11]	Pupil S.:	Now I need to get rid of that. [She indicates the first square of the second row] We just have to multiply by two. [S. adds the rods; then S. takes off the rods]
[12]	Pupil F.:	Done.
[13]	Pupil S.:	Well, we can divide in the usual way, six, we have two. [S. takes off the rods] The grain is one here and two here.

Let us now briefly discuss pupils' behaviour. The first stage (steps 1-2, with a brief period of impasse, one minute without utterances) is interesting: there is a conflict between the situation and the necessity of taking off as many rods as possible (as pupils did in the previous resolution: see Figure 2). This initial conflict led F. to a wrong conclusion. But the utterance [3] by S. is very important:

[3] "Alright, we must take off the rods, but if we now increase them… using the other rule, we can multiply these, that is… yes… until there are enough of them".

These last words are meaningful: it is necessary that the counting-rods are "enough". The teleological structure of the action has a primary role. When pupils correctly used the rule (1) allowing the multiplication of all the terms of the second row by $k = 3$, F. realized that her previous conclusion ("this time the exercise is impossible") was wrong, she indicated the different squares and noticed:

[6] "Oh yes, yes, we must make this equal to these".

It is worth noting that the pupil connected the application of the considered rule to the counting-rods arrangement and to its effect upon this arrangement: F. was finally aware of the necessity of the absence of rods in a square in order to solve the problem, and she realized that in order to reach this situation it is necessary that two squares referring to the same type of bundles contain the same number of counting-rods. So solving strategies are connected to a combination of enactive skills (e.g. spatial awareness) and iconic skills (e.g. visual recognition and ability to compare: Bruner, 1966).

DISCUSSION

In the experience considered, pupils approached an algebraic procedure without using the typical algebraic signs. They effectively solved a couple of simultaneous linear equations by using counting-rods, with reference to the secondary artifact expressed in chapter 8 of the *Jiuzhang Suanshu*. By that we are not suggesting introducing systems of linear equations in grade 6 (in many countries, e.g. in Italy, this is done in grade 8, or in grade 9); nevertheless the idea of using the ancient Chinese methods for solving linear equations for introducing the topic in school teaching can be interesting.

According to Bruner (1987), developmental growth involves the enactive, iconic and symbolic modes, and requires the ability to translate between them: the experience considered provides us with interesting examples of translation (see moreover the examples discussed in: Bagni, 2006). An effective translation from the enactive to iconic mode (the frequent use of deictic expressions and of gestures, e.g. in steps 1, 5 and 6) and, in addition, a first approach to the symbolic mode can be seen in the pupils' behaviour.

Pupils gave a preference to the rule that is based more directly upon the concrete presence of counting-rods on the counting board (in the first resolution, see Figure 2). Indeed, when they apply rule *zhichu* ("direct reduction"), which allows term-by-term subtractions of two rows, they consider two quantities that they can see and touch; whereas when they apply rule *biancheng* ("multiplication throughout"), which allows the multiplication of all the terms of a row by a number *k*, this number *k* cannot be referred to the concrete presence of counting-rods. So, according to the empirical data, we can state that using (original) primary artifacts with reference to (original) secondary artifacts can be relevant to the introduction of some methods; more generally, the crucial point is that *the considered method is based upon the "positional" character of ancient Chinese algebra*, according to which a particular place in the board must be always occupied by a particular kind of number, e.g. a particular coefficient. This "positional" character cannot be pointed out in our basic algebraic European tradition.

Let us point out an important element: the artifacts considered are "non transparent" (in L. Meira's words; see: Meira, 1998). In particular, this means that there is no way of knowing, for instance, why some entries in the given rectangular array should be added together, or why the entries in the last column would represent totals, or why the given rules would preserve relevant features of the situation represented by the rods arrangement. Nevertheless, pupils effectively used a representation including signs, spatial relations and embodied rules with reference to a context having some typical features of a *game*. So an important path to follow can be related to the role of the game: this concrete context may allow a first construction of meanings that can be referred to abstract algebraic representation.

It is worth noting that the secondary artifact introduced is not strictly necessary in order to allow a physical action with the primary artifact. From this point of view, the introduced rules can be considered conventional, arbitrary (the original secondary artifact can be simplified: counting-rods can be arranged in very many ways; of course the intercultural aspect leads us to make reference to the original dispositions). So pupils made reference to a particular algebraic “language” that is not just a code, whose power can be referred to its syntax; its creative power lies in how the language itself is embedded into the rest of pupils’ activities (Steinbring, 2006), and, in the case considered, into a *game*, a *new game* to be explored and played (we are dealing with a game making reference to a very different cultural tradition): and this can be *useful in order to give sense to the algebraic procedure considered*.

Of course it is important to investigate the conceptualization of the experience: so further research will be devoted to the study of the educational possibilities connected to activities similar to the one considered (although, in our opinion, the main point cannot be summarised in the possibility of a complete derivation of an argumentation or a mathematical proof).

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Patterns and generalization: the influence of visual strategies

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This paper gives a description of an ongoing study focused on pattern exploration and generalization tasks and justifies that study with respect to the literature. It promises to make a contribution to appreciation of the ways in which visual strategies can be used to enhance and enrich learners' experience of generalization. The main purpose is to analyse the strategies and difficulties presented by grade 6 students when solving these activities, along with the role played by visualization in their reasoning. Preliminary results indicate that, in general, students prefer analytic approaches over visual ones and that, among the group of students that are most successful, the majority chooses a mixed strategy.

THEORETICAL FRAMEWORK

A quarter of a century ago, problem solving became a focus of school mathematics. According to recent curricular guidelines of several countries, one of the main purposes of mathematics learning is the development of the ability to solve problems. In spite of the growing curricular relevance of this theme over the last few years, several international studies (SIAEP, TIMSS, PISA) have shown that Portuguese students have low results when solving problems is required (Ramalho, 1994; Amaro, Cardoso & Reis, 1994; OECD, 2004). Pattern exploration tasks may contribute to the development of abilities related to problem solving, through emphasising the analysis of particular cases, organizing data systematically, conjecturing and generalizing. The *Principles and Standards for School Mathematics* (NCTM, 2000) acknowledges the importance of working with numeric, geometric and pictorial patterns. This document states that instructional mathematics programs should enable students, from pre-kindergarten through grade 12, to engage in activities involving the understanding of patterns, relations and functions. On the other hand, Geometry is considered a source of interesting problems that can help students develop abilities such as visualization, reasoning and mathematical argumentation. Visualization, in particular, is an important part of mathematical reasoning but, according to some studies, its role hasn't always been emphasized in students' mathematical experiences (Hadamard, 1973). According to Dreyfus (1991) visual reasoning in mathematics is important in itself. Therefore it's necessary to give increased status to purely visual mathematical tasks. The usefulness of visualization and graphical representations is being recognized by many mathematics educators. However more research is still necessary concerning the role mental images play in the understanding of mathematical concepts and in problem solving. Research is also needed to ascertain when visualization is more useful than analytical methods (Gutiérrez, 1996).

The purpose of this study is to analyse the difficulties and strategies used by grade 6 students (11-12 years old), when solving problems involving pattern search, and the role played by visualization in their reasoning. The tasks that will be used in the study involve pattern generalization. Students of this age have not yet had formal algebra instruction, thus the importance of analysing their approaches. This study attempts to address the following research questions:

1) Which difficulties do 6th grade students present when solving pattern exploration tasks?

2) How can we characterize students' strategies?

3) What's the role played by visualization in the mediation of students' reasoning? Will it simplify the path to solution or will it act as a blocking element?

Patterns and generalization

Using patterns to promote and provoke generalization is seen by many as a pre-algebraic activity (e.g. Mason et al., 1985; Mason, 1996; Lee, 1996). The focus on pattern exploration is frequent in the recent approaches to the study of algebra. The search for regularities in different contexts, the use of symbols and variables that represent patterns and generalization are important components of the math curriculum in many countries. Portuguese curriculum recommends that students should develop the predisposition to search and explore number and geometric patterns throughout elementary and middle school (DEB, 2001).

Research on students' thinking processes in generalization

There are now several studies about the analysis and development of pattern-finding strategies with students from pre-kindergarten to secondary school.

Stacey (1989) focused her investigation on the generalization of linear patterns, with students aged 9-13 years old. A significant number of the subjects used an erroneous direct proportion method in an attempt to generalize. Stacey also reported some inconsistencies in the strategies used by students in *near generalization* (activities that can be solved by the use of a drawing or the recursive method) and *far generalization* (the strategies stated before are not adequate to these kind of activities, they imply the finding of a rule) and concluded that drawing had a major influence on their approaches, although she didn't explore this theme further.

García Cruz & Martinón (1997) developed a study, with 15-16 years old students, aiming to analyse the way they validate results and to ascertain if they favoured numerical or geometric strategies. The research showed that drawing played a double role on the process of abstracting and generalizing. It represented the setting for students who used visual strategies in order to achieve generalization and, on the other hand, acted as a means to check the validity of the reasoning for students who favoured numerical strategies.

Orton & Orton (1999) focused their investigation on linear and quadratic patterns with 10-13 years old students. They reported a tendency to use differences between consecutive elements and its extension to quadratic patterns, by taking second differences, but without success in some cases. They also pointed to students' arithmetical incompetence and their fixation on a recursive approach as some of the obstacles to successful generalization.

Sasman et al. (1999) developed a study with 8th grade students, involving generalization tasks with variation of the representations. Results showed that students used, almost exclusively, number context, neglecting drawings, and favoured the recursive method, making several mistakes related to the erroneous use of direct proportion.

Mason et al. (2005) promotes the use of the strategy of 'Watch What You Do' as learners draw further cases of patterns and attend to how they naturally draw the pattern efficiently. Each such efficient drawing method offers a potential generalization when expressed as instructions as to how to draw the pattern.

Visualization

The relation between the use of visual abilities and students' mathematical performance constitutes an interesting area for research and does not achieve consensus. Many researchers recognize the importance of the role that visualization plays in problem solving, while others claim that visualization alone isn't enough, that it must be used as a complement to analytic reasoning. According to Presmeg (1986), teachers have a tendency to present visual reasoning only as a possible strategy for problem solving in an initial stage or as a complement to analytic methods.

Thornton (2001) points to three reasons to re-evaluate the role of visualization in school mathematics: (1) math is currently identified with the study of patterns and that, together with the use of technology, has the power to demean the difficulty of algebraic thinking; (2) visualization can often provide simple and powerful approaches to problem solving; (3) teachers should recognize the importance of helping students develop a repertoire of techniques to approach mathematical situations.

Different students can use different strategies when solving the same problem. Some prefer visual methods, others are in favour of non-visual ones. Krutetskii's (1976) study with mathematically gifted students showed that they use different approaches to problems, leading to the following categorization concerning reasoning: analytic (non- visual), geometric (visual) and harmonic (use of the two previous types of reasoning).

In spite of the preference for the use of numerical relations as a support for reasoning, in part due to the work promoted in the classroom, some studies indicate that most are more successful when they use a harmonic or mixed approach (Moses, 1982; Noss, Healy & Hoyles, 1997; Stacey, 1989).

METHOD

The sample used on this study consists of three classes of grade 6 students, from three different schools, aged 11-12 years, corresponding to a total of 54 students. The study is divided in three stages: the first corresponds to the administration of a test that focuses on pattern exploration and generalization problems; the second stage involves the implementation of tasks, of the same nature, to all students, in pairs; on the third, students will repeat the test in order to examine changes in the results. The second stage of the study is, at the present time, in the beginning. All students will be involved in solving 10 tasks over the school year and two pairs of students from each school will be selected for clinical interviews. These sessions will be videotaped for further analysis in order to investigate students' mathematical reasoning, in particular the strategies used to solve each of the problems posed, as well as the difficulties they experienced on that activity.

There will be qualitative and quantitative data. To gather the quantitative data a scoring scale for the test was developed in order to compare the two applications. Qualitative data will be collected from the interviews with the elements of the pairs and from the analysis of the strategies used in the tests.

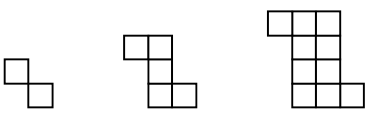
PRELIMINARY RESULTS

At present time only the first stage of the study is concluded. Students were given a written test with pre-algebraic questions. The test contains sixteen introductory questions consisting of visual and numerical sequences **(see a)**, for an example**)**, followed by two more complex tasks involving *near* and *far generalization* **(see b) and c))**.

a) Examples of introductory questions:

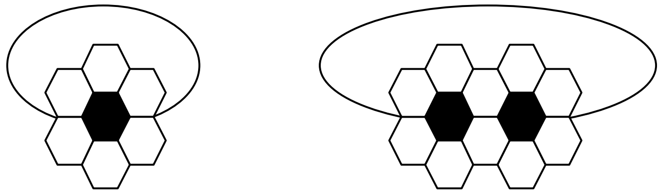
1. *Complete the following sequences indicating the next* ***two*** *elements:*

1.2: *2, 5, 8, 11, 14*

1.13: 

b) Second task:

2. *Joana likes to make necklaces using flowers. She uses white beads for the petals and black beads for the centre of each flower. The figure below shows a necklace with one flower and a necklace with two flowers, both made by her.*



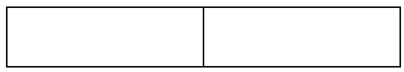
2.1. *How many white and black beads will Joana need to make a necklace with 3 flowers?* ***Explain your conclusion****.*

2.2. *How many white and black beads will Joana need to make a necklace with 8 flowers?* ***Explain your conclusion****.*

2.3. *If Joana wants to make a necklace with 25 flowers, how many white and black beads will she need?* ***Explain your conclusion****.*

c) Third task:

3. *On the following figure you can count three rectangles.*



Consider the figure below:

3.1. *How many rectangles of different sizes can you find?* ***Explain your reasoning****.*

3.2. *If you had 10 rectangles in a row, how many rectangles of different sizes could you count?* ***Explain your reasoning****.*

The test was constructed with the purpose of analysing students' abilities when performing pattern seeking and generalization tasks and of studying their problem solving approaches. It was validated by a panel of teachers and researchers in mathematics education. It was also solved by 5^{th} and 6^{th} grade students of different schools before its implementation.

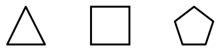
The application of this test with the sample used in this study made it possible to collect some preliminary data about students' thinking processes and most common difficulties.

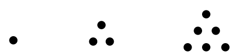
Thinking strategies that emerged from the application of the test

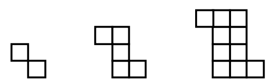
In spite of being given an image of the first two elements of the sequence, in the second task students rarely use drawing as a solving strategy, they favoured a numerical approach. Some students made a drawing to solve the first two questions and applied direct counting to determine the number of beads, but they weren't able to solve the last question by the same method, since it involved *far generalization*, so they left it in blank or presented a feeble attempt to solve it. The few students that have successfully solved this task used a mixed strategy, presenting numeric relations and referring to the visual structure of the sequence.

Third task was considered by the students as the most difficult. No one could reach a solution. Some students identified the existence of different rectangles but, as they didn't found an organized way to approach the question, they couldn't find all the cases. In the second question of this task no figure was given. Most students started by representing the situation, but in the end they weren't able to discover the pattern due to the application of inadequate strategies like direct counting or the use of a confusing diagram.

Difficulties emerging from the application of the test

The greatest facility was achieved on the first task of the test, possibly because they had prior experience solving this type of tasks. Nevertheless they showed some difficulties that should be pointed. Some of the sequences were interpreted, by several students, as repetition patterns, both on visual and numerical contexts. The two most frequent cases happened with the numerical sequence *1, 4, 9, 16* and the visual  sequence . Students continued the first by adding 3 to 16 and 5 to 19, instead of continuing the sequence of squares of whole numbers. In the second case, we expected to get a hexagon and a heptagon and some students presented a triangle and a square, repeating the sequence. The majority of students achieved better results on the questions involving numerical patterns than on those involving visual patterns. They presented very low scores in completing the following two sequences, whose nature was purely visual:





In our opinion, the first one caused some difficulties possibly due to the triangular shape of the elements of the sequence and the second one due to the simultaneous variation of length and height.

On the second task there's a general tendency to the erroneous use of the direct proportion method. This indicates that students didn't analyse properly the structure of the sequence, thinking of each flower as a disjoint unit. Most of them considered that each flower had six white beads and one black, so a necklace with eight flowers would have forty-eight white beads and eight black and a necklace with twenty five flowers would have hundred and fifty white beads and twenty five black. These students didn't notice that consecutive flowers had two white beads in common. They would easily see the error if they checked the rule with a drawing. On the last two questions of this task, which involve the use of the recursive method or the finding of a general rule, scores were very low. In our opinion, students' tendency to manipulate numbers only, may have contributed to enhance the difficulty of finding the pattern in question.

The last task of the test was the most difficult for students to solve. The majority identified only the smaller rectangles and the bigger one, possibly influenced by the example given in the problem. In some cases, they used the direct proportion to determine the number of rectangles, similarly to the previous task, considering that if they had ten rectangles in a row, then they would have to duplicate the result obtained in the first question.

DISCUSSION

Work with patterns may be considered a unifying theme of mathematics teaching, appearing in different contexts and contributing to the development of several concepts (NCTM, 2000). In this research, the use of pattern exploration tasks has the main purpose of setting the environment to analyse the impact of the use of visual strategies in generalization.

Some studies indicate that students prefer analytic approaches to mathematical activities, converting into numbers even problems that have a visual nature. At this point this research is one more contribution to this view. Research on visualization and on the role of mental images in mathematical reasoning has shown the importance of representations in conceptual development (Palarea & Socas, 1998). Our expectations, at this moment, are that, on the second stage of the study, with the implementation of the tasks, students will then use more frequently visual or mixed strategies and develop a higher competence in solving pre-algebraic activities.

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MATRICES AS PEIRCEAN DIAGRAMS: A HYPOTHETICAL LEARNING TRAJECTORY

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The notion of diagrammatic reasoning (Ch. S. Peirce) contends that perceptive observation of transformations of diagrams (iconic signs) is at the centre of mathematical inferences and constructions. In the form of a narrative about an hypothetical learning process concerned with matrices an exemplary interpretation of Peirce's position is offered. At the centre of this interpretation is the view that diagrams in the sense of Peirce are not just the means but rather the very objects of mathematical activities. The empirical and observational quality of those activities with diagrams is exemplified by looking at matrices as diagrams, i.e. as structured inscriptions with rule governed transformation rules.

Introduction

In the broader public and in the scientific communities of mathematics education and mathematics as well there are widespread but scarcely reflected and mostly implicit presuppositions: mathematical activities (thinking and reasoning) are predominantly mental and mathematical objects are abstract (immaterial) and ideal. This basic ontology and epistemology has far-reaching consequences on the roles which are attributed to external signs, notations and symbol systems which commonly are subsumed under the rubric of (external) representations. In this view, representations are primarily a means for learning and understanding but they do not become a genuine and autonomous topic of it. As the term "representation" already suggests, their main purpose is to express and communicate something which is essentially different and separated from them. This, for instance, is reflected in the strict separation of number and numeral and in the supposed primacy (genetically and epistemologically) of the former over the latter. The purported just auxiliary and secondary quality of (written) mathematical signs is reflected by the position that those signs always signify something else to which they permit cognitive access. For all of this and the following please refer to Dörfler (2006). The main intent of this paper to the contrary is to offer a view which attributes a central and primary role to the signs (without pre-given referents) as being constitutive, and the very objects, for all kinds of mathematical activities. Thereby the common representations turn into genuine but then perceivable mathematical objects the properties and structures of which are to be investigated and learned. A similar view is taken by Rotman (2000) who speaks about the inseparable unity of "scribbling/thinking". The relevance of those considerations for mathematics education as a practice in the classroom possibly could be to reduce the widespread anxiety of and unwillingness to learning mathematics. I suggest that those detrimental phenomena are partly connected to the strong belief that mathematics is highly abstract (which is often equated with difficult and inaccessible) and therefore on part of the learner necessitates a special ability. The common discourse as it also guides the writing of schoolbooks suggests that the mathematical signs and notations refer to something abstract which is essentially different from them but which is at the core of learning mathematics. For that goal the (external) representations are considered to be an unavoidable means but most learners are stuck with mechanic and algorithmic manipulations of the various representations. This is reflected in the notorious complaint that our students do not use representations as tools for problem solving. This could partly be due to the fact that they are not sufficiently familiar with those representations (visualizations) since they never had studied them in their own right beyond being a pathway to the abstract realm. I contend that the representations or as I will call them the diagrams according to the semiotics of Peirce (see below) can and should be much more than just a means to construct mental representations.

To make the intention of this paper transparent some more remarks are appropriate. Inscriptions as diagrams (something written or drawn on paper, in the sand or on a screen together with writing/rewriting rules) play an important role everywhere in maths and maths education. This is certified by the comprehensive literature on (external) representations, visualizations and the like. Here the ubiquitous discourse suggests that those representations by embodying properties of abstract (mathematical) objects permit the learner to construct mental (internal) representations of those objects. The representations are taken to mediate between the mathematics and the mind of the learner. The Peircean notion of diagram focuses the interest on the inscriptions themselves and their properties and relationships, irrespective of what they might (re-)present. Here it is appropriate to say that Peirce himself was rather a realistic philosopher who attributed an important role to the objects within his triadic conception of signs. But for diagrams which according to Peirce are icons of relational structures he explicitly admits the variant that a diagram stands for a possibility which might be realized only in the future. This can also be interpreted in the way that for diagrams a kind of identification of object and sign is conceivable which is discussed also by Stjernfelt (2000). In this sense then the diagrams will be not just a means for investigating an object different from them. The diagrams turn into the very objects of investigation and this latter one is carried out by manipulating (and inventing) diagrams and detecting recurring patterns within these actions with diagrams. Those patterns can then be formulated as theorems or by formulas which again are diagrams. Those ideas are not alien to maths education where (predominantly for the lower grades) many proposals for investigating different kinds of diagrams have been presented (number squares, number walls, polyominoes that is squares of equal size arranged with coincident sides, geometric figures, etc.). Here, the case of matrices taken as diagrams is to show that this view extends to higher mathematics as well. The inventiveness and creativity necessary for successful diagrammatic reasoning are discussed in much detail by Hoffmann (2005b).

Diagrams and Diagrammatic Reasoning

In the past few years the notions of diagram and diagrammatic reasoning as espoused by Ch. S. Peirce have received increased interest in mathematics education (e.g. Hoffmann, 2005a; Dörfler, 2004, 2005; Marietti, 2005; Otte, 2005). Peirce (1976) himself gives among many others the following description:

By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, performs experiments upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses this in general terms. This was a discovery of no little importance, showing, as it does, that all knowledge without exception comes from observation. (Peirce NEM IV, 47 f.)

For our purposes here consider a diagram as a type of inscriptions with a definite (imposed) structure and a set of precise rules for the transformations and manipulations of those diagrams. Other examples besides the matrices considered here are: decimal numerals, algebraic terms, polynomials, function graphs, geometric figures. Thus the Peircean notion of "diagram" is a wide one and differs from the more common use of that word. It is related to the Kantian and Piagetian concept of "schema" but without their idealistic or cognitivistic aspects. For more details refer to Dörfler (2004, in press). Outstanding features of diagrammatic reasoning are:

- it is based on perception, observation and pattern recognition
- it is inventive, constructive and creative though following precisely defined rules
- it takes the diagrams/inscriptions as its objects of investigation
- being successful needs much foregoing experience with diagrams, precision and attention, persistence and experimental behaviour (e.g. to ask "What happens if I do that?")
- it needs memory and recall of previous diagrammatic reasoning
- it detects properties of and relationships between diagrams as regularities and invariants when those are manipulated according to the rules
- it needs a language in which to talk about the diagrams and their properties and to interpret and apply them (giving referential sense to the diagrams)
- it is more like a handicraft based on writing and observing through which and in which the reasoning is inextricably carried out (a similar view takes Rotman, 2000)
- it is neither mechanistic/algorithmic nor formal but content-rich because of the very many properties of the diagrams
- it invents and constructs diagrams of different sorts, for different reasons and purposes
- it is not an individualistic study of abstract objects but a social practice on perceivable, material and manipulable inscriptions.

The main purpose of this paper is to demonstrate those features on the chosen example of matrices.

Starting the Process

The following main part of the paper presents a hypothetical learning process based on diagrammatic reasoning, i.e. a kind of thought experiment about what a learner might do, observe, note, memorize, discuss with others. This hypothetical course of actions/reflections can also serve as guidance for somebody (a teacher) who wants to initiate and to keep progressing a process of that kind. As diagrammatic objects I have chosen matrices (of, say, whole or real numbers) for the following reasons: they are discrete objects with a concise (well observable) structure and simple operation rules; though they originate from various problems outside of mathematics, and can be applied in a manifold of ways, matrices can be studied as "pure" diagrams disregarding their referential interpretations (compare Dörfler, 2004); thus working with matrices perfectly exemplifies the above made remarks on diagrammatic reasoning, i.e. this can be taken as a paradigmatic case. Thereby the usual notation of a matrix, i.e. an inscription, is taken as a diagram and not as a representation of whatever other entity. Those matrix-inscriptions and their (rule-governed) transformations are themselves to be investigated. The readers are invited to carry out themselves all the indicated transformations and observations; and thereby to experience the materiality and concreteness of the hypothetical learning process which mainly consists in the realization of structural and operational regularities and invariants (theorems about matrices). The mathematical content in all of that is standard knowledge. The issue in the following rather is how this content is developed/ constructed by the learner and what are its essential features and characteristic aspects from the semiotic point of view taken here.

The following must be considered as a thought-experiment about an imaginary learner L whose potential activities with matrices are described. As any hypothetical learning trajectory (cf. Simon, 1995) it is a theoretical model for possible and potential learning processes. Any such real process will deviate in many places, of course, and the possibility of realization is an empirical question.

The imagined learning process starts in a context where the learner who we call simply L knows what a matrix and its entries are: a rectangular array (inscription) of numbers; and L has already written down quite a few of those matrices which we consider as diagrams according to Peirce. Thereby L might be in upper secondary grades or an undergraduate in the first year. L finds it convenient to view a matrix to be built up either from its single elements, its lines or its columns and can switch between those views which correspond to ways of looking and observing. Possibly at some time L needs to be told how by convention an arbitrary matrix is noted, i.e. as the reader will know in the form $A = (a_{ij})_{mxn}$ or more explicitly as

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

By organizing the writing and reading of matrices she observes the alignment of the numbers (elements of the matrix) either in lines or in columns, a structure which she imposes on the array. Similarly, diagonals and submatrices are for L modes of writing and observing/seeing which she can describe verbally, by gestures and schematically in various ways. Matrices turn thereby into bodily recognizable objects. There again a lot is to be observed in a truly empirical way: how the indices are written, which index varies in a line/column, extreme cases (m=1, n=1 or even m=n=1). Thus our hypothetical learner again meticulously analyzes the matrix-diagram and builds up a reliable (iconic) memory or mental image of it which comprises well all the modes of writing/seeing/reading. L knows now how to write and read an arbitrary line/column and recognizes them as submatrices (with m or n equal to 1). This poses some demands to her abilities of pattern recognition and perceptive parsing which is strongly supported by communicating and talking about her experiences and views with others (learners and/or teachers). This means that a shared practice of working with matrices develops which is strongly based on writing/reading.

An important form of diagrammatic activity is the design of new diagrams like the building (writing) of the transpose of a matrix: write the lines as columns and observe what happens: Where will you then write, say, the old a_{32} in the new matrix? What are the lines of the new matrix? The latter can be observed in the strict sense of the word from the obtained diagram starting from A above:

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{mn} \end{pmatrix}$$

L literally sees: the lines of A^T are the columns of A. This also works in any specific case, of course. Here the previous observations about line index, column index and the memory of them guides the visual scanning carried out by L. This perfectly well matches the cited paragraph by Peirce.

All this and the rule for the elements of A^T our learner L will verbalize and imagine iconically as well, and she will consider extreme cases again (m=1, n=1, m=n, etc): She now is good friend with the diagram-matrices and has a reliable bodily, imaginative "feeling" for matrices due to many writing/reading/observing experiences.

Observing Matrix Products

We will now skip the time when L starts to add matrices and to explore the respective operation rules (element-, line- and column-wise) by diagrammatic reasoning of a similar kind. We return to the process at the time when the product of matrices is proposed by the teacher and she does this in the traditional way (motivated by mathematizing certain economic situations): the (i, j)-element of the product C=AB is obtained by adding the element-wise products of the i-th line of A and the j-th column of B, as a formula

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The reader is urged to make a thought experiment as well by imagining being in the role of the novice learner L who now starts exploring this new diagrammatic operation what very likely is initiated by questions and suggestions on part of the teacher. What can L observe, what can L do with matrices, what can be varied, which recurring relationships can be detected, what can be generalized? A host of diagrammatic reasoning activities is conceivable for being executed by L!

Just by looking how she writes and calculates product matrices our hypothetical learner L will see, for instance, that she needs as many columns in A as lines in B for making AB. Similarly, by inspection of her inscriptions she realizes that AB has as many lines as A (for the columns likewise with B). To repeat, all this is a (reflected) perceptive activity the outcomes of which are described verbally and schematized in various ways. From previous experiences and experiments L knows the value of "extreme cases". Thus she investigates the cases "one line times one column" and "one column times one line" and very likely will be a little surprised by the very different outcomes as for instance

$$\begin{pmatrix} 1 & 2 & 8 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 19 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 40 \\ 0 & 0 & 0 \\ 1 & 2 & 8 \\ 2 & 4 & 16 \end{pmatrix}$$

To obtain especially the latter result L has to keep precisely to the rules. She has experience that to recognize regular relationships often the use of variables helps a lot and thus she writes the following diagram. I leave it to the reader and her observations to speculate what L can see in this diagram and which general conclusions in the vein of Peirce's dictum L will draw. Namely, that what she has observed (for instance, the structure of the lines of the product) will hold in all conceivable cases:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} u & v & w \end{pmatrix} = \begin{pmatrix} au & av & aw \\ bu & bv & bw \\ cu & cv & cw \\ du & dv & dw \end{pmatrix}$$

The manifold success when reflecting on and attentively looking at the products of her writings is a strong motivation for L to continue in this manner even if some trials and manipulations or calculations appear to lead to nowhere. Sometimes guiding questions by others will give new orientation. One such hint makes possible the detection (by inspection of the calculation) that for the first line, say, of AB, only the first line of A is used. L describes this empirically gained insight as a general rule like: line i of AB is line i of A times matrix B (whereby she considers lines as submatrices). The way how she calculates convinces her that this rule ("theorem") is of general validity. This experience is in accordance with Peirce who says:

It is, therefore, a very extraordinary feature of Diagrams that they show, - as literally show as a Percept shows the Perceptual Judgement to be true, - that a consequence does follow, and more marvellous yet, that it would follow under all varieties of circumstances accompanying the premises (Peirce, NEM IV, p. 318).

As comes clear from the Peircean point of view, diagrams have to be invented and designed to serve certain purposes and fulfil specific intentions. This is a creative act which possibly is beyond the reach of the learner. For instance, L wants to express the insight she just has gained for the lines, and then quickly transferred to the columns, in a diagrammatic form, i.e. as a "formula". For this she needs a notation which exhibits, say, the columns of a matrix like the following:

$$B = \left(B_1, B_2, \ldots, B_n\right)$$

If this diagram is available the observed regularity can be expressed as:

$$A(B_1, B_2, \ldots, B_n) = (AB_1, AB_2, \ldots, AB_n)$$

Diagrammatization for L and everybody else is a precondition for mathematical reasoning. The above diagram-formula exhibits the possibility to further analyze the product of matrices by studying the special case of AB where B is a single column.

The reader should pay attention to the pivotal role of the diagrams, their structure and manipulations, and their interpretation. The latter for L is warranted through the close connection between her calculations and diagrams which describe her observational experiences with (other) diagrams. The reader is also invited to speculate how L could write the corresponding formula for the lines of a product AB.

In this hypothetical narrative about L and her experiences with matrices as diagrams we cannot include everything what L possibly could do. Thus, let us imagine that L was asked to write down as explicitly as she wants the product AB for a column B. With the appropriate notation she obtains the following matrix (again a column) or something like it ($b_1,\ldots,b_n$ are the components of B):

$$\begin{pmatrix} a_{11}b_1 & + & a_{12}b_2 & + & a_{13}b_3 & + & \ldots & + & a_{1n}b_n \\ a_{21}b_1 & + & a_{22}b_2 & + & a_{23}b_3 & + & \ldots & + & a_{2n}b_n \\ a_{31}b_1 & + & a_{32}b_2 & + & a_{33}b_3 & + & \ldots & + & a_{3n}b_n \\ \ldots & & \ldots & & \ldots & & \ldots & & \ldots \\ a_{m1}b_1 & + & a_{m2}b_2 & + & a_{m3}b_3 & + & \ldots & + & a_{mn}b_n \end{pmatrix}$$

For L (and very likely for the reader) certain structural features catch the eye when one scans this diagram. It is the regular occurrence of $b_1, b_2, \ldots, b_n$ in a column-wise way. And further, L literally sees that with b_1 stand the elements from the first column of A, and for the other b's likewise. This is a great discovery as the informed reader will know with far-reaching consequences (e.g. for solving systems of linear equations). L can for sure express her discovery verbally but she is aware of the importance and usefulness of diagrammatic expressions. Now she already is versed in adding matrices of all kinds and knows about number times matrix. Thus she can parse or visually slice the above diagram to see that it is – by those conventions – equal to:

$$b_1\begin{pmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{pmatrix} + b_2\begin{pmatrix} a_{12} \\ a_{22} \\ \ldots \\ a_{m2} \end{pmatrix} + \ldots + b_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \ldots \\ a_{mn} \end{pmatrix}$$

Again L will describe her diagrammatic experiences verbally and might even write a more condensed formula. To feel the concrete materiality of those reasoning processes the reader is invited to convince herself of the validity of the corresponding formula for a product “line times matrix”. Of course we could continue for long with the narrative of L's hypothetical, imagined activity with matrices as diagrams but we close with a final but even more instructive case of diagrammatic reasoning by L. She was very impressed by what she has found out about “column times line” and thus has a very good intuitive iconic feeling for implied relationships (this is my fantasy – not empirical data!). Now she looks again at the formula for the elements of AB

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

and recognizes that, say, $a_{i1}b_{1j}$ is the (i,j)-element of the product "first column of A times first line of B". A visual hint for that was the empirical observation of the regularity of the indices 1,2,…,n with which they occur in all such sums for any i and j. And L already knows from experience that such structural features point to "theorems", i.e. general properties of the diagrams and of manipulating them. What L by inspection of the above diagram has found and will after some trials formulate is that AB equals the sum: (column 1 of A times line 1 of B) plus (column 2 of A times line 2 of B) plus… And L is surprised and exhilarated by her discoveries and strongly determined to continue her explorations of mathematical diagrams.

Conclusion

This fantasy-narrative is intended to exemplify the viewpoint taken by Peirce on mathematical reasoning, namely that it is based on observation of the outcomes of manipulating diagrams, interpreting and describing them again by diagrams. My contention is that analogous narratives can be told for other mathematical topics as well starting with natural numbers and number sense. Matrices furnish a particular transparent case since there are no ontological qualms and the diagrams can be viewed in a "pure" way without considering potential referents of the signs (compare Stjernfelt, 2000). All this turns mathematics form a mental and abstract activity into a rather material, external, perceptive and empirical one. In a way it is based on a shift of focus by which the mathematical inscriptions no longer function as signs with referents but are treated as very objects of investigation and exploration. They lose their mediating function, between learners and objects and between sign and objects as well. Mathematics turns thereby into sort of a symbolic/ diagrammatic handicraft preserving all its creativity and inventiveness. And for education a broad consequence would be to organize learning as the progressive participation in a practice of diagrammatic activities in the vein as proposed here.

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DO YOU SEE WHAT I SEE? ISSUES ARISING FROM A LESSON ON EXPRESSING GENERALITY

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Whilst working on a study designed to explore ways in which teachers' use of language contributes to students' expression of generality, I became increasingly aware of the potential of algebra as a tool for expressing mathematical generality, and of the corresponding potential of activity that involves the expression of mathematical generality as an opportunity for students to develop, and practise using, algebraic notation. In this paper I explore some of the issues connected with using algebra to express generality in a secondary mathematics classroom. This brings to prominence the issue of how classroom based experiences and findings that influence my thinking can also be illuminating for others.

INTRODUCTION

This paper focuses on what can be learnt about students' use of algebra to express generality from reflecting upon lesson observations. Through offering a description of a complete lesson, the complexity of the process of expressing generality is revealed. The paper contains a brief description of the wider study, of which the lesson discussed here forms a part. It then discusses some issues involved in describing lessons for others. The lesson in question is then described, with reflection on the interpretive tensions experienced during its writing. I conclude that the role of algebraic notation as a tool for expressing generality is a complex one, and that greater appreciation of this complexity by both teachers and students would contribute to student understanding of algebra. The paper also demonstrates how explicit and shared consideration of methodological concerns can influence and inform research findings.

RESEARCH BACKGROUND AND THEORETICAL FRAMEWORK

The wider study describes ways in which teachers' use of language can affect students' appreciation of generality in mathematics. The importance of students' appreciation of generality has long been understood. When examples are used to illustrate a whole space of possibilities, it is important that students appreciate this general space, and are able to generalise. Krutetskii (1976) observed that the more successful mathematics students in his study were those who could generalise on the basis of analysis of just one phenomenon. Following this argument, it would seem desirable for mathematics classrooms to be places where all students become better able to see the general through the particular (Whitehead, 1932). The discussion of generality is thus a central part of mathematics education, and the role of the teacher in promoting and guiding such discussion is worth serious consideration.

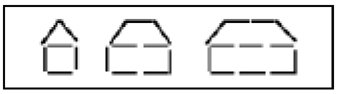
The broader study, of which this paper reports a part, asks what generality is observably present in whole class discussions, and how language (algebraic or otherwise) structures students' awareness of generality. In the lesson described in this paper, teachers work with students on using algebra meaningfully, to express generality.

METHODOLOGY

In striving to answer the questions outlined above, my approach was ethnographic. Having completed the first phase of my observations (where the intention was to minimise intervention, and over fifty lessons were observed and recorded), I offered the teachers in my study the task shown in Figure 1, and asked them to plan and teach a lesson based on it.

Task 3.3.4a Multiple Expressions

Take a picture sequence and imagine building each picture from sticks, perhaps like the one shown here (all sticks are assumed to be the same length).



In how many different ways can you work out how many matchsticks will be needed to make the p^{th} picture? How many sticks will be needed to make just the perimeter?

Figure 1 (Mason, 2005: 56)

The intention was to observe 'normal' classrooms where there was potential (as a consequence of the task on offer) for algebra to be used meaningfully to express generality. The transcripts of the resulting lessons were analysed using a grounded theory approach (Strauss and Corbin, 1990), with some techniques from discourse analysis (Parker, 1992).

It is the intention of my research design that my study benefits from my own total submersion in the data, presence in lessons, and transcription. The process of analysis of the transcripts of classroom discussion is influenced by my experiences in the classroom as an observer. The conclusions reached through this analysis are intended to be directly attributable to, and justifiable by recourse to, the data, so that someone else could understand the conclusions. Naturally, that person may not have reached the same conclusions themselves. It is the customary for readers, perhaps especially those who are practising teachers, to compare the data with their own experiences and to consider aspects not intended by the researcher when collecting or describing the data.

It was these dual aims of wanting my analysis to benefit from my immersion in the field, whilst being able to share my findings as fully as possible, that led to my desire to write 'thick' descriptions (Geertz, 1973 and earlier Ryle, 1949) of the lessons I observed. A description of human behaviour is considered to be 'thick' if it explains not just the behaviour, but its context as well, such that the behaviour becomes meaningful to an outsider. Morrison (1993: 88) argues that by "being immersed in a particular context over time not only will the salient features of the situation emerge and present themselves but a more holistic view is gathered of the interrelationships of factors". Such immersion facilitates the generation of descriptions which lend themselves to richer insights and interpretation of events. Of course, as such salience is relative to the observer, thick descriptions can be problematic.

It is important to acknowledge that, when creating 'thick' descriptions of classrooms, researchers must go some way towards analysis. Even before the start of the lesson, my interpretation of the lesson discussed in this paper may have begun. My reading of relevant research literature, my teaching experience and my research design contributed to the way in which I read the teacher's lesson plan, and doubtless affected my expectations of the lessons. These implicit expectations resulted in certain student actions being particularly noteworthy (because they illustrated or conflicted with my expectations). Although, as an ethnographer, I aimed to be open to all possibilities, I acknowledge the impossibility of eliminating all conjecture and expectation. Better, I believe, to work hard at being open minded, whilst being aware of possible structures that I may be imposing on my experiences. As Gadamer (1976) argues, prejudice is the necessary starting point of our understanding. The critical task lies in distinguishing between "true prejudices, by which we understand, from the false ones by which we misunderstand" (Gadamer, 1976: 124). After the lesson, interpretation again comes into play. Even if I were to record my experiences fully, in order to share my data with others I would need to condense it. Before the experiences can be analysed they must be rendered analysable – tapes transcribed, experiences noted in journals and written up in detail. Choices must be made about when the exact words used are important, and must be quoted, and when the speaker can be paraphrased. Paraphrasing often makes the lesson easier for readers to access than if it had been written as an annotated transcript. Interpretation is involved again; the way I choose to structure my data affects subsequent analysis.

Describing the lesson in such detail enables the readers to make sense of the data for themselves. It may also persuade the reader of the veracity of the researcher's explicit and implicit conclusions.

THE LESSON

In the process of writing the lesson description that follows, and reflecting on interpretive choices (these interpretive reflections appear in italics in this section), one possible unifying theme emerged from the data, which provides me with a possible lens with which to view the complexities of using algebra to express generality. In line with my desire for readers to have the opportunity to interpret and reflect on the lesson in accordance with their current thinking, I restrict my analysis here to separate incidents, rather than explicitly draw this connecting thread between them. Readers who prefer to read the data with a preconception of my interpretation *will find my discussion of how it influenced my own thinking in the conclusion section.*

As students entered the room at the start of the lesson, the tables were arranged for groups of 4 students so that most chairs were sideways to the board. There was an A3 sheet of questions on each table. The students were told that they could choose who to sit with, although the teacher intervened in deciding that a 'fifth' person on one table should move to join a table with only three students.

This teaching decision suggests that the teacher prioritises discussion and students' expression of mathematical ideas. Group work offers its own opportunities and challenges, the exploration of which is beyond the scope of this paper.

The register was taken, with students asked to respond to their name with the counting numbers (first student responding 'one', second student 'two', third student 'three', etc. They were also asked to give special answers according to two criteria. If their name's position in the register was in the sequence $2n$, they were to reply 'buzz'. If their 'number' appeared in the sequence $3n + 1$, they replied 'fizz'. The answers of the first 8 students in the register are shown below, for illustration:

Student	1	2	3	4	5	6	7	8
Response	'1'	'buzz'	'3'	'fizz buzz'	5	'buzz'	'fizz'	'buzz'

They then discussed how frequently a student would be both of these things. Reference was made by students to multiples of six.

In this segment of class discussion, the teacher speaks to and with the whole class about a selected phenomenon, and leads individual students to explore it verbally with the rest of the class. Segments such as these leave both teacher and observer unsure about the consequences of this 'discussion' for the majority of students who merely listen. Does one individual student's expression of a generality support other students' understanding? If a student does not follow the reasoning expressed by a fellow student, it is possible that this might put them at an affective disadvantage for the remainder of the lesson.

The temptation in writing about the register activity is to share the full transcript of the discussion, so that the reader might have their attention directed towards particular areas of tension and possibility within it. Constraints of space necessarily restrict 'thick' descriptions, and in this case I believe exploration of this issue deserves separate attention elsewhere. I am also confident that the reader will be able to draw on their own experiences of such discussions. Such self-illustration might not be so easy for later issues, which are consequently described in more detail.

The teacher then introduced the main theme of the lesson.

Teacher: We're going to be looking at sequences today, ok, and in particular we're going to be looking at picture sequences. It's easy enough for most of us, especially if you've had somebody tell you how to do it, to work out *n*th terms for things and work out what sequences should look like. What's a much harder skill, that we love to develop in you, is to be able to look at pictures, and try to ascertain sequences from the pictures.

Students were invited to work in their groups, and given around ten minutes, subject to what the teacher could see and hear happening in the groups. The teacher told them that he was very interested in what ideas they would come up with:

Teacher: I'd love you to feed back your ideas to me as I'm coming round as well, so, maybe, you know, as I'm approaching your table you can speak up that much louder or say one of your very good ideas, as a group, ok.

This teacher comment can be interpreted as contributing to a conjecturing atmosphere in which students' ideas are considered to be worth listening to. The comment could also be interpreted as a behaviour management technique, with students not being trusted to talk about the task in their groups unless they thought they were to be 'checked up on' by the teacher. It seems feasible that students, and the teacher, believe that the way they express their mathematical ideas to each other in groups should be different from the way they would express them to a teacher. It is possible that, for some students, this teacher comment leads to their attempting to clarify or formalise their expressions in preparation for explaining them to the teacher.

Most groups quickly began lively discussion, which generally appeared to be related to the task. One student in each group seemed to take on a 'chairing' role, reading out the questions and writing down the answers. The layout of the tables in squares appeared to make it difficult for some of the other group members to see.

The questions on the sheet were as follows:

1. How many matchsticks are needed for each diagram?
2. Which matchsticks are added on each time?
3. How many matchsticks are needed for the 50^{th} diagram?
4. What is the rule for finding how many matchsticks are needed for the n^{th} diagram?

With the permission of one group, I left a recorder on their table. When listening to their discussion later, while they were deciding which matchsticks were 'added on each time', I was struck by how limited their descriptions were.

Student 1: It's these
Student 2: Just these, yes.
Student 3: I think it's that one, that one and that one. Those three.
Student 4: I think its,
Student 1: No, I want that one.
Student 3: It's that one, that one and that one.

The students seem to be relying on pointing to matchsticks, rather than describing them verbally. Given that some group members were finding it difficult to see the sheet, this seems restrictive and prone to misunderstanding. There was a contrast here with the teachers' discussion during the INSET session, where they used expressions such as 'the walls', 'the roof' to make clear what they meant. This made me question the extent to which unambiguous communication was valued by the students. Perhaps they felt they benefited from 'hedging' their suggestions in this way (Rowland, 2005), leaving them less exposed to criticism by their peers.

After fifteen minutes of group work, several students came up to the board to describe their 'way of seeing'.

In the last five minutes, the teacher invited the class to consider four expressions (taken from Mason, 2005: 57) that could be used to describe the number of matches in the *n*th pattern:

$$2 + (p - 1) + (1 + 2p + 1); \qquad 2 + (3p - 1) + 2; \qquad 4 + 3(p - 1) + 2; \qquad 3(p + 1)$$

In their groups, they were asked to choose an expression and think about what 'way of seeing' it related to. Some groups drew the '*n*th' house in stages using longer lines as groups of *n* matches, and shorter lines as individual match sticks.

One group considered $2(2 + n) + (n - 1)$. Their work is shown in figure 2 below.

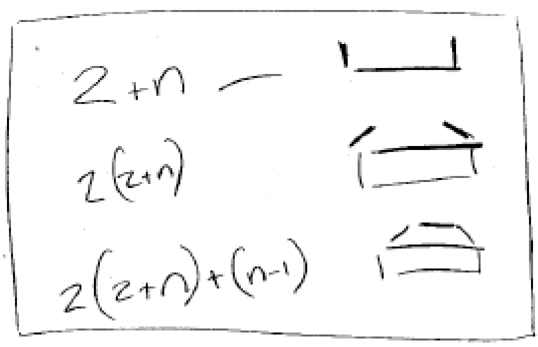


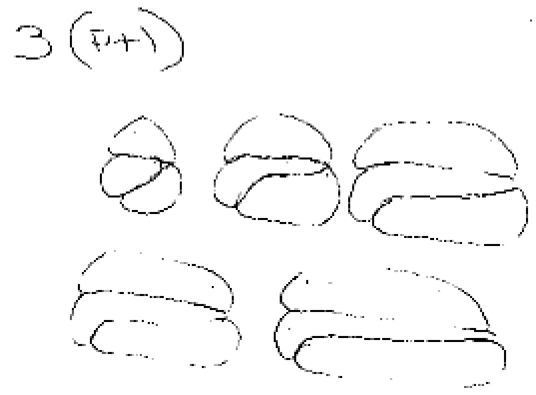
Figure 2

The first diagram, $2 + n$, shows a line indicating n matches along the base of the house, with a single match on each side. In the second diagram, the expression is doubled, and written as $2(2 + n)$, and the diagram shows this doubling, with the addition of another line of n matches along the base of the roof, with a single match on each side. The third diagram shows the addition of $(n - 1)$ to the expression, with the roof drawn on.

The students' diagram does not seem to provide a convincing explanation for why the total length of the roof is $n + 1$. The line at the top of the roof (which has a length of $n - 1$) is of ambiguous length in the diagram, only slightly longer than the single match and about half the length of the line used to denote *n* matches. This may be because this had already been discussed as a class, and was taken as a given.

When describing students' discussion, or written work, there is an interpretive tension concerning the extent to which it is desirable to use language that the students did not use themselves. To do so might give the reader the impression that the students' appreciation or use of algebraic notation is more accomplished than they offered evidence for in the observed lesson. It is important to clearly distinguish *between what the students actually wrote or said, and my own interpretation of the possible underlying generalisation.*

The second way of directing attention towards the general rule, or locating the *n*, proved even more popular: looping the sections corresponding to the variable. Asked to justify others' rules, many groups worked on $3(p+1)$. One group's work on this expression is shown in Figure 3. Thus, the first diagram has 3 groups of 2 looped, the second has 3 groups of 3, the third 3 groups of 4 and the fourth 3 groups of 5.

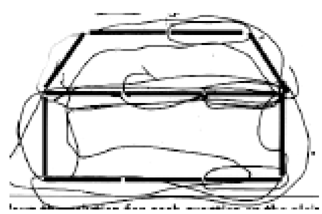


The difficulty here is giving the impression that all communication was successful. There is a tension in trying to give an accurate description of all that happened without offering every student's work in full. Arguably, if the description is intended to show what is possible, it is acceptable to focus on the positive. This may, however, lead to readers developing unfeasible expectations that may lead to feelings of disillusionment, if they try the task with students themselves.

Figure 3

The 'looping' technique became confusing when students chose to use the same set of diagrams for all their general rules as shown on the right, below. This illustrates the dual, potentially conflicting, roles of language (in this case the language of 'looping') as both a way for individuals to work something out for themselves, and a tool for the communication of their findings to others.

The inclusion of this diagram here results in the reader experiencing 50% successful and 50% unsuccessful examples of looping. A judgement needs to be made at this stage about students' understanding, so that the description allows the reader to have some idea of how representative is each piece of work offered.



The looping technique enabled students to express and share their general understandings without the algebra, as shown by the work on the right, below.

The following two paragraphs offer alternative descriptions of what is shown by the students' work in figure 4. In the first paragraph, I set out to describe only what could have been observed by the students. In the second paragraph, I intend to describe the work as clearly as possible to a reader with some background in algebra, and to indicate the generalities observable in the work.

Description 1

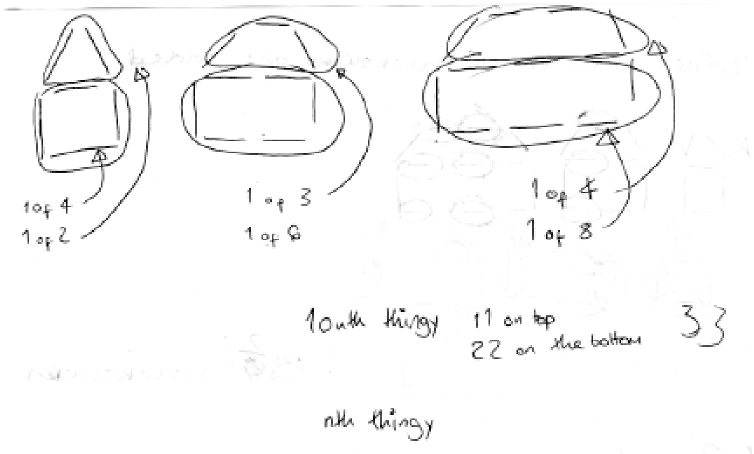


Figure 4

The students' work shows the first three houses with the roof and the main body of the house separately circled. They have recorded the number of matchsticks in each looped section. Underneath this it is written that the '10th thingy' would have '11 on top' and '22 on the bottom'. There is a large '33' that appears to be their calculation of the total number of matchsticks used in the 10th diagram.

Description 2

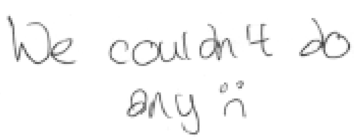
In the students' work, three diagrams have been drawn, resembling the 1st, 2nd and 3rd diagrams (i.e. where $n = 1$, 2 and 3). The students here have indicated that the roof of the house consists of $n + 1$ matches, and the body of the house consists of twice this amount. They have circled the roof section, and written the corresponding value of $n + 1$, and circled the body of the house, writing the corresponding value of $2(n + 1)$.

Again, in describing the students' work, my attempt to create an 'account of' suffers from interpretative tensions. My desire to offer an adequate description of the lesson leaves me tempted merely to offer the students' work with no explanation. As a reader, however, I know I gain more from the data by reading a description, or even analysis, of it, as it provides a structure around which I can build my own 'story'. As long as I remain aware of the distinction between the two, there is a role for both 'account of' and 'accounting for' here.

The objective of the students' work is unclear during this section. As they have written 'n^{th} thingy', it seems that at least part of the aim is to 'write it as algebra'. What is less clear is what the 'it' might be in this situation. It is possible that, rather than the algebra providing a useful tool for the students to communicate 'what is happening in general', the goal of communicating has been superseded by the desire for correct use of algebraic notation, as a goal in itself.

$$2 + (p - 1) + (1 + 2p + 1); \qquad 2 + (3p - 1) + 2; \qquad 4 + 3(p - 1) + 2; \qquad 3(p + 1)$$

Not all groups were successful in explaining the different 'ways of seeing' that had resulted in the four expressions. A student in one group wrote the comment on the right.



The choice to include the student note above in my description illustrates that the tools offered for understanding the generality had not been successful for all students. It also enables me to point out that this was unusual – this was the least expressive and all other students achieved more.

I felt that the students were confident and honest, working in a conjecturing atmosphere. The apparently willing acknowledgement of not being able to 'do any' suggests that the students did not feel constrained by pressure to succeed (Boaler, 1997). It is also interesting to reflect upon what it was that the student who wrote this believed they were trying to 'do'. It is quite possible that these students were aware of (and, perhaps, informally expressing to each other) patterns in these four expressions but took the view that their solutions were not what was required by the teacher.

CONCLUSION

A key noticing that appears to connect all the lesson segments described in the previous section, was that of *why* the generalities were being expressed, and *for whom*. The role of language shifts during an activity, as the participant moves between talking for themselves and talking for others. It is important that teachers and students are aware of the purpose of their language use – on a spectrum between personal exploration and communication – and the relative merits of the different available registers (ordinary language, the mathematics register, algebraic notation, looping) to fulfil this objective. This need for students to be aware of the purpose of the activity they are carrying out has been previously emphasised (e.g. Pimm, 1987: 34; Ainley, 1996: 406-7), and seems to hold a particular significance when considering the development of algebraic notation as a tool for expressing generality. Experienced mathematicians are aware of the advantages of using algebraic notation, in appropriate situations, both for developing new ideas, and for communicating those that have already surfaced. The wider study seeks to explore what teachers do to make students aware of algebra's potential and use as a language for expressing generality.

This paper has also been concerned with the numerous interpretative tensions involved in writing 'thick' descriptions of lesson observations. I have shown how research might benefit from increased openness regarding the interpretations and choices made in data descriptions. I note tensions in describing a lesson vividly and convincingly without conveying value judgments about the effectiveness of the teaching and learning. I also question the extent to which it is appropriate or justifiable to use language to describe student work that the students might not themselves have used. The answers to such questions will vary according to the purpose of the research. By being open about the tensions I have experienced in describing observed lessons, I found myself making more considered and informed choices. Wolcott (1990) argues that 'writing is thinking', and points out the value of writing during the process of qualitative research. I would go further than this, and argue that writing about decisions made while writing is also of significant value. In this study, data description and data analysis were intrinsically linked. The act of carefully considering what to describe from the lesson, and how, and the resolution of tensions between telling 'a lot about a little' or 'a little about a lot', *became* a process of analysis through which a meaning emerged from the lesson which contributed to my research questions.

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SIGNS USED AS ALGEBRAIC TOOLS – A CASE STUDY

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In mathematics signs play an important role both for gaining new knowledge and for communicating knowledge. In an empirical study students were asked to write short essays on the mathematical structure of the set of cosets of 5Z. Some essays displayed inventive ways of sign usage. One of these essays is discussed in this paper. The focus will be on the way the student uses signs for representing and communicating her ideas of Z/5Z. The analysis of the essay reveals hints that in the process of communicating her ideas these ideas are further shaped.

INTRODUCTION: FOCUS OF THE PAPER

How are abstract mathematical objects "learned", i.e. mentally constructed or reconstructed by mathematical novices? Mathematical signs provide useful tools in this endeavour. This paper considers an example of a student struggling with the problem of adding sets of numbers as elements of Z/5Z. The focus is on the way she uses signs to present this addition in a short essay. An analysis of this essay discusses hypotheses on possible meanings she has in mind for the signs she uses.

Sáenz-Ludlow & Presmeg (2006) ask if learners can be challenged to use signs in creative ways as tools to express their personal mental ideas of mathematical conceptions. The presented essay will be an example where a student does so.

THEORETICAL FRAMEWORK

Research on the epistemology of mathematical concepts both in the historical development and in the cognitive (re)construction of students many times is concerned with the use and meaning of representations. Cobb et al. (1997) propose the use of signs – either conventional or unconventional – as a didactical means to support the emergence of mathematical meaning with learners. They argue for an invention of signs which can serve two intentions: First, such a sign can be used as a model of a starting situation that is familiar to students. Second, as the learning process progresses, the sign can serve as a model for mathematical activity or mathematical objects on a higher level.

The special role signs play in mathematics is partly due to some characteristics of mathematical objects. Sfard (2000) holds the view that mathematical objects have no "actual reality" but are of purely "virtual reality". Yet they are treated by experts as if they were of actual reality. Sfard claims that the only possible access to a mathematical object is via representing signs which themselves are not identical with the object. She describes the following as one way for learners to mentally construct mathematical objects. They may start with signs introduced by teachers, regarding these signs as mathematical objects. The signs are treated according to certain given rules. After some time of using these signs learners will eventually change their viewpoint and no longer regard these signs as objects themselves but as representations of virtual objects. This change is marked by a different behaviour and attitude. A hint for this change would be when a learner refers to two representations of the same mathematical object as the same. This happens when, for example, the graph of a function and the equation of the function are regarded as representing the same object, and statements on one of them are considered to correspond to statements on the other.

Steinbring (2005), too, emphasises specific characteristics of mathematical objects. He claims that mathematical objects always are relations between other (mathematical or non mathematical) objects. He describes the way in which learners (re)construct a mathematical concept as follows. Signs that will later be representing the concept are used in contexts familiar to the learners. Within these contexts, the learners will develop the abstract relation the concept stands for. One difference between Steinbring and Sfard is the way they interpret the original meaning of signs for learners: Whereas Sfard assumes that the signs are treated as objects themselves with no referential meaning and meaning is developed by the way the signs are used, Steinbring postulates that signs are always interpreted with referential meanings, but the meanings are changed during the learning process.

My own conviction is that learners or researchers in mathematics do not always use signs with referential meaning. In particular, I agree with Sfard that new abstract objects may be mentally constructed during a phase of getting used to certain rules on certain signs. However, I am also convinced that abstract mathematical objects can be constructed by using signs referentially within concrete contexts and changing referents in a process of abstracting general features from different contexts. Which path a learner will take may depend on his or her personal disposition as well as on the learning environment.

In the second part of this paper the use of signs as means for constructing an abstract mathematical object in a student's essay will be discussed, both in respect to meanings given to the used signs and to operations used on these signs. The analysis will neither postulate the position of Sfard nor of Steinbring but be opened both to sign usage purely according to given rules and to signs used as referring to mathematical objects.

RESEARCH METHODOLOGY

Research plan

The aim of the research study was to find out about students' strategies of constructing and representing the abstract mathematical objects "residue classes". A sub point contributing to this concern seemed to be the way students use signs. In this paper I will discuss the use of formal mathematical signs in a student's essay with the research question:

"How does the student use signs as tools for communicating and – possibly – thereby refining her ideas of residue classes?"

The concept of the study was to construct a learning environment in which students would experience different characteristics of residue classes. After this a writing task was planed in which they were to elaborate on their ideas of residue classes.

Proceedings

In their first weeks at university students of a linear algebra course worked in small discussion groups, in class, on problems leading to the construction of the group (Z/nZ,+). When needed they could ask the teacher for help. After each set of problems there was a presentation and discussion of the work done in the small groups. The first set of problems asked them to discuss the role of remainders in respect to division by 2 or by 3 or by another given natural number n when two natural numbers are added. In the next set of problems they had to consider operations on the set of the first n natural numbers conforming to their previous observations. In the third set of problems the whole numbers on the number line were coloured with four different colours, where each fourth number got the same colour. The numbers of the same colour were combined in one set and an addition was defined on these sets. In the last set of problems they were asked to discuss a formal definition of cosets of 4Z – defined as "classes of congruencies modulo n" – and the group (Z/4Z,+). They also had to compare the three different ways of constructing a cyclic group of n elements. In class, different types of signs for denoting residue classes were used. These were signs like $\bar{1}$, 1+4Z, and in the third set of problems these were the names of colours and descriptions like $\{4z+1 | z \in Z\}$.

After about four weeks the students had worked through these problems. Until now no data had been collected. In the following lesson the students were given the writing task for the empirical study. In this investigation 39 students in two classes took part. They were asked to take 20-25 minutes time in class for writing the essays and they delivered answers of a length between a few signs and two handwritten pages. They were given the task:

> The following table is meant to illustrate the set of classes of congruencies modulo five. Explain in a short essay. Consider:
>
> - elements
> - addition
> - connection with Z

⋮	⋮		⋮	
		⋮		⋮
⋮	⋮		-2	
		…		-1
0	1		3	
		2		4
5	6		8	
		7		9
1	1		1	
0	1	12	3	…
⋮	⋮	⋮	⋮	⋮

Before we discuss one of the essays let us have a look at the task and some possible answers. The general ideas of these answers are found in various ways in the responses of the students.

Discussion of solutions

The task asks for an interpretation of the table as representing Z/5Z, the set of residue classes modulo five [1]. We find the five elements of this set represented by the columns. In numerating the elements of the residue classes, the table emphasises the perception that the elements of Z/5Z are sets, not numbers. A solution of the task could be: The columns represent the five elements of Z/5Z. Each of these elements is a subset of Z. Addition of two columns (i.e. residue classes) is done by the following procedure: Choose an arbitrary number in each of the two columns. Add these two numbers. Find the column in which the sum of the two numbers lies. This column is the sum of the two original columns. The resulting column is independent of the choice of the two numbers, because they are unique except for multiples of five and adding a multiple of five to any number will not change the column.

If a student has not yet mentally constructed the abstract idea of sets being mathematical objects that can be elements of a set and can be added, he or she may find an interpretation of the table by focussing on the remainders: The table arranges all numbers with a common remainder modulo five in one column. This column represents the remainder (named in the third line of the table). We add the remainders and calculate the remainder of the sum.

A third interpretation may focus on the whole numbers: The table represents the set Z of the whole numbers. They are arranged in the following way: In each of the five columns we find the numbers with a common remainder when divided by five. If you add two numbers their sum will always be in the same column as the sum of their remainders.

The task to interpret the table and relate it to Z/5Z asks for assignment of meaning of the components of the table. Thus, it does not encourage a treatment of signs without referential meaning. However, answers may take this viewpoint in illustrating the rules of addition in Z/5Z with some signs without committing themselves to referents, i.e. without a definite decision if the signs used for this illustration refer to numbers, remainders, subsets of Z, or if they are mathematical objects in their own right without referent.

Among the 39 essays there were only 9 interpreting the table as an illustration of Z/5Z as a set of residue classes. Most essays were not clear on the referents of their signs or were concentrating on the whole numbers and residues or residue classes as instruments of classifying whole numbers. Some of them seemed to be on the verge of viewing residue classes as abstract mathematical objects. These are of special interest in respect to the research question, because the essays may give some hints on the cognitive conflicts connected with this change of perception and attempts to overcome them. One of these is Maria's [2] essay.

MARIA'S ESSAY

Maria wrote the following [3]:

> "The set of classes of congruencies modulo five (Z/5Z) can be subdivided in 5 classes. In the following way all elements of Z can be assigned to these classes:
>
> Group 1: 5x
>
> Group 2: 5x+1
>
> Group 3: 5x+2
>
> Group 4: 5x+3
>
> Group 5: 5x+4
>
> By this arrangement all elements of Z are included and grouped.
>
> The classes of congruencies can also be composed. In the addition of two elements of (Z/5Z) you add only the "remainders" and assign the new remainder to one of the groups. Example: (5x+2)+(5x+3)=5x+5=5x, therefore the result of the addition of elements of group 2 and 3 is in group 1."

An analysis of the essay

In her first sentence Maria treats Z/5Z in a way appropriate for Z. Does she identify the two sets? In her following subdivision she obviously makes a difference between the two: The residue classes each contain certain whole numbers, which she refers to by formal notation. I suppose that "5x" in her table is meant as a description of the multiples of five. She does not define "x". It may stand for one whole number or for any whole number. Since she introduces her subdivision of Z by saying that <u>all</u> elements of Z are put in one of the classes, it is more likely that she is thinking of x as any whole number [4]. In this case the numbers in her five "groups" are identical with the five columns of the given table. It should be remarked that the five subsets which Maria calls "groups" are not numbered according to the remainder of their elements (modulo five), but from 1 to 5.

Now Maria announces the composition of two residue classes. She then explains how this addition can be carried out: Only the remainders are added and the remainder of their sum is assigned to one of the “groups”. She puts the word “remainder” in quotation marks. This may indicate: she does not regard residue classes as numbers and wants to remark that they do not have remainders in the strict sense. Besides the numbering of the “groups” this is a second hint that she does not identify the residue classes with remainders.

Maria demonstrates the addition with an example: She adds the formal terms “5x+2” and “5x+3” which she has used before for description of the numbers in “groups” 3 and 4 respectively. Why does she use these signs? Does she refer to something by this notation, and to what?

(a) If Maria means x to be any whole number as I have supposed before, then 5x+3 denotes the number following 5x+2. She reduces this sum: (5x+2)+(5x+3)=5x+5=5x in a surprising way. According to conventional use of these signs we would have (5x+2)+(5x+3)=10x+5 and 5x+5=5(x+1). It is not likely that Maria thinks that she conforms to the common rules, i.e. that 5x+5 is equal to 5x in usual terms. Rather, there seems to be a scheme leading her to make these “mistakes”: She wants to express some idea with this notation concerning something different from whole numbers.

(b) Perhaps this idea is not specified enough for her to be clear about the meaning of her signs. It may be that she simply wants to demonstrate the rules valid in this setting concerning “remainders”.

(c) If “5x” is meant for “a multiple of five” then “5x+2” would be “two more than a multiple of five” and “5x+3” would be “three more than a multiple of five”. Then the equation can be translated: “The sum of a number that is two greater than a multiple of five and a number that is three greater than a multiple of five is a number that is five greater than a multiple of five, and this is a multiple of five.”

(d) If we let Maria’s explanation on addition guide our interpretation, we find that she means to add residue classes. Then the first summand “5x+2” denotes what she has named “group 3”. She says that only the remainders are added, and this is what she does: She adds 2 and 3 – the numbers carrying the characteristic information – and copies the term “5x”. In the next step she assigns one of her “groups” to the result: 2+3=5 has the remainder 0, so the resulting group is “5x”.

Maria’s final conclusion is concerned with the addition of numbers: If you add a number of “group” 2 and one of “group” 3 [5], then you get a number of “group” 1. In this interpretation of her own formal notation she no longer speaks of addition of residue classes, but of numbers. This again implies that her summands are meant as numbers like in the interpretations (a) and (c).

It seems that an interpretation that complies with the complete essay would have to take into account a change in meaning, possibly also an unspecified use according to the rules of addition valid within the context “remainders”. The following table gives an overview of possible referential or nonreferential use of “5x+2” in the course of her writing. The different referents are labelled with R1, R2, and R3, nonreferential use with nR:

	Activities of Maria:	Two possible developments of meanings of 5x+2:	
(i)	Writing down the table, which is giving account of the arrangement of the whole numbers in five subdivisions	R1 any or all whole numbers with remainder 2	R1 any or all whole numbers with remainder 2
(ii)	Interpreting the table, in which 5x+2 appears quasi as a formal term for the “group” 3	R2 the residue class of “group” 3	R3 or nR something unspecified that is associated with remainder 2 or is to be treated as remainder 2
(iii)	Starting with the example for addition of residue classes	R2 the residue class of “group” 3	R2 the residue class of “group” 3
(iv)	Manipulating the formal terms, ignoring conventional usage for number variables	R2 the residue class of “group” 3	R3 or nR anything unspecified that is associated with remainder 2 or something that is to be treated as remainder 2
(v)	Interpreting the result of the manipulation	R1 any whole number with remainder 2	R1 any whole number with remainder 2

Maria assigns meanings to her signs by announcing her intentions. But we see that the assignments are not consistent. Eventually, she either acts according to different meanings or according to rules associated with her signs without thinking of objects her signs refer to. We have at least two changes in the meaning of “5x+2”: from R1 to R2 and back. They may be induced by associations with the configuration of the written signs. In both cases the missing of a declaration of the meaning of this algebraic term makes a reinterpretation possible.

According to the interpretation given by the second column of the table, Maria's behaviour conforms to the Steinbring's postulation of signs always having referential meaning. If we follow the interpretation in the third column of the table, Maria's operating with the term "5x+2" in step (iv) signals a sign usage, where she does not consider ontological characteristics of the object "5x+2" or its referents but where she simply follows the rules valid in an environment where only the remainders of numbers are of interest. This would be a sign usage as described by Sfard (2000). However, different from Sfard's description, this sign usage does not mark the beginning of Maria's considerations. Instead, she is slipping into this attitude in between attempts of assigning referential meaning to "5x+2".

Maria's cognitive representation seems to be a step ahead of her ability to communicate it. We see that she has an understanding of residue classes that goes beyond the idea of classifying whole numbers. She is trying to demonstrate an addition defined on residue classes. She is acquainted with all components necessary for carrying out the process of addition: Picking elements of each summand, adding only the remainders, choosing the correct residue class. Her personal, unconventional way of sign usage offers inventive opportunities to express her ideas of character and treatment of residue classes. Thus she is able to make use of the general idea that the treatment of the classes is derived from the rules of operating with the numbers. The same is true for the general idea of the class being a way of identifying and representing numbers with a common remainder. The problem Maria is facing is how to state – or grasp – her ideas more precisely. Instead of naming the residue classes by signs, distinguishing them from numbers, she uses each sign for several alternative meanings: the class as set of all numbers fitting the description, one particular number fitting the description, and any number fitting the description. She does not discuss questions like: What names do I choose for my different objects? What objects do my different notations refer to? And, as a consequence, what rules for operating with these signs are valid? Maria's use of formal signs gives her a great flexibility. She is actually able to denote the addition of two classes. However, in the end, Maria is trapped by her inconsistency in reference of her signs: In her final interpretation of her equations she does not hold on to her achievement but goes back to the idea of having added numbers. Yet perhaps this (probably unconscious) strategy of changing interpretations of signs – taking steps towards new abstract objects and retreating to the safer ground of familiar objects – is a helpful strategy in collecting experiences with the new objects, getting used to them, and constructing mental representations of them. Similar strategies are displayed in the concept of the "chain of signification" in Cobb et al. (1997). For Maria a sign like "5x+2" has double (or even multiple) functions. On the one hand it can be described as a model of the numbers with common remainder two (modulo five), as it concentrates on their common characteristics both in numerical features and in the way they are operated upon. On the other hand it is a model for the residue class of two, as it is used to subsume the idea of combining the all numbers with remainder two and to treat this consumption as an object of calculating. Thus it serves as a step in between two levels of abstraction.

CONCLUSION

The example presented in this paper shows a flexible and creative way in which a student uses formal mathematical signs for communicating and possibly constructing or advancing her mathematical ideas. In spite of her mistakes, her way of dealing with the subject shows a great deal of consideration and meaningful action that seems to be helpful in the process of mentally constructing the abstract objects. How can teaching support students to use signs in such a creative way? Further investigations on students' potentials of using signs as tools for representing and developing their ideas should be done. These could help us to design applicable learning environments.

NOTES

1. In class these three names "Z/nZ", "set of congruencies classes modulo n", and "set of residue classes" had been used synonymously.
2. The name has been changed.
3. This is a translation of the original essay, which is written in German. It is published in Fischer (2006).
4. She may also think of x as a variable running through the whole numbers subsequently. Then, too, 5x covers all multiples of five.
5. She makes a mistake in the numbers of these groups.

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PROBLEMS OF A LINEAR KIND: FROM VALLEJO TO PEACOCK[1]

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This work is concerned with epistemological conceptions linked to a historical and pedagogical alternatives and approximations in the evolution of teaching of algebra, with respect to the solving of linear problems through algebraic method.

Same of these alternatives are epitomized into two books published at about the same time: Vallejo's Treatise (1812-1841) and Peacock's Arithmetical Algebra (1830-1842). We analyze the proposal for teaching these problems reflected in the first book (Vallejo's), taking into account the historical and cultural background in which it appeared, and then, we contrast this with the proposal for teaching these problems that appears in the second (Peacock's).

Such analyses shed light on pedagogical approaches, which helps us understand the historical roots of the organization of elementary school teaching of algebra, with relation to this topic.

AIM AND METHOD

For a very long time mathematicians have solved a great variety of problems whose common unifying nexus is the theme of linearity. Also, secondary school students advancing from arithmetic to algebra find a wide variety of problems of this kind, with differing teaching focuses, and alternative algebraic and arithmetic methods for solving them.

It is believed that the solution of these problems is elementary and has already been dealt with. However, when studied in depth, it can be seen that in the same way that there is not only one way of solving a linear problem, neither is there only one way of teaching it. The idea of linearity can be treated from different perspectives: through proportionality, through first-degree equations, and through linear function[i]. These three foci correspond to epistemological conceptions linked to a variety of historical and pedagogical alternatives and approximations that reveal the wealth of mathematical thought lying behind such an apparently simple topic.

The historical change from one conception to another caused changes with repercussions in teaching them. One of these occurred on substituting arithmetic methods for algebraic ones to teach and solve linear problems. Another had to do

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with the way of organizing these in textbooks. These changes in focus, in textbooks in the first half of the 19th century, are epitomized in the treatises of Vallejo and Peacock. The former is a key piece of work, due to its influence, for studying the incidence of these changes in mathematical teaching in Spain, and represents a traditional textbook. The latter represents a forefront textbook, an Arithmetic Algebra, this being a bridge between arithmetic and symbolic algebra.

The main goal of this research was to analyze the pedagogical approaches for teaching and solving linear problems, as reflected in Vallejo's Treatise (1821, third edition). For this analysis two main features have been considered: a) the conceptions reflected in Vallejo's text with respect to mathematical concepts linked to algebra, and the way of classifying/organizing/solving the problems of first degree equations; b) its relation to the historical and cultural background in which it appeared, and the proposal for teaching the first degree equations problems that appear in Peacock's *Arithmetic Algebra (1845).*

THE TREATISES OF VALLEJO AND PEACOCK IN THEIR HISTORICAL CONTEXT

At the moment in history when the Treatises of Vallejo and Peacock appeared, an algebra concentrating on finding solutions to equations was giving way to an algebra aimed at studying the conditions necessary for these solutions to exist. Arithmetic began to take on the kinds of approach typical modern or structural of algebra, such as the horizontal language of equalities and parentheses. At the same time the school system began to be reorganized itself into a general, centralized, state system of education.

The *Elementary Treatise of Mathematics* by José Mariano Vallejo (1779 - 1846) was first published in 1813. Written in 5 volumes, it underwent four re-editions, the last being published in 1841 after Vallejo's travels round Europe, for political reasons, which served to give him knowledge with which to broaden and perfect his work. In this period, interest in the study of mathematics in Spain was found not so much in civil society as in military institutions. Although the mathematicians who gave classes in them did not bring anything new to mathematics in itself, one could say that they did add to the teaching of it in that they updated mathematics textbooks, thus renewing the Spanish mathematics bibliography. These authors produced encyclopedic works in which they presented mathematical advances of the 18th century which had not yet been published or widespread. They did this with two aims in mind: to give a general view of mathematics, and to be used as textbooks for official institutions. In these works they synthesized and adapted previous prestigious authors, taking care to produce a strict, intrinsically coherent presentation of ideas that, together with a good job done in reproducing them, made them invaluable and popular.

Vallejo's text on algebra appeared at a transitional moment in the history of 19th century mathematics when steps were being taken towards a general theory of algebra, or an abstract algebra, whose interests lay more in algebraic structures (groups, rings...) than in the theory of equations. These steps were directed at finding and stating the conditions under which solutions to a specific equation or system of equations exist, rather than determining what those solutions are. It seems that this change of direction was influenced by the demonstration that in general no algebraic method exists to demonstrate a fifth degree equation, though Vallejo made a great effort to find one, eventually even believing that he had managed to do so.

Arithmetical Algebra, as Peacock called it, forms part of *A Treatise on Algebra* in its two versions of both 1830 and 1842. This work announces the existence of two different bodies of algebra, "Arithmetical Algebra" and "Symbolical Algebra". In the second version of his *Treatise* Peacock dedicated a volume to each one of these two domains. While Peacock may be considered the driving force behind the birth of current-day Modern Algebra, the most interesting side of him with respect to this study is that of a 19th century author concerned with teaching algebra to students, and in this sense one can take his *Arithmetical Algebra* (Gallardo y Torres, 2005). On reading this (chapter V, "On solving equations") one finds elements of analysis with respect to the way of organizing the teaching of that part of algebra dedicated to solving problems modeled on first-degree equations.

THE CONCEPTIONS REFLECTED IN VALLAJO TEXTBOOKS WITH RESPECT TO MATHEMATICAL CONCEPTS LINKED TO ALGEBRA

Vallejo's textbooks organized the algebra into two parts. In the first, one learns the syntax of algebra; and, in the second, one learns how to use this calculating system to solve problems, by means of equations.

In the first part, we find an explanation of what means a negative solution in an algebraic problem, where the epistemological conception of that period in time are reflected. Vallejo states that although negative it is a correct solution algebraically speaking, it represents a mistake in the original problem statement, because the difference between two numbers must always be less than the greater of the two. "So that if we wish the result to be in positive numbers we must vary the way of working out the problem" (op. cit., p. 251). In stating that the difference between two numbers must be less than the greater of the two, he is not saying anything unusual in the context of the concept of numbers in his time. At that time, a pair formed by a number with its sign was not considered anything more than an adjectival quantity, and when this quantity's adjective meant "opposite" it was called a negative quantity which, considered on its own, meant nothing more than the answer to an opposite question or the objective contrary to that for which the calculation was performed. Specifically, he speaks of negative quantities and not negative numbers, the former being algebraic quantities which arise from algebra and which have not been needed in arithmetic because, there, in solving problems “everything is substituted by words” (op. cit., p. 173). On the other hand, on stating that the solution −6 represents a mistake in the definition of the problem, he is echoing the idea that D’Alembert expressed in the article “Negatif”, which appeared in Diderot’s *Enciclopedia*.

In the second part of the algebra, Vallejo tries to teach how to solve problems by means of what is called “analysis”. Throughout the text Vallejo unveils what “analysis” is for him, following steps in this order:

> Assume that we know that which we are trying to discover in order to find it afterwards. Express the quantities in letters, such that the known values are distinguished from the unknowns. To express through equations the conditions that must be met by the aforementioned quantities. This is called “stating the problem”. Determine what known quantities are equal to the unknowns, by means of operations that are performed to leave just one member remaining. This is what is known as “clearing away the unknowns”. Translate the formula obtained by clearing away unknowns into ordinary language, so as to obtain the practical rule that can be applied to all similar problems.

These steps expressed the logic that underlies the Cartesian Method. The main features, as Puig y Rojano (2004) have pointed out, are the actions of analysing the statement of the problem and translating it into equations which express, in algebraic language, the relations among quantities.

In Vallejo’s opinion, there are no general rules for stating problems, since this depends on the talent of the calculator (op. cit, p. 230), such that his only advice is to ensure a diligent translation from ordinary language to algebraic language. There are two ways to formulate equations from verbal data: either by direct translation of the key words to symbols or by trying to express the meaning of the problem. These ways are referred to in current literature as *syntactic or semantic translation* respectively (MacGregor & Stacey, 1993). Vallejo showed his preference for the former, agreeing with Newton[ii], and the more frequently used method for formulating equations. Here, he coincides with Peacock but this author goes further in his suggestions:

> In some case the conditions may be symbolized in the order in which they present themselves in the problem, by an immediate translation of ordinary into symbolical language; in others they will be involved in such a manner that the discovery of their relation and succession and their consequent symbolical expression will present difficulties, which can only be overcome by close attention and a clear insight into the relations of the numbers and magnitudes which they involve; for such cases general rules are nearly useless, and the student must trust to the diligent and patient study and analysis of examples alone for the acquisition of those habits of mind which will guide his course in their symbolical enunciation (op. cit., p. 250).

In this second part of the algebra, also we find a notion of equation and an algebraic concept of variable.

Equation was related to the comparison of quantities, which in turn was related to relations of equality and inequality. Comparison by equality led to the idea of the equation, and comparison by inequality to the idea of reason. Through this concept, the equation was seen as a way of thinking that need not be restricted to algebra, since an arithmetic equality such as “1+1=2” was an equation as was the algebraic equality “ax+b=c” (Brooks, 1880, p. 194). And need not be a conditioned equality, which is only verified by certain values – the roots of the equation -, since it includes when the two expressions, connected by the sign “=” are identical or are reducible to identity, such x=x, x+5=x+5, … , 3x+4x=7x. Vallejo adopted this broad concept of the equation although he was aware of (and also explicitly noted in a footnote) the more restricted position of other contemporary authors who differentiated between equation and equality (the expression of two quantities separated by an equals sign). The restricted definition of equation to a conditioned equality, is the one that finally prevailed in school textbooks but, as happens all too often in such books, no justification was given as to why one position should be justified and not another.

Although in general the notion of an unknown refers to an unknown number, it can also refer to a variable. This happens when, on defining the problem, indeterminate equations are obtained – those that have fewer equations than unknowns. This is because when we have, for example, “an equation such as: ax + bz = c, there is no other way of finding any of the unknowns x, z than giving values to the other; and since for each value given to z, for example, a different one will appear for x, it is deduced that in an equation of this kind the quantities indicated by the last letters of the alphabet are called variables, because within the same problem there may be as many values as one likes; and those that are indicated by the first letters are called constants because they can have only one value” (op. cit., p. 375).

CLASSIFYING/ORGANIZING THE PROBLEMS TO BE TAUGHT

It is reasonable to suppose that Vallejo aimed to encourage the acquisition of habits that guide the students in stating the problems that lead to equations, and help in familiarizing them with their rules to clear away unknowns. To illustrate these principles and tips, and to familiarize the student with their use, Vallejo sets out and solves a collection of problems. However, in the text there are no comments to help deduce the criteria he followed in their choice and sequence, other than putting them under three different headings, interwoven among others, whose titles are:

> On algebraic analysis and the resolution of first-degree equations. On the rule of three and others that depend on it, such as exchange, fellowship, alligation, false position, etc. On indeterminate first-degree equations.

Note that Vallejo showed his preference for the ancient practices to classify problems according to certain characteristics related to the context, or method. Whereas Peacock classifies the problems choosing a particularly pertinent point of view, according to whether they involve one or more unknowns and the relations between them:

> We shall now proceed to the consideration of the general rules for the symbolical enunciation of problems, in which the unknown number or numbers are more or less involved in the conditions which are required for their determination, and it will tend to facilitate this enquiry if we classify, very generally, the problems which present themselves for solution, with reference to the unknown number or numbers which they severally involve (op. cit., p. 249).

Along these lines, he says:

> There are three great classes of such problems to be considered
> First Class. Problems which involve one unknown number only, which is throughout the subject of the conditions proposed.
> Second Class. Problems which involve two or more unknown numbers, which are so related to each other by the conditions of the problem as to be expressed or immediately expressible in terms of one of them only.
> Third Class. Problems which involve two or more unknown numbers which are immediately expressible in terms of one of them, but require to be denoted by distinct symbols" (op. cit., p. 249).

This classification, of problems, implied a change of perspective with other systems for classifying problems, according to their method of solution[iii], given that in many cases there is no single method of solving the same linear problem; or according to certain characteristics of the statement of the problem related to the context[iv], given that problems of a different appearance can have the same structure and methods of solution.

It is clear that some of these were soon displaced by the method of equations, but the way of organizing problems would resist Peacock's proposed change, despite its advantages in methodological clarity and generality, the preference being for the traditional teaching of collections of methods and "curious" problems with no apparent unifying nexus. In any case, the generalization of algebraic method was to bring with it the loss of arithmetic methods based on the study of numerical relations, methods eclipsed by the dominant idea that it was enough to study only one method and this should obviously be the best one. And this was the algebraic one, since it was the most general.

SPECIFICS PROBLEMS OF FIRST DEGREE EQUATIONS

Under the heading dedicated to "solving first-degree equations" Vallejo sets out 18 problems. The order in which they are presented does not follow a flow in keeping with Peacock's three classes; they appear to be shuffled up. It seems clear that Vallejo uses other criteria to organize them. Nor is there a criterion for distributing the problems according to an increasing level of difficulty with respect to the data (whole numbers, fractions), the complexity of the problem's conditions, or the operations or extensions to the rules for clearing away unknowns. Rather, it seems that he wants to attend to some of the elements that constitute the analytical spirit.

In the first one, a Diophantus problem: “To divide a proposed number into two parts whose interval or difference is given”, Vallejo introduce and shows how algebra is a language which enables us to move towards generality by means of literal representations and the quantities involved, and how it helps create statements of general rules and formulae for solving families of problems. In the second problem, an abaci problem[v] expressed verbally, with almost nothing else than numbers and abstract relationships of quantities, he carefully explains the syntactic translation process to algebraic symbols. The third problem is an example in which we need to designate all the conditions of a given problem by an equal number of equations. The four following problems show the special interest of Vallejo in enigmatic problem statements[vi], where the difficulty seems to lie more in the language distracters present in the text. The last nine problems are supplementary or for revision. Most of them are problem statements that can be found in texts by contemporary or previous authors, all involving one unknown number only.

Peacock distinguishes two cases of these classes of problems. In one, the conditions can be symbolized in the same order in which they appear in the problem by means of immediate translation from ordinary to symbolic language. In the other, the conditions are so mixed up that it is difficult to discover their relationships and how they succeed one another, and consequently it is also difficult to define them symbolically. Vallejo presents problems of both kinds, without paying much attention to this fact. On the other hand, he stresses the importance of a diligent syntactic translation, and thus shows in the first problems how he substitutes expressions from symbolic language to algebraic language. Among these problems we can highlight one of “God Greet You Problem”[vii], another of “people who buy one thing together”[viii] - both linked traditionally to the rules of false position; and two more with travelers which correspond to the cases: - “they go to meet another”, and “they go after another to catch up”.

SPECIFICS METHODS TO SOLVE PROBLEMS OF FIRST DEGREE EQUATIONS REFLECTED IN VALLEJO’S TREATISE

The algebraic method enables one to define and resolve rule of three problems in a different way from the traditional one based on the old theory of reason and proportion. To do this, one only has to set about solving the problems of proportionality considering the reasons as fractions and the proportion as an equation. This is what Vallejo does, without abandoning the old theory, to which he dedicates two sections within the algebra, before the rule of three. These sections are: “On reasons and proportions”, and “On the transformations that can appear in a proportion while maintaining the existence of the proportion, which is how the analysis was performed in ancient times”. Here, as well as the standard problems, Vallejo includes a problem of “cisterns” and one of “clocks”. He solves the former through algebraic method and the latter with proportions.

The “faucets filling a cistern” can be seen as an example of “Co-operative work problems” that were very popular in texts of the ancient and medieval world (Kangshen, et al, 1999, p. 338).

> There were two types of such problems: (i) The days required to complete a task by A or B (or more than two) is known and one is asked to find the days required to complete the task by A and B (or more than two) together. (ii)) The number of task of type A or of another task of type b (or more than two task) that one individual can complete in *a* day is known, and it is asked to find the number of sets of task (containing one each of types A and B, etc.) that one individual can complete in a day (Kangshen, et al, 1999, p. 337).

The solution of the first type of problems is obtained through comparison with unity in two ways by using unitary fractions or the minimum common multiple. In the former case, the method taken up by Vallejo, one must consider two parts, and in both a reduction is made to unity: first, the amount of the tank filled in one hour or day is calculated; then knowing this one can calculate how much time it takes to fill a complete tank. In both parts a rule of three is used[ix]. The other case also has two parts. First the minimum common multiple is calculated to find the effect of the taps together. Knowing this, a reduction is made to unity as before[x].

In the “clock” problem[xi], Vallejo does not strictly use the algebraic method, but the one of proportions. This deals with finding at what time the minute hand is above the hour hand, starting from a given time of day. To solve this, he establishes the relationship between the distance covered by the hour hand, x, and that covered by the minute hand, a+x. Then he establishes the proportion at the moment they coincide: 12 : 1 :: a+x : x., reducing through the properties of the proportions and clearing away algebraically, x=a/11. The problem is similar to that of two travelers that reach each other, which has been solved in a previous section, but Vallejo makes no mention of this similarity. Moreover, the method for solving the problem is not strictly algebraic.

Of the three problems that Vallejo solves with the two rules of false position, the first one is of two deficits, while the second and third have one deficit and one excess, and so they are similar. One of these problems is a repetition of another that he already solved by algebraic methods in a previous section[xii].

Thus, one may conclude that once again his criteria for choosing the problems is nearer that of the ancient practices, as have been commented before. Also one might think that he separates the problems of syntactic translation from the problems that are not of this type, when the circumstances of the stated question (or problem) appear unsuitable to be translated directly.

SYNTHESIS AND CONCLUDING REMARKS

Generally speaking, we have seen a conception of the teaching of elementary algebra, reflected in a Spanish textbook that was used as a reference in the first half of the 19th century. The way in which this conception is manifested in questions solved with first-degree equations is contrasted with Peacock's proposal.

Both positions show an interest in pointing out that the general method, the algebraic one, is the best method. But, one way of teaching this topic is to present a collection of apparently arbitrary, but traditional problems, that have more or less interest from the point of view of developing analytical skills, and to organize them in agreement with this. (e.g. distinguishing those that have syntactic translation into algebraic terms from those that do not); without abandoning the ancient practices (Specifics problems and methods). As opposed to this, there is an organization of the questions following a precise, well-defined classification according to the number of unknowns and the relations between them.

In addition, we can look into technical and pedagogical guidelines, sometimes underlying in Newton's tradition, for the teaching process of analysing and solving linear problems. Vallejo, especially offers us points of discussion and ideas quite enlightening related to: The negative solutions when problems are solved by means of equations, the notion of equation and its relation with identity, the notion of variables in algebra and its relation with the unknown quantities, examples of the process of syntactic translation to write equations, the idea of algebra and its relations with equations, letters and unknowns (involves only activity with).

The challenge now is to deduce how these positions and guidelines have been incorporated into subsequent texts, in order to try to understand the roots of the present curriculum. In future, it will be necessary to evaluate how general methods fit in with specific personal ones, and how the powerful tool of algebra sits with the legacy of arithmetic, in relation to new approaches and aims in the teaching of current mathematics.

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[i] From a historical point of view, functions are not formally a part of algebra. So we shall not tackle here.

[ii] "for the solution of questions whose preoccupation is merely with numbers, or the abstract relationships of quantities, almost nothing else is required than that a translation be made from the particular verbal language in which the problem is propounded into one (If I may cal it so) which is algebraic; that is, into characters which are fit to symbolize our concepts regarding the relationships of quantities" (Newton, "Newton's Lectures on Algebra during 1673-1683". Whiteside "The Mathematical Papers of Newton", Ed. D. T. Whiteside. Cambridge. Cambridge University Press. 1972. Vol. V. p. 133).

[iii] reduction to unity, simple and compound rule of three, rules of false position, rules of per cent, of interest, discount, alligation, proportional sharing and fellowship, bartering and exchange

[iv] travelers that split up to meet each other later, taps or cisterns that fill up or empty, clocks that go fast or clock hands that coincide, co-operative jobs, …

[v] "There is asked a such number, which if to the quintuple of the number are added seven times the twelfth part of the same number, and take from of all 17 units, turns out to be 17 added to 203 units" (Vallejo, op. cit., p. 242).

[vi] "On being asked how old Alexander the Great was, Artemidoro the philosopher gave the following reply, according to the bishop Caramuel:

On asking Diodoro
Ambassador of the Prince of Egypt
The Age of the undefeated Macedonian,
Artemidoro
Answered him ingeniously
Two years more has the bellicose
King than his comrade
Efestion, whose father
Four years more than the two he counted,
And the father of Alexander
When ninety six journeys of Apollo
Were all the years these three counted". (p.244).

[vii] God greet you with you 100 scholars! We are not 100 scholars but our number and the number again and its half and its fourth are 100. How many are we? (Kangshen et al., 1999, p. 161).

[viii] x people buy an item costing y coins. If each one pays a_1, there is an excess of c_1; if each one pays a_2 there is a deficit of c_2.

[ix] One task completed by A and B in p, q days (respectively), is transformed to 1/q, 1/p task respectively in one day. Knowing this one can calculate how much time it takes to complete the whole task with a rule of three (1 day is to 1/q task like x days is to x/q). Therefore we can find the solution solving the equation $x/q + x/p = 1$.

[x] A completes one task in p days, while B takes q days. Suppose they work together for pq days, then A will complete q task, and B, p task. Therefore they will complete p+q task in pq days, i.e. together they will complete one task in pq/(p+q) days.

[xi] Knowing that the handle of a clock is between a given hour, to find what hour will be when the handle of the minutes is on the hours.

[xii] A father in order to stimulate his son whom he studies, says to him: for every day that you know the lesson I give you 10 coins, but for every day that you do not know you have to give me 4; after 15 days the father had to give him 66 coins; he wonders, how many days he studied and how many not?

DEVELOPMENTAL ASSESSMENT OF ALGEBRAIC PERFORMANCE

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We report a study of 12 to14 year old pupils' performance on algebra, using three diagnostic age-standardised assessment tests. A difficulty scale, using the Rasch methodology was built. The items form a 3 level hierarchy describing algebraic performance. Key errors were also incorporated in the hierarchy. The items were divided in two categories according to their context: symbolic and word/story tasks. Previous literature indicates that teachers consider story problems and word equation problems to be more difficult than symbolic problems, whereas pupils find symbolically presented problems more difficult. Our analysis shows that, according to pupils' achievement, there is no clear precedence of the word/story over the symbolic tasks. Items in both categories appear within the same Level, finding that may have a considerable impact on learning sequences and teaching approaches. Errors concerning algebra were consistent with previous literature. However, by comparing the performance of our pupils on items which were used in the CSMS study, we found that some symbolic tasks concerning 'combining of like terms' behaved in a different way.

BACKGROUND OF THE STUDY

Although it has been found that teachers' beliefs about their pupils' ability have great influence on their instructional practice, several studies (Alexandrou-Leonidou and Philippou, 2005; Hadjidemetriou and Williams, 2002; Nathan and Koedinger, 2000a and 200b) indicate that teachers fail to identify the correct (according to their pupils' ability) difficulty of mathematical tasks. This may have an impact on teachers' pedagogical content knowledge. Shulman (1987) stresses that understanding of what makes particular tasks easy or difficult is an important component of this knowledge.

The debate about the 'concrete' and the 'abstract' is a frequent topic in mathematical studies. It plays a significant role on the way teachers introduce concepts to their students; it relates powerfully to learning and instructional sequences. Cooper and Dune (2000) cite a variety of quotes, spanning over a period of 80 years, to demonstrate the view that mathematics was perceived as the study of the 'abstract' which had to be made comprehensible to all the students. The application then of mathematics to everyday situations used to be, and still is, a major component of the UK National Curriculum. Cooper and Dunne moreover show how the context of an assessment task can be a serious puzzle particularly for students who take the context more seriously than the mathematical symbol 'game'.

The research of Nathan and Koedinger (2000b) examined teachers' predictions about the development of algebraic reasoning and looked for discrepancies between teachers' perceptions of item difficulty and pupils' actual performances:

> The most salient discrepancy is teachers' predictions that story problems and word-equation problems would be more difficult than symbol equation problems, whereas students found symbolically presented problems most difficult (p. 180).

In order to find a possible source of this discrepancy, Nathan and Koedinger (2000b), examined the content of two textbooks. Their analysis confirmed that the arithmetic computations in symbolic form preceded the application of these computations to stories. Nathan and Koedinger (2000a) state that a teacher with symbol-precedence view may withhold story problems from a student until he can demonstrate fluency with symbolic problems. Such decision may restrict the teacher's beliefs about pupils' algebraic development and also does not provide the opportunity to the teacher to explore other possible instructional sequences. Pupils in another study (Hadjidemetriou and Williams, 2002), found some contextualized tasks (based on a story) easier than symbolic/abstract items, whereas teachers ranked those items in a different order of difficulty, with symbolic items appearing lower on the difficulty scale.

Many studies focused on the difficulties pupils experience with algebra (Booth, 1984; Küchemann, 1995; Macgregor and Stacey, 1993). The CSMS (Concepts of Secondary Mathematics and Science) study by Küchemann (1995), which investigated the performance of school students aged 13-15 on items regarding the use of algebraic letters in generalised arithmetic, found that many students were unable to cope with items that required interpreting letters as generalised numbers or specific unknowns.

Herscovics (1989) found that although most students were able to recognise a pattern presented in a table of ordered pairs connected by simple rules, the majority were unable to express these rules algebraically. Pupils find it difficult to generate algebraic rules from number patterns. When they have to generate the n^{th} term of a sequence, pupils seem to focus on the difference between successive values of each variable. They look for a recurrence rule that will help them predict the next number from the value predecessor without focusing on the functional relationship that links the pairs of the numbers (Arzarello, 1991; Macgregor and Stacey, 1993).

In this study we limit our scope on the aspects of algebra as they relate to generating expressions from word problems, generating and generalizing linear patterns, combining of like terms, substituting and moving between different representations of an equation (symbolic and graphical). We analyzed data obtained from a large scale national-representative survey. We report empirical results concerning the performance of 3302 pupils on symbolic and word/story tasks in order to uncover pupils' actual difficulty on such items. The method we employed also allows us to rank not only the items (in terms of difficulty) but also the most common errors. This methodology is a useful tool for teachers who need to know not only their pupil's most common errors and misconceptions but also at which stage of their development they will most probably exhibit them.

METHOD

Data and sample

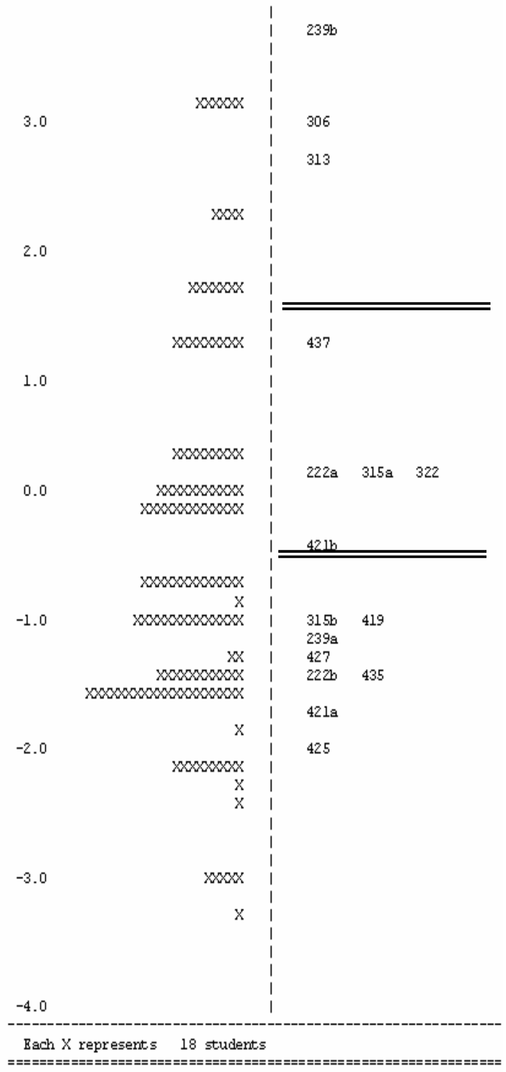
Diagnostic assessment materials for ages 5-14 were developed by the Mathematics Assessment for Learning and Teaching (MALT) team from the University of Manchester with collaboration with the publishers Hodder Murray (Williams, Wo & Lewis, 2005). The items of the assessment materials were developed from the research literature to suit the UK National Curriculum. These items formed 10 papers (years 5 to 14) without any common items between papers. There was, though, vertical linking with a subset of students sitting two papers. A total of more than 14 000 pupils took part from a nationally representative sample of schools from England and Wales.

For this piece of research, we selected only the 16 items that belong to 'algebra' across the years 12 to 14. We aimed to examine the developmental difficulty of the items as experienced by 3302 pupils and compare the results to previous studies.

Analysis

The data of this research were analysed using the one-parameter Item Response Theory (IRT) model or Rasch model (Hambleton et al., 1991). The mathematical models of IRT calculate the probability of a correct response to an item as a function of the subject's ability, the item's difficulty and some other characteristics (depending on the relevant model).

IRT analysis was selected in order to develop a scale measuring pupils' algebraic ability, which will incorporate common errors and misconceptions. The Rasch model calculates the 'item difficulty' and 'pupil ability' estimates, which have the same measurement unit. This allows us to plot the items on a hierarchical scale according to their difficulty and locate pupils on the same scale according to their 'ability'. This analysis allows us to average the ability measures of the pupils who exhibit a particular error response. By using the average ability parameter of the pupils who made the error as a proxy difficulty parameter we can scale the pupils, items and errors on the same scale. The hierarchy then shows which items a pupil of a given ability is likely to achieve but also which are the most likely errors he/she will exhibit. Such a hierarchy then entails information about curriculum designs, teaching sequences, and possible hypothetical learning trajectories



Description of Items	Errors
LEVEL C (1.5, ∞)	
Find the rule of the nth term (2) Substitution (multiplication and exponent) Solve of equation by trial and error	
LEVEL B (-0.8,1.5)	
Linking a graph to its equation Substitution (rational expression) Simplify LAE* by combining terms Add 6 to x+3, add 5 to n+3 Find the rule for the nth term (1)	Wrong substitution: $(at)^2$insetead of at^2 (0.99) Correct slope but wrong intercept for an equation of a given graph (0.0) Horizontal perception of pattern (-0.10)
LEVEL A (-∞,-0.5)	
Word problem-express in symbolic form (20-n) Simplify LAE* by combining terms (add 3 to 7y) Generate terms of a sequence Word problem-express in symbolic form 20n Simplify LAE* by combining terms (add 5 to 3n, 3+6y+1+5y) Generate terms of a sequence Word problem-express in symbolic form (x+3)	X+3+6=9x Non acceptance of LoC* (-0.51) 7y+3=10y Non acceptante of LoC* (-0.53) Number instead of an expression for the nth term (-1.00) Confuse Gradient and intercept when choosing an equation of a line (-1.00) Horizontal perception of pattern (-1.08) 3n+5=8n non acceptance of LoC* (-1.23) 3n+5=8 number (-1.33) Write next number in an arithmetic sequence instead of the requested one(-1.44) Produce a formula that works for one term of the sequence(-1.48)

***LoC**: Lack of Closure
***LAE**: Linear Algebraic Expressions

Table 1: Algebra and Graphs hierarchy

RESULTS

16 items were sorted into 3 levels according to their difficulty (Table 1). The fit statistics indicated that the data fitted the model well. There were no items or children who behaved in an inconsistent way with Rasch measurement prescriptions. This also suggests that the scale is unidimensional and measures a single ‘trait’, what we call here pupils’ algebraic ability.

Table 1 shows the resulting detailed hierarchy of children's performance. It is an empirical hierarchy that describes pupils’ attainment. Each X on the figure represents 18 pupils. The higher the place of the X on the scale the higher the ability of those pupils. On the right hand side each number corresponds to a mathematical task. Difficult items are located at the top of the scale and easier ones at the bottom.

The level boundaries (see Table 1) were drawn according to two criteria: (a) the statistical analysis (where groups of items seem to cluster on the difficulty scale) and (b) on a qualitative analysis based on homogeneity of item content within the levels.

Level A includes items that deal with generating expressions from story problems, generating a later term of a linear sequence and combining like terms. There is a progression of these story problems starting from those which involve addition (i.e. Tom scores x points in a quiz. Greg scores three more points than Tom. Write an expression for the number of points Greg scored), moving on to the multiplication ones (20n) and then to subtraction tasks (20-n). Another progression in difficulty is evident for the items concerning combining like terms, with the easiest being the ones where pupils have to ‘add a number to an algebraic term’ (add 5 to 3n, add 3 to 7y). It is interesting to note that both 12 and 13 year old pupils found it easier (Level A tasks) to ‘add a number to an algebraic term’ (two unlike terms together) than ‘adding a number to a sum’ (i.e. add 6 to x+3). There is a gap of 1.63 logits for 12 year old pupils between the two types of tasks, and 1.22 logits for the 13 year old pupils between the difficulty estimates of the two tasks (Table 2). Given the distribution of this sample, it seems that this gap is likely to be very significant educationally, since one logit equates to approximately one standard deviation of one years population.

Item	Year	Difficulty
12q22a: add 5 to n+3	12	0.23
13q15a: add 6 to x+3	13	0.21
13q15b: add 3 to 7y	13	-1.01
12q22b: add 5 to 3n	12	-1.40

Table 2: Pupils’ performance by year on the ‘combining terms’ tasks

According to the CSMS study, these items (add 5 to 3n, add 6 to x+3) fall into the category ‘Letter as specific unknown’ where ‘*children regard a letter as a specific but unknown number and can operate upon it directly’* (Küchemann, 1995, p.104). However, in their study, it was found that 14 year olds found it easier to ‘add 4 onto n

+5' (which was an item belonging to Level 2 of their hierarchy) and more difficult to 'add 4 onto 3n' (which belonged to Level 3 of their hierarchy). The CSMS team also acknowledged this order of difficulty as surprising:

> It may seem surprising that 4(ii) [*Add 4 onto 3n*], in particular, turned out to be quite so difficult (Level 3). The answer 3n+4 appears to be very simple, but for this reason it is rather unsatisfactory. In a sense nothing has been done with the 3n and the 4 to arrive at the answer but since n is an unknown, children have to recognise that this is all that *can* be done to combine the elements. Many children were unable (or willing) to do this, and instead gave the answer 7n or just 7 in which the elements that were meaningful (the numbers 3 and 4) were 'properly' combined but the letter was simply left as it was or ignored entirely (p.108).

In our study, the opposite occurs with both year groups. A simple way to interpret this is to just accept that pupils find it easier to 'add 5 to 3n' since this is 'an one step question' given that pupils accept the 'luck of closure', whereas the other task (add 6 to x+3) requires one additional step, to combine the like terms, which also makes use of the commutative property of addition. If we analyse this task according to Sfard's theory of process-object duality, where at first we deal with a process (operational stage) on already established objects, then this process becomes an 'autonomous entity' and transforms to a 'more compact, self-contained whole' (Sfard, 1992, p.64), we can argue that pupils have to conceptualize 3n as an object in order to be able to act on it (i.e. add a number to this object). The next step is to conceptualize the idea of a sum (such as 3n+4) as a whole entity in order to be able to act on it operationally (add a number to 3n+4). In the past two decades, a substantial body of research on the learning and teaching of algebra has been established, considerable emphasis was given to early algebra education and generalized arithmetic which seem to have helped pupils achieve higher success rates on similar items.

Level B includes items that involve simplifying of LAE (Linear Algebraic Expressions), substitution and the linking between a graph and its equation. It also includes an item where pupils have to generate a formula for a given pattern.

Pupils achieving Level C can solve an equation by trial and error, can find the value of an algebraic expression by substitution and can generate the rule for the nth term of a sequence.

The hierarchy shows that there seems to be no clear story/verbal over symbolic precedence of items. Both forms of items (story/verbal and symbolic) seem to have similar difficulty estimates; generating expressions from word problems appears in Level A and simplifying symbolic expressions (add 3 to 7y) also appears in the same Level. According to previous research, teachers usually have a Symbolic Precedence view of algebra. More recently, research by Nathan and Koedinger (2000a,b), and Hadjidemetriou and Williams (2002) claims that pupils find word/story problems easier than symbolic and abstract ones. This analysis goes even a step further to indicate that pupils of a certain ability are able to solve story problems and at the same time combine like terms (symbolic). More specifically the hierarchy indicates that a pupil of a certain ability can solve both of the following items: 1) *Tom scores x points in a quiz. Greg scores three more points than Tom. Write an expression for the number of points Greg scored*, 2) *Add 5 to 3n*. Pupils with ability within the level A (which also includes the above items) can also solve the following story problem: *A packet of rulers has N rulers each costing 20 pence. Write an expression for the total cost of the number of rulers*.

Items at Levels B and C are mainly symbolic, fact which strengthens the pre-existing idea that pupils experience difficulties with purely symbolic items.

Errors

The data drawn from the sample were also used to scale the 'common' errors on the same scale, using the average ability parameter of the children who made the error. The most common errors are shown in the third column of Table 1. Although the errors found confirm previous literature we go a step further to plot the mean ability of the pupils producing a particular error on the developmental map.

One error which appears persistent is the 'lack of closure' where pupils exhibit difficulty in accepting that adding 3 to 7y does not equal 10y. Misunderstanding of the order of operations might be another possible cause of this error (i.e. pupils may apply addition first and then multiplication).

In the CSMS study (Table 3) an answer such as 7n (for the item 'add 4 onto 3n') was given by 31% of the 14 year olds and an answer such as 7 (a number) by 16% of the 14 year olds. In our study, the answer 10y was given by 27% of the 13 year olds and just 'a number' by 8%. Only 18% of the 12 year old pupils gave the answer 8n, and 12.5% gave just a number. According to our developmental map, these two errors are exhibited by 12 year old pupils of very similar mean ability (-1.23, -1.33 respectively). Results show that in our study, pupils in general had higher success rates than the CSMS study. 13 year olds had a high percentage of correct responses (47%) comparing to the 13 year olds of the CSMS study (22%). Additionally, a smaller number of our pupils performed the common errors.

Year	**Item**	**Study**	**% correct**	**Error**	**%**	**Error**	**%**
12	add 5 to 3n	MaLT	39	8n	18	Number	12.5
13	add 3 to 7y	MaLT	47	10y	27	Number	8
13	add 4 onto 3n	CSMS	22	-	-	-	-
14	add 4 onto 3n	CSMS	36	7n	31	Number	16

Table 3: Comparison of Pupils' performance and errors (1)

As far as the items dealing with adding 'a number to a sum' (Table 4) are concerned, a common but not very frequent error was to add the two numbers together and then just write the variable next to the number. Hence, for the item 'add 5 to n+3' and 'add 6 to x+3', 8% of the 12 year old pupils and 10% of the 13 year old pupils gave the answers 8n and 9x respectively (mean ability -0.51). Additionally, 9% of the 13 year olds gave the answer 6x+3. Here, again the percentages of the error ‘number’ of the younger pupils were a lot lower than those of the CSMS study (14.5 and 10 comparing to 20).

Year	Item	Study	% correct	Error	%	Error	%	Error	%
12	add 5 to n+3	MaLT	23	8n	8.3	Number	14.5	-	-
13	add 6 to x+3	MaLT	40	9x	10	Number	10	6x+3	9
14	add 4 onto n+5	CSMS	68	-	-	Number	20	-	-

Table 4: Comparison of Pupils’ performance and errors (2)

In two other tasks, in an attempt to generalize a pattern, a significant number of pupils tended to use the difference between two consecutive terms in a sequence to produce the nth term (Figure 1, questions b). Item 39 (Z counters) was given to the 12 year olds and item 21 (Toothpicks) to 14 year olds.

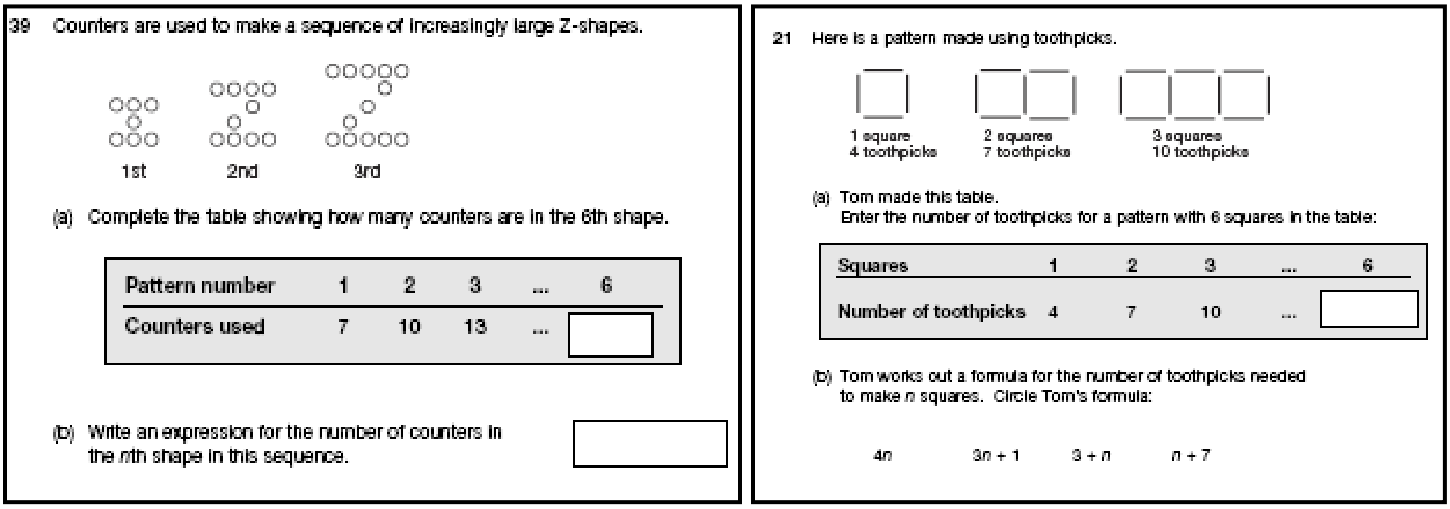


Figure 1: Pattern Generalizing

It is interesting to note that the answer to the first part can be found by horizontal reading of the ‘Counters used’ or the ‘Number of toothpicks’ rows respectively. Counters/toothpicks go up by 3 each time. This part does not help pupils to answer the second part of the item where they have to read the table vertically and make the connections i.e. 1 to 7, 2 to 10, 3 to 13 and so on. A common mistake then is to give the answer n+3 or +3 (Table 5), which is also supported by the picture provided (every next Z or square has three more counters/toothpicks than the previous one). This ‘unidimensional reading of patterns’ is a well known error. Here we provide evidence which indicates this error is usually exhibited by 12 year old pupils of high mean ability (-0.1) and by very low ability 14 year old pupils (-1.08). A significant number (21%) of 12 year old pupils of average ability of -1 also gave ‘a number’ for the nth term, instead of an expression.

Year	Item	% correct	Error	%	Error	%	Error	%
12	Z counters	2.5	n+3, +3	11	number	21	NA	37
14	Squares	42.5	3+n	30.5	4n	14	NA	6

Table 5: Generating Patterns-Pupils' performance and errors by year

CONCLUSION

In this paper we have described an empirical hierarchy of the development of algebraic performance of 12 to 14 year old pupils, in order to examine whether teachers' beliefs about the so-called Symbolic Precedence View of algebraic tasks (Nathan and Koedinger, 2000b; Hadjidemetriou and Williams, 2002) is valid. The data were obtained from a large scale national-representative survey in England and Wales. The hierarchy was divided into 3 Levels and items within each level were categorized as 'symbolic' or 'word/story'. The research by Nathan and Koedinger, (2000b) on algebraic development showed that pupils find symbolic items more difficult than story related items. Our analysis shows that pupils do find symbolic items difficult (since many symbolic items were located towards the top of the difficulty scale), however the order of the difficulty of the items at the lower level does not indicate clear precedence of the word/story over the symbolic tasks. It indicates that pupils of a given ability are able to answer correctly both categories of items with the same ease.

A hierarchy of performance illustrates one possible trajectory of the way pupils think and perform. We share the view that there may be serious dangers if teachers use such tools as recipes of instructional and learning sequence (Brown, 1989; Noss et al, 1989). However, we do not claim that this is the only possible learning sequence for pupils.

The 'abstract' versus 'concrete' issue has a wider importance in mathematics education. It is not a debate which relates only to the teaching and learning of algebra. The findings of this study indicate that pupils tackle word/story problems and symbolic problems with similar ease. We need to investigate whether this is result is true for other mathematical domains as well and make the relevant curriculum modifications to support our students' learning.

Additionally, a comparison of our results to the CSMS results showed an increase in success rates on similar items. This indicates that in the last two decades there have been changes which we have to research and find ways to reshape the algebra curriculum in order to have more immediate value to our students. We need to identify explicit examples, in meaningful contexts, which will be more easily adopted by the pupils.

We believe that it is very important for teachers to know what makes a particular task easy or difficult. The 'same mathematics' might be easier in one task than in other. Thus, if the teachers' perception of task difficulty is misconceived (because it might be based only on the concept of the task and not on the surrounding context) then their pedagogical approach is likely to be flawed.

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INTEGRATING THE LEARNING OF ALGEBRA WITH TECHNOLOGY AT THE EUROPEAN LEVEL: TWO EXAMPLES IN THE REMATH PROJECT

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The ReMath project is a European project that addresses the task of integrating theoretical frames on mathematical learning with digital technologies at the European level. A specific set of six dynamic digital artefacts (D.D.A.) is currently developed, reflecting the diversity of representations provided by ICT tools. Here we consider two D.D.A.s related to the learning of algebra a) the 'Cassiopee - calculator of functions' developed by a French team, b) ALUSNET, a set of closely integrated components (Algebraic Line, Symbolic Manipulator and Cartesian Plane) by an Italian team.

The two D.D.As are developed for students at upper secondary level. The two teams share a common concern that secondary students generally lack of proficiency in algebraic representation. They start from a common analysis of students' difficulty in algebra. They nevertheless reflect specific orientations and decisions with regards to algebraic representations, students' algebraic activity and the help that technology can provide.

STUDENTS' DIFFICULTY IN ALGEBRA

In recent years several researches have interpreted the students' difficulty in algebra as a lack of capacity to control the sense and the denotation of algebraic expressions or of algebraic propositions. We can make reference to the Frege's triadic semantic model to explain what we intend for sense and denotation of algebraic expressions and propositions. The Frege's model distinguishes between sense (Sinn) and denotation (Bedeutung) of an expression (Zeichen): the denotation is the object which the expression refers to, while the sense is the way in which the object that is denoted is presented to the mind by it or is understood by the mind. In elementary algebra "Zeichen" is an algebraic expression or an algebraic proposition. An algebraic expression is a writing composed of numbers and/or letters connected by the symbols of mathematical operations (addition, subtraction, multiplication, division, power of a number, extraction of root); this expression indicates the result of the operations performed in sequence. Two algebraic expressions can have different senses and the same denotation. For example the two expression 2x+1 and x+(x+1) considered in N denotes the same function that associates the result of the operations indicates in their form to each value of the letters but have different sense (for example the first one presents the odd numbers as the successive of an even number while the second one as the sum of two consecutive numbers). Algebraic transformations change the sense of an algebraic expression preserving its denotation. An algebraic proposition is a writing that expresses a relationship between two algebraic expressions by means of the signs =, >,<. It denotes a truth value (true/false). For example the proposition "2x-5=x-1" denotes "true" for x=4 while it denotes "false" for all the other numerical values of x).

The students difficulties in algebra emerge in two distinct circumstances: when they write or interpret an algebraic expression or proposition in order to solve a specific situation, and when they transform an algebraic expression in order to highlight a specific property of that situation or to find the solutions for it. Using Frege's model we can explain these difficulties in terms of lack of control of the sense and of the denotation of algebraic expressions or propositions.

Students defined by Sfard (1991) as pseudo-formalists are unable to imagine the intangible entities that underlay the use of algebraic writing. For these students, algebraic expressions and propositions are "things" in themselves that do not stand for anything else. The students are unable to grasp the invariance of the denotation as regards the changes of sense. They are unable to recognize that a syntactic structure of an expression reflects in a iconic way a property of a numeric set, namely they are unable to recognize the sense of an algebraic expression in relation to the aim for which an algebraic expression or a proposition is constructed or transformed. They show a sort of rigidity and they operate as if there were a one-to-one correspondence between sense and denotation. Their algebra is purely syntactic; for these students a mathematical expression denotes itself. In contrast other students are unable to use the algebraic language and the methods of algebra for tackling the assigned tasks. They prefer to elaborate solution strategies using different representations such as the verbal language and the method of arithmetic. In both the two cases there is not a development of a genuine algebraic thinking.

With the two DDAs we are developing within the ReMath project we intend to make available new effective tools to overcome some of the difficulties evidenced.

THE CASYOPÉE PROJECT: LINKING ENACTIVE EXPERIENCE OF FUNCTIONS WITH ALGEBRAIC REPRESENTATIONS

Motivation

The D.D.A. Casyopée is built as an open problem-solving environment with the aim to provide students a means to work with algebraic representations, progressively acquiring a control of the sense of algebraic expressions and of their transformations. Functions are the basic objects in Casyopée. Using this tool, students can explore and prove properties of functions.

Casyopée takes into account the potentialities that Computer Algebra Systems offer to teaching/learning: going beyond mere numerical experimentation and accessing the algebraic notation, focusing on the purpose of algebraic transformations rather than on manipulation and connecting the algebraic activities. A major choice in the Casyopée project is to develop a software environment embedding algebraic transformational knowledge by way of a 'state of the art' symbolic kernel and facilities to help build and write a proof.

Exploring and proving seems to us particularly relevant when students work on algebraic models of domains where they can experiment 'enactively'. We expect that students will make sense of algebraic representations by linking these with representations in these domains. For instance, a geometrical figure can be a domain for experimenting dependencies between measures. Then an algebraic model can be built choosing one of the measures as an independent variable and the other as the dependant variable. Properties of the dependency can be conjectured and proved: they take sense both in the algebraic and in the geometrical settings.

In its present state, Casyopée is limited to algebra. Thus the algebraic work is separated from the enactive exploration and of the modelisation. It would be better if students could go forth and back between algebra and a domain of enactive exploration, building and testing models. Our goal in the ReMath project is to include this possibility in Casyopée. We choose the domain of dependencies between measures in geometrical figures that we gave above as an example. Casyopée is then being extended to include a 'dynamic geometry' (DG) module and facilities to help students to link geometrical dependencies and algebraic representations.

In order to explain how the present version of Casyopée works, the motivation for this extension and how it will be implemented, the next section offers an example of a scenario of use.

An example of an evolutionary scenario

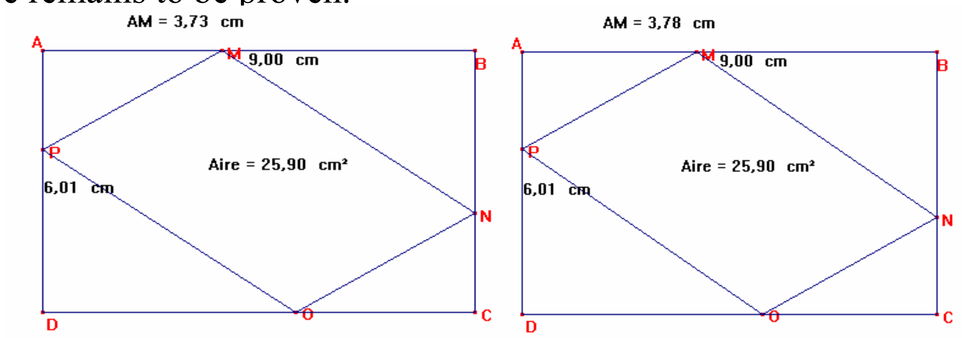
The problem

> *ABCD is a rectangle, M is a point on [A,B], N on [B,C], O on [C,D] and P on [D,A] such as AM = BN = CO = DP. How does the area of the quadrilateral MNOP vary when M is moving on [AB]? Is there a minimal value?*

This problem can be found in French textbooks for the 10th grade under various forms. Generally, values are given for the lengths of the sides of the rectangle. Analysing this problem, Artigue (2005), distinguishes first an 'initial question', where values are given for the lengths of the sides of the rectangle, and a 'new development', where a general conjecture, following the study of the initial problem can be proved.

The initial question

Let us consider the rectangle ABCD with AB = 9 cm and BC = 6 cm. Starting from AM=0, it is clear that the area of MNOP becomes smaller when AM increases and also that it becomes again bigger when AM increases towards AB. A first conjecture could be that a minimum occurs when M is at the middle of [A,B]. Exploring this situation with dynamic geometry software, confirms the intuition that the area decreases, goes through a minimum then increases again. It also invalidates the first conjecture because it appears that a minimum is reached for AM between 3.5 cm and 4 cm and not for AM= 4.5 cm. We can correct the initial conjecture by noticing that 3.75 cm, which is the 1/8th of the perimeter, is at the middle of these two values. This conjecture remains to be proven.



To achieve this proof, we have to define a function whose independent variable (x) is the length AM, and the dependent variable ($f(x)$) is the area of MNOP. Various computations give $f(x) = 54 - x(6-x) - x(9-x)$.

The minimum of f can be proved using an algebraic technique currently taught at the 10th grade. If 3.75 is the value where the minimum of $f(x)$ is reached, then the difference $f(x)-f(3.75)$ has to be positive for every x in [0;6]. Actually, we find that $f(x)-f(3.75)= 2x^2 - 15x + 225/8 = (4x-15)^2/8$.

A new development

In the above rectangle, the minimum is at 1/8th of the perimeter. For a mathematician, it is natural to wonder whether this is a general property. Building an algebraic model is particularly relevant for this generalisation. We have to introduce the lengths of the rectangle and to do the calculations as above. We get $f(x)=$ AB.AD $-x$ (AB+ AD) + 2 x^2 .

The minimum for $x=$ (AB+AD)/4 can be proved using the above technique because $f(x)-f(($AB+ AD$)/4)= (4x-($AB+ AD$))^2/4$

Experimenting with the existing Casyopée

This problem was given to 10th grade students of a teacher participating in the project, during two one-hour sessions. Before that, students had had a session using Casyopée to learn about linear functions and algebra. At this point in the academic year, they had just started to learn about quadratic functions and algebraic methods.

Consistantly with or our focus on modelling we wanted that students explore the geometrical situation before considering an algebraic model. Then students had to prepare at home a geometrical study showing that MNOP is a parallelogram, and to find a method to compute its area. The first session with Casyopée was devoted to the conjecture In our meaning, studying a numerical case like in the 'initial question' above was not enough to justify the use of algebraic methods. A decision was then to consider the general case (the 'new development' above) as the main goal and to make students explore numerical cases, as a preparation of a general conjecture. In order to reduce complexity, we chose to have only one parameter a, and values of the sides (AB=4 and BC= $4a$) giving a minimum for AM= a+1. This we expected to be easily conjectured by students.

Using Casyopée, students had to create a parameter a and functions corresponding to the areas of AMP et BMN and then of MNOP. They had to observe the minimum values of a from 1 to 4.75, by steps of 0.25 and to try to think of a general conjecture. Casyopée was expected to help them to switch from geometry to algebra, and to search for a conjecture by exploring numerical cases and generalising.

In this first session, the students' main obstacle using Casyopée, was that they had to find and enter a domain for each function. They were used to domains for functions, but in problems they encountered before, this domain were given relatively to the algebraic conditions of existence. Here they had to find it from the geometrical situation. This difficulty is clearly linked to insufficient understanding of the links between the geometrical situation and the algebraic model.

When studying the minima, it was difficult for them to engage in a conjecturing process. Instead they looked closely to each numeric case, refining the values as much as possible, but not trying to make sense of what they found. This is evidence that students cannot find a conjecture by looking at one isolate numerical case, as a mathematician would do. The teacher draw a table of x and a for each numeric case, and students could get the conjecture by pattern recognition. Again, the obstacle was in the difficult link between the figure and the algebraic model.

The second session, three weeks later, was devoted to proof. Because of the difficulties in the first session, we thought it necessary that students make sense of the conjecture in the geometrical situation before building an algebraic proof. That is why the teacher prepared a dynamic geometry figure displaying the rectangle with the parallelogram, and a curve of the values of the area. With this figure, the students could animate the variable *x, modifying the* rectangle and moving the current point on the curve. They could alternatively animate the parameter a, modifying the rectangle and the curve. Thus they understood better the dependency between the minimum and the parameter a.

Students had to build a proof using a technique similar to what was outlined above. They had seen this technique before. The goal was not that they become skilled at this technique, but rather that they get notions about algebraic treatments and recognise their power. Students easily made sense of the equivalence between $s(x) \geq s(a+1)$ for every x in $[0;a]$ and the sign of the function $x \to s(x) - s(a+1)$ defined on $[0;a]$.

After entering this function in Casyopée, they tried the entries of the 'compute' menu (expand, factor, normal…) and choose the factored form because they thought that it was simple and therefore more easy to use. They then used the 'justify' menu, which produced elements of the proof in the Note Pad. After that they edited these elements to write a proof. In contrast with paper/pencil situations, it was a very active and satisfactory part of the work, thanks to the help of Caysopée.

A scenario for the extension

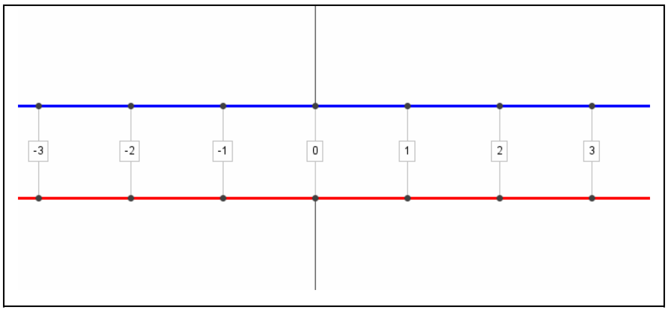
The above report on these two sessions of experiments using the existing version of Casyopee is evidence of its support to students' algebraic activity, but also highlights their difficulty in making sense of the links between the geometrical situation and the algebraic model. In the second session, we made students use the dynamic geometry software before using Casyopée so as to help them to make this link, but this was limited because they could not go back and forth and move objects from one environment to the other.

With the extension planed in the ReMath project, students will be able to create the rectangle and the points in a dynamic geometry (DG) module, M being a free point on [AB]. The lengths AB and AD will be Casyopée parameters (like *a* above) and the students will be able to change them by animating the parameters. Casyopee will then provide a means for defining and computing symbolically the geometrical functions. Students will be able to choose an independent variable (for instance the length AM or BM) and a calculation involving lengths as a dependent variable. They will be then able to build a function for the area of the parallelogram. The DG figure and the functions will be dynamically linked. Animating the parameters will change the rectangle as well as the representations of the functions (expressions, graphs, tables…) Dragging M will move the trace on the graphs. Conjecturing and proving the minimum will be done by steps similar to the situation experimented without the extension. However, we expect that these steps will involve more exploration with the geometrical situation in mind.

A NEW OPERATIVE AND REPRESENTATIVE ARTEFACT FOR ALGEBRA LEARNING: THE DIGITAL ALGEBRAIC LINE OF ALNUSET

In order to mediate and to support the development of the capability to control the sense and denotation of algebraic expressions and propositions, in the ReMath project a working group of ITD-CNR is developing ALNUSET (ALgebra of the NUmerical SETs). In this document we consider only a component of ALNUSET, namely the Digital Algebraic Line (DAL).

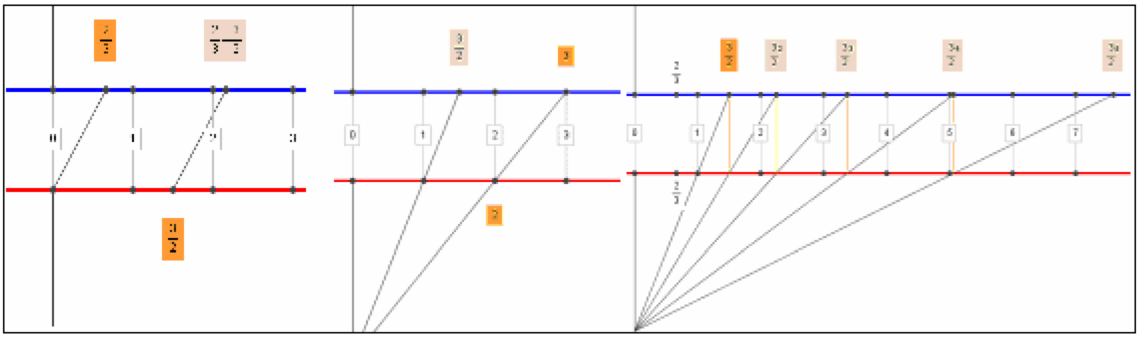
The DAL environment of ALNUSET comprises two parallel lines intersected by a perpendicular line. The two points of intersection define the points 0 on the two algebraic lines, which are characterized by the same dynamic unit of measure. A post-it is associated to each point represented or constructed on the DAL: the content becomes visible when the mouse pointer moves close to



it. The DAL is based on a representation built by mathematicians in the preceding centuries, the numbers line. In the ReMath project with the design and the implementation of ALNUSET we have exploited the possibility of visualisation, interactivity, dynamicity and computation offered by technology to transform the numeric line into a *digital algebraic line*, making available a new operative and representative possibility, not available on the numbers line: to use mobile points on the line marked by letters to express, in general form, the generic element of the considered numeric set. The mobile point is a useful metaphor to conceptualize the notion of algebraic variable and it is a useful operative tool to geometrically construct algebraic lettering expressions on the line and to externalize what they denote. We do believe that the Algebraic Line can exist only as digital artefact because its algebraic nature is due to the introduction of mobile points on the line and this characteristic is possible only through digital technology. The new operative and representative possibilities offered by the DAL of ALNUSET allow the user to perform the following four algebraic activities: Construction of algebraic expressions and their representation on the line; Externalization of what variables and algebraic expressions depending on them denote; Research of real roots of polynomials with integer coefficients; Identification of the truth set of algebraic propositions. In the following we will present the first two activities.

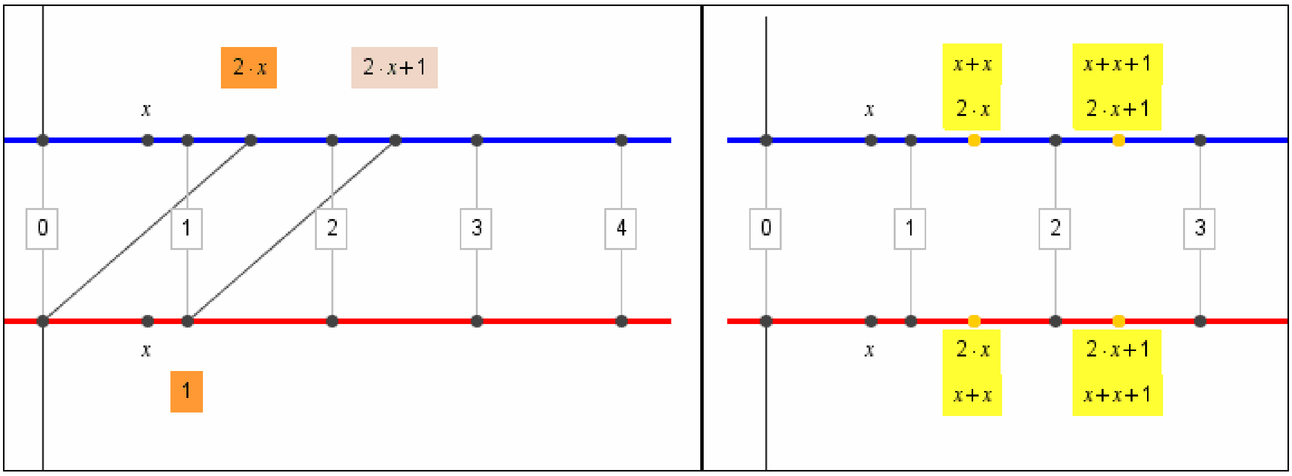
Activity 1: Construction of algebraic expressions

The DAL is an operative and representative environment for the construction of algebraic expressions involving integers and letters defined on a specific numeric set (natural integers, relative integers, rational numbers, rational numbers extended to rational powers). Three geometrical models are available to join numbers and letters by means of the symbol of the mathematical operations and to construct algebraic expressions: a model for addition/subtraction, a model for multiplication/division, a model for integer power/rational power. Each of these three models can be used to perform both the direct operation and the inverse one. In the figure three examples concerning the model of the operations are reported.



In the first case the model of addition is used to perform 2/3+3/2; in the second case the model of the division is used to construct the fraction 3/2; in the third case the model of power is used to construct until the fifth power of 3/2. As a basis for constructing new mathematical expressions, the DAL makes available a predefined range of integers on the lines and provides the opportunity to use letters as names of mobile points represented on them. Integers represented on the lines, letters corresponding to mobile points and mathematical expressions already represented on the DAL can be joined up by means of the geometrical models of mathematical operations in order to create new mathematical expressions. Every new mathematical expression constructed in this way is associated both to a point that indicates the result of the operations performed in sequence and to a post-it that will contain all the equivalent expressions constructed by the user denoting that point. Some examples of mathematical expressions:

- "2/3" is a mathematical expression produced by combining the integers 2 and 3 with the operation of division. The expression created in this way is associated to a point on the line and it is inserted in the post-it associated to this point. Any equivalent expression of 2/3 constructed by the user denotes the same point and it is inserted in the same post-it. We note that the point constructed on the line is a efficient metaphor for the notion of rational number while the post-it associated to it that contains all the fractions that are equivalent to 2/3 is an efficient metaphor of the notion of "class of equivalence".
  The DAL makes available an important conceptual metaphor for the construction of the concept of algebraic variable: the algebraic variables are mobile points on the DAL that denote numbers. The variable associated to a mobile point can be exploited to generalize properties of the numerical sets. For example, once the variable n is represented on the line the expression *(n*2)/(n*3)* can be constructed and it is possible to verify that such expression is associated to the same point of 2/3 and it is contained in the same post-it, whatever the value of n is, namely whatever the position of the mobile point n on the line is.
- 2*x+1 is a mathematical expression produced through two constructive steps: construction of the expression 2*x by combining the integer 2 and the letter x by means of the operation "*"; construction of the expression 2*x+1 by combining the previous expression 2*x and the integer 1 by means of the operation "+"



We note that the expression x+(x+1) is an equivalent expression of 2*x+1, i.e. it makes reference to the same point on the DAL, and so it appears in the same post-it of that point. Activities of this type can be of great importance to construct the idea of what an algebraic expression denotes or to understand what means that two expressions are equivalent or to learn how to build algebraic expressions addressing specific aims.

Activity 2: Externalization of what a mathematical expression denotes

Once a mathematical expression that contains letters in its structure has been constructed and represented on the line, it is possible to drag the variable points on the DAL referring to those letters. When a mathematical expression depends on a variable, the dragging of the variable point on the DAL will refresh the expression depending on it. It is not possible to show on a static support the dynamic effect produced by the drag of the variable point on the DAL. Referring to the previous figure the reader has to image to drag the mobile point x and at the same time to see the movements of the points that are associated to the expressions depending on x (i.e. moving the "x" point will make the "2*x+1" and “x+(x+1) points move accordingly). Hence, the dragging of variable points on the DAL is a way of externalizing what a mathematical expression denotes (e.g. in the previous example, the set of odd numbers) and the sense incorporated in its syntactical structure (odd number as successive of an even number or as sum of two consecutive numbers) Moreover the drag action can be used to highlight different meanings of the use of letters and of mathematical expressions in algebra. For example, when the user drags the variable x point to verify what the expression 2x+1 denotes, s/he uses the letter as a means of numerical generalisation.

When the user drags the variable x point searching for the value(s) for which two points relating to the A(x) and B(x) expressions coincide, s/he uses the letter as an unknown and this meaning is reified in the way s/he uses the drag. A variable point assumes the meaning of a parameter when the drag is used to instantiate only some specific values of the associated letter with the aim of evaluating the corresponding results of a mathematical expression containing that letter.

CONCLUSION

Casyopée and ALNUSET share a common motivation: to use technology to provide students a means to access *existing* algebraic representations, built by mathematicians in the preceding centuries. It is based on a transpositive perspective: mathematicians did not abandon the algebraic notation, but rather developed specific computer artefacts (symbolic calculation, mathematical writing systems…) to help in their everyday practice of this notation.

Researchers in both teams also think that technology must be used to enrich the existing algebraic representation. In ALNUSET new cultural artefacts embedding new ways to represent mathematical object are made available. Caysopée focuses rather on new opportunities for operating upon algebraic representations and on new possibilities for linking algebra to other domains.

These new operative and representative characteristics exist thank to the possibility of visualisation, computation, dynamicity and interactivity offered by the technology. They can facilitate the user to interpret algebraic phenomena or to recognize specific mathematic meanings in what the representation exhibits in the interaction. For instance the possibility to construct mobile points on the line in ALNUSET is a concrete way to reify the idea of letters as *unknowns* for the aim to use them to build algebraic expressions and polynomials in a constructive way. As another example, modelling a geometrical dependency into an algebraic function in Casyopée helps to give a meaning to letters involved in the algebraic definition and treatments.

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RESEARCH AND PRACTICE IN ALGEBRA: INTERWOVEN INFLUENCES

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I report on the development of curriculum materials in which pedagogy, didactics and mathematics are interwoven. The course was intended for practising teachers whose own perceptions of and competence in algebra is limited and whose choices when teaching are therefore circumscribed. The materials were informed by research, and designed to influence practice. Some of the principles underpinning the development of the materials are described, and observations are made about the kinds of research which proved informative and influential.

BACKGROUND

The Open University has a significant history spanning nearly 40 years in the presentation of distance learning materials for teachers at all ages and stages of their careers. In response to evidence that a significant number of teachers in secondary school have limited, and usually very traditional views about mathematics, it was decided to prepare courses contributing to an Advanced Diploma which would broaden their perception of what algebra is and could be about, and which would also enrich their access to informed choices when planning and leading lessons in algebra.

In the event, probably due to lack of funding for teachers to undertake professional development once they are qualified as teachers, and probably also due to severe pressures on their time and their energies, the audience is rather varied. Some 50% of the people taking the course are people who wish to qualify as a teacher, and who therefore require an undergraduate degree with substantial mathematics in it before they can go on to become qualified teachers. Many of these are already employed as teaching assistants in schools. Some 40% are actually practising teachers, and some 10% are parents, consultants, and other people with an interest in mathematics education.

The course, called *Developing Thinking in Algebra* is based on a book of the same name (Mason *et al* 2004) and is a companion to similar courses in *Geometry* and *Statistics*, with an introductory course on *Developing Mathematical Thinking* to round out the Advanced Diploma. Each course is considered to involve 200 hours of study. Assessment is by tutors who contact the students by phone and who mark their four assignments. There is also a website where issues and concerns can be raised and discussed by staff and students. Typically 150 students take each course each year, though even if there were 500 on each, the national need would not be met.

CORE OF THE COURSE

The algebra course is based around the notion of *expressing generality* as being the essence of school algebra. Historically, algebra is usually seen as arising through a desire to be able to solve problems involving some unknown number or numbers. As Mary Boole (Tahta 1972) put it, by 'acknowledging your ignorance' you can denote what you do not know with a letter, and then manipulate that letter as if it were a number in order to express relationships and constraints arising from the problem. Support for this view can be found in the use by early authors of the term *cosse* ('thing') as the 'as-yet-unknown'.

At the same time however, there is a pervasive historical thread by authors wanting to solve every problem, or trying to indicate that the solution to a particular problem was to be seen generically as a method for solving a whole class of similar problems. Authors used a variety of means for informing the reader of the 'general rule', in words, and through the use of examples. Newton may have been one of the first to use letters to denote as-yet-unspecified parameters so as to solve a problem 'in general'.

There is however a conceptual commonality between the use of a letter to stand for an as-yet-unknown and the use of a letter to stand for an as-yet-unspecified parameter: both depend on the person to be stressing the letter as label rather than as the value signified. Flexible movement between attending to the label and attending to the content (syntax and semantics of expressions) is the essence of working effectively with expressions of generality.

Clearly every discipline involves expressing generality about something: geometry for example is about relationships and properties to do with shape and space, or as Gattegno put it, 'about the dynamics of the mind [mental imagery]'; algebra arises from expressing generality about properties of and relationships between numbers. Gattegno (1970 p. 26) goes further and suggests that

> Algebra is present in all mathematics because it is an attribute of the functioning mind.

He was convinced that awareness, recognition, and explicit evocation of the powers of the mind, could make mathematics much more meaningful to learners. In a sense, algebra arises when we transcend the particular, when we refer not to the actual but to the possible. In algebra we refer to quantities and relationships which cannot be seen or touched because they exist only in our minds as potential.

Core Principle

The conjecture and core principal underlying the course materials is that where learners have been explicitly involved over a long period of time in becoming aware of and expressing generality involving quantities, the manipulation of algebraic symbols and the use of algebra to resolve problems is relatively straightforward. However, where learners are thrown straight into the manipulation of symbols 'as if they were numbers', the whole process becomes mysterious, the purpose unclear, and the practice merely routine; motivation and performance suffer, and interest in mathematics itself declines.

It is not easy to point to specific empirical research to validate the core conjecture using statistical studies. However, taking the core conjecture as a stance, reading through the literature confirms that where learners have struggled with algebraic thinking, generality, and manipulation, they have usually had a very limited explicit exposure to these ideas. I am however not aware of research which has been able to pinpoint the source of algebraic success to expressing generality specifically. Fairchild (2001) did show significant improvement in scores for learners enculturated over a year into expressing generality using the visual metaphor of area.

Course Structure

The course is divided into three blocks of four chapters, with three final chapters summarising and collecting in one place various remarks about the pedagogical principles employed in the construction of the materials, and which can also inform and underpin pedagogical and didactical choices teachers can make when preparing for or participating in lessons. Each block has a chapter which develops the theme of expressing generality; a chapter which develops the notion of learners' powers to make mathematical sense, and the notion of pervasive mathematical themes; a chapter on the role and use of symbols, including symbol manipulation; and a final chapter on representations and the use of ICT (graphics calculators, spreadsheets and graphing software in the three blocks, respectively).

Thus the course design follows a form of Bruner's spiral curriculum, seeking to deepen and enrich learners' sense of the topics by returning to them repeatedly in more detail. Our own version of the spiral curriculum is expressed in terms of 'frameworks' or 'pedagogical constructs', including a spiral of personal development and insight summarised as *Manipulating–Getting-a-sense-of–Articulating* by drawing attention to the nature of activity which contributes to effective internalisation and hence learning. As articulations become succinct, they in turn become manipulabel components for further use. The framework See–Experience–Master acts as a reminder not to expect instant mastery or evidence of 'learning' when introduced to a new way of thinking or a new concept. Often considerable re-experiencing is required before fluency and facility (components of competence or mastery) are achieved.

An unusual feature of the book is the large number of tasks on which the reader is invited to work, with, in many cases, only passing indirect reference to 'answers'. This is in order to emphasise that the purpose of working on tasks is to generate experience from which to learn, rather than to obtain answers to check at the back of the book. There are far more tasks than anyone can work on much less explore fully, because the book is seen as a lifetime resource, not simply a once-off course. While the tasks are aimed at people studying the course, most have been adapted for use in classrooms by ourselves or others. The course is explicit about providing *not* readymade tasks for use in classrooms, but rather principles for augmenting and modifying tasks so as to suit specific learners in particular situations.

Another unusual feature of the book is the interweaving of suggestions about pedagogical strategies and suggestions about specific didactical tactics and devices (which are at least algebra specific if not topic specific within algebra). This, like the high density of tasks, has been a hallmark of CME materials since the centre began in 1982. We have always seen working on mathematics for yourself as an essential component of preparing to teach others, and that in order to sensitise yourself to learners it is vital to become more aware of your own experiences and propensities, your own use of your powers and your own encounters with significant and pervasive mathematical themes.

MATHEMATICAL POWERS

Concomitant with the core conjecture is the observation that every learner who gets to school has displayed the powers necessary to think mathematically, and certainly algebraically. What is needed therefore is not 'instruction in algebra' but rather the evocation and application of those powers to number relationships and properties. The issue is how to get learners to use, develop, refine and hone their powers rather than have them usurped by textbooks (as is usually the case) and teachers. When teachers and texts do the specialising and the generalising, the conjecturing and even the convincing for students, they enculturate students into parking their own powers at the classroom door as 'not wanted here'.

Fundamental among the powers referred to explicitly in the course is *imagining & expressing*, which may make use of a variety of modes including movement, gesture, pictures, words and symbols. Expression includes Mary Boole's 'acknowledging your ignorance' and so denoting that which is as-yet-unknown or un-specified by some sort of a symbol (a little thought bubble, a box, an acronym, a letter). A particular feature of imagery associated with algebra is seeing through the specific to something more general, to seeing the particular as exemplary rather than simply as singular, to seeing the particular as indicative of possible variation. *Specialising & Generalising* are powers which are hardwired in people in order to cope with the myriad of sense-impressions which impact them at every moment. These powers have been referred to explicitly by many, including Whitehead (1911) and Polya (1962) and reiterated in our materials since 1982.

Conjecturing & convincing are core human powers, since every action is a conjecture, no matter how confidently asserted. Justifying actions is what is meant by 'being responsible' (from the verb *spondere*). Western culture has amplified two aspects of convincing: explaining our actions (as if 'to mummy') and justifying our actions (as if 'to daddy'), hence the continuing confusion between *proof* and *reasoning*. Mathematics involves, among other things, being enculturated into the practices of convincing others, not on the basis of emotion or tradition, but on the basis of mathematical structure, on the basis of mathematical reasoning itself. This can provide an important emotional experience for troubled and turbulent adolescents looking for solid ground in which to anchor themselves, by providing experience of a way of recognising cause-and-effect in relation to their actions outside classrooms.

Organising a complex of objects by isolating certain properties and sorting out relationships between classes, is primal human activity. It is one of the major contributions of language, and involving as it does the *characterising* of objects which belong in a specific class. Without organisation into classes and groups, every sense impression, every encountered 'object' would be individual and unique. Much of mathematics concerns organising problems into problem-types resolvable by the using the same technique(s) or method(s), and specifying properties which characterise the objects belonging to a class (Lakoff 1987).

These powers are only some of the most pervasive, significant and effective human powers which can be, which need to be exploited and developed in order to make sense of algebra and to make algebraic sense.

MATHEMATICAL THEMES

Pervasive themes which serve to integrate apparently disparate mathematical topics are interwoven in the course materials. For example:

Freedom & Constraint: Every mathematical task and exercise is an example of constraints placed upon freedom. Think of a pair of numbers (bcome aware of the freedom of choice available to you: did you consider rationals? irrationals?); add the constraint that their sum must be 10; (what is the effect on the freedom available now?); add the constraint that the difference is 3 (what has happened to freedom now?). Not only can this perception of tasks be liberating, but it can also offer an approach to solution: try to express the most general possibilities at each stage, adding the constraints one by one, rather than trying to deal with all the constraints at once. This theme mirrors adolescent experience of institutions, and if made use of in mathematics, can serve as a model for how to deal with imposed constraints in the social dimension.

Doing & Undoing: Every time you find yourself 'doing something' in order to get a result, you can ask yourself whether you could go backwards from the result to the given. Put another way, what sorts of actions can be undone uniquely, or even at all? Much of the power of mathematics derives from the creative undoing of routine doing (Melzack 1983, Groetsch 1999, Gardiner 1992, 1993)

Invariance in the Midst of Change: Many mathematical theorems are statements of some invariant relationship. But invariance only makes sense and is only detectable when there is variation. So any theorem stating an invariant also has to state what is permitted to change, and in what ways. When two things are considered to be the same in some respect, then an invariance has been detected, and it is useful to consider what changes will preserve that invariance. This is what Marton (see Marton & Booth 1997) refers to as *dimensions of variation,* which Watson & Mason (2002, 2005) extended to the domain of mathematical pedagogy by referring to *dimensions of possible variation* and *ranges of permissible change*. This language is used in the course to draw teacher attention to the importance and pervasiveness of this mathematical theme.

PEDAGOGICAL STRATEGIES AND CONSTRUCTS

A number of distinction-triples are proposed in the book, as frameworks on which to hang examples drawn from personal experience which can act as reminders to choose to act in a particular way in the future. Thus the more someone can relate their own experience to a label, the more likely they are to find a relevant distinction come to mind when planning or conducting a lesson. These constructs were used to inform the writing of the book and the course.

Enactive–Iconic–Symbolic: These modes of (re)presentation identified by Bruner (1966) provide a framework-label not simply for different modes of (re)presentation, but as reminders about different worlds which people occupy at different times, and a reminder that many people learn more effectively if they are given support for working in different modes, as well as prompts to make transitions from one mode to another. Bruner paradigmatically refers to using apparatus, having it present but slightly out of reach, then present but requiring an effort to use, and finally being weaned off the apparatus altogether. This is also a paradigmatic example of scaffolding–and–fading which provides an alternative and complimentary label for access to the same awarenesses when teaching.

Do–Talk–Record: Introduced in Floyd *et al* (1981), this construct can act as a reminder that it is not enough simply that learners be ‘doing something’, but rather that what they are doing supports and promotes desire to articulate what they are doing in preparation for making written records. Forcing learners to record in symbols, or even in pictures, before having sufficient time to develop some facility in the doing and some fluency in the talking, can inhibit rather than support learning. Trying to articulate can clarify actions; trying to record can clarify both doing and articulating. All three together contribute to learning.

See–Experience–Master: In order to make more precise the notion of spiral learning, Floyd *et al* (1981) introduced these distinctions as a reminder that early encounters with an idea, technique, concept, method, way of thinking etc.. is more a matter of ‘seeing something go by’ than of ‘taking on board’. With ongoing experience of (re)encountering the same idea, perhaps in fresh contexts, its significance and utility can begin to be appreciated. Masterful proficiency cannot reasonably be expected without continued exposure and learners realising that it is indeed re-exposure to things they have encountered previously.

Manipulating–Getting-a-sense-of–Articulating: This framework-label was also introduced in Floyd *et al* (1981) to complement and precise aspects of the other frameworks. Here emphasis is on the purpose and intention of specialising: not simply to 'do some particular cases' but to 'get a sense of' underlying structure, of what is permitted to change (and in what way) and what is invariant. Hence it supports the process of generalising. It also acts as reminder that talking about what you are doing, if only in your own head, is an important contribution to the processes of assimilation (what is the same about this and other things experienced previously?) and of accommodation (what is different about this situation and what was expected based on past experience, and how can these be reconciled?).

Watch What You Do: A really useful strategy when specialising (working on particular cases in order to locate and express a generalisation) is to watch what you do as you 'do' the particular or special case. Often your body and-or your subconscious will display patterned behaviour that indicates how to generalise your actions to other cases or situations.

Say What You See: Given a picture, drawing, diagram, figure, set of exercises, an algebraic expression etc., it can be very helpful to get into the habit of rehearsing to yourself (be articulate and explicit about) what strikes you. This can be developed by getting people to say what they see in small groups, where, starting very simply, people discover details to discern that they may have overlooked, and relationships they may not have contemplated.

Structure of Attention: We have never been reticent about putting forward conjectures in our courses which have not been validated by large scale empirical studies. Quite the contrary. We have found that distinctions which we find fruitful are likely to be fruitful to others. To this end we included in both the algebra and the geometry course, suggestions about the way in which attention is structured. The idea is that if teacher and learner are attending to different things, or if they are attending to the same thing but in different ways, then there is likely to be confusion. By becoming aware of what they are attending to themselves, and how they are attending to it, teachers can be sensitised to consider what it is that learners are attending to and how. This in turn may suggest actions so as to try to bring these into closer alignment.

The structure of attention proposed in the course is based around terms which are closely related to the van Hiele levels in geometry (van Hiele 1986, Usiskin 1982) but makes use of the observation that often people experience rapid fluctuations between the different states of attention, while at other times the states seem to be quite stable. The states identified are: gazing or *holding a whole*; *discerning details*; *recognising relationships* between discerned details; *perceiving properties* of which the relationships are particular cases; and *reasoning on the basis of* agreed explicit properties, rather than simply on the basis of any properties that come to mind.

While it is never appropriate to label people with levels, it can be helpful to be aware of the structure of your own attention, so that you can take action to direct the attention of learners appropriately.

DIDACTICAL TACTICS

Didactical tactics are specific to particular mathematical topics. For example, there are a number of tactics which support the expression of generality and the uncovering of structural properties. Here is one such.

Tracking Arithmetic

A specific strategy promoted in the book is called *tracking arithmetic.* It is not 'new', but expressed as a didactical choice, it seems to strike many people as 'so obvious, why did I not think of it?'. THOANs (Think Of A Number games) provide a rich context for engaging learner interest:

> Think of a number. Add one. Multiply by 3. Subtract the number you first thought of. Add one. Divide by 2. Subtract the number you first thought of. Your answer is 2.

By undertaking to do the calculations on a specified number BUT to carry out only those operations which do not actually involve your starting number is to *track your number* through the calculations. Experience suggests that it is useful to *track* an unfamiliar number like 7 rather than a familiar number like 2 or 3 which is more likely to crop up as part of the structure. A useful side-awareness is that not all occurrences of a number have the same status: in other words, the same number can have several roles in the same expression, a structural role as well as an incidental role.

Stepping back and asking yourself what happens if someone else starts with a different number, most people can see what will be the same and what will be different about the calculations. Once fluency and facility is gained in 'seeing' the effect of tracking a particular number, it is a short step to denote the tracked number by some other symbol such as a little cloud (the number I am currently thinking of or starting with) or even a letter or acronym such as SN (starting number) or S (for Starting).

Note that this tactic makes use of a pervasively fruitful pedagogic strategy of asking yourself (inviting students to ask themselves) what is the same and what is different about two or more objects. (Brown & Coles 2000) This itself makes use of learners' powers and plunges to the heart of mathematical thinking, and particularly algebraic thinking when concerned with relationships between numbers.

INFLUENTIAL RESEARCH

Research which has influenced the development of our materials is invariably research which highlights or brings to our notice a useful distinction where previously there was little or no discernment of difference. In other words, distinctions which promote noticing and becoming aware of finer distinctions than had been made previously, or which highlights and sharpens a distinction which had been made but which had not yet been articulated. However, distinctions themselves are not enough. To be useful, a distinction has to help make sense of previous experience, and-or has to be associated with tactics or strategies which can be used in the future when the distinction comes to mind.

The professional development issue is then how to arrange that distinctions come to mind when they could be useful. In order that they come to mind in the future, there must be a sense of possibility, of appropriateness or fit with current ways of working. The issue is thereby transformed into working on ways of presenting possibilities to teachers so that they can readily imagine themselves acting in some new way in their own situation, motivated to do so because they perceive more details, more relationships than previously, and because they feel that alternative actions on their part could improve the learning of their students. The various ways that were developed in our centre for helping people have something come to mind when they wanted it to has been elaborated as the Discipline of Noticing (Mason 2002).

Research that did not have significant influence on our materials was empirical research which at best illustrated or exemplified a distinction in transcripts and other material put forward as data. Sometimes the illustrative feature could be used, but only where the distinction was considered informative and insight-generating. Empirical 'findings' on their own are not something that, in my experience, fully convince practicing teachers. The claim that some percentage of some particular subjects scored better on some test than did others is not sufficient evidence to change my teaching practices much less my way of perceiving and construing the world, unless of course the change is something I wanted to do anyway, or at least fits within my theoretical perspective (however implicit or explicit). Convincing evidence supports my intuition or challenges it in some acceptable manner. By contrast, trying to get people to act differently against their better judgement is extremely difficult, and fraught with danger because when a practice is carried through mechanically rather than generatively, it is very likely to fail. This is where education differs substantively from natural sciences: cause-and-effect does not apply to actions initiated by wilful agents such as human beings in the same way that cause-and-effect applies to mechanical actions like the tightening or loosening of a nut.

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DISTINGUISHING APPROACHES TO SOLVING TRUE/FALSE NUMBER SENTENCES[1]

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This paper focuses on eight-year old students' ways of approaching true/false number sentences. The data presented here belongs to a teaching experiment in which the use of relational thinking when solving number sentences was explicitly promoted. The study of the use of this type of thinking and of students' structure of attention allow us to make distinctions between strategies which are more and less conceptual and to provide a description of the variety of approaches available.

As a mathematic learner, when working with numeric or algebraic expressions, I have always enjoyed looking for ways to simplify the expressions before and during my manipulation of them. To me it is fun doing it and it also helps to save some work. From another point of view, as a mathematics teacher, it is quite frustrating when students embark on quite difficult computations before looking at the expressions they have to work on and getting a sense of their structure, so missing the opportunity to choose the best approach or to simplify the work to do before starting to operate. Other researchers have commented on the occurrence of this type of event at university level when working on other contexts such as integral calculus, by claiming that often students' mathematical knowledge seems to be only mechanical (Hejny, Jirotkova, and Kratochvilova, 2006).

TWO DIFFERENT APPROACHES

When working on true/false number sentences such as $257-34=257-30-4$ or $27+48-48=27$, whose design is based on some arithmetic properties, in general there are two different approaches to follow: doing the computations on both sides and comparing both results, or looking to the whole sentence, appreciating its structure and making use of relations between its elements as well as of knowledge of the structure of arithmetic to solve it (Carpenter, Franke, and Levi, 2003; Koehler, 2004; Molina and Ambrose, in press). Similarly when solving equations such as $\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1})$, students may proceed by operating on the variables and the numbers on each side as well as regrouping them, or they pay attention to its structure and appreciate that this equation is equivalent to $-x=5$ by noticing that $\frac{1}{4}-\frac{x}{x-1}$ is repeated in both sides (Hoch and Dreyfus, 2004).

We identify the first approach as what Hejny *et al* (2006) call a "procedural meta-strategy" and the second one as a "conceptual meta-strategy". The main distinction between these two types of strategies is that the first one is based on the student activating some procedures in his/her mind after having identified the area to which the problem belongs, while, in the second one, the student creates a image of the problem in his/her mind as a whole, analyzes it to find its inner structure, and looks for some key elements or relations to construct a solving strategy. While the first process leads students to become more skillful in problems of the given type, the second one leads them towards a higher level of understanding of the situation in question (Hejny *et al*, 2006)[2].

NUMBER SENTENCES AND RELATIONAL THINKING

Our interest is focused on the use of these two types of strategies when working on solving number sentences, especially sentences whose design is based on some arithmetic properties. We choose this context because of its potential for integrating the learning of arithmetic and the development of algebraic thinking. Number sentences are frequently used to introduce students to equations by drawing a figure or a line instead of a variable, as in $\_+4=5+7$ (Radford, 2000). Discussions about these equations and the properties that they may illustrate can help students to learn arithmetic with understanding and to develop a solid base for the later formal study of algebra by helping them to become aware of the structure underneath arithmetic (Carpenter *et al*, 2003; Hewitt, 1998; Kieran, 1992; Resnick, 1992). Carpenter *et al.* (2003) illustrate the potential of number sentences to work on the development of generalizations of arithmetic relations and their symbolic representation.

When students solve the sentences by using conceptual meta-strategies, we say that they are using relational thinking (a term introduced by Carpenter *et al*., 2003) or analyzing expressions (as expressed in Molina and Ambrose, in press), as their thinking makes use of relations between the elements in the sentence and relations which constitute the structure of arithmetic[2]. Students who solved number sentences by using relational thinking (RT) employ their number sense and what Slavit (1999) called "operation sense" to consider arithmetic expressions from a structural perspective rather than simply a procedural one. When using relational thinking, sentences are considered as wholes instead of as processes to carry out step by step. For example, when considering the number sentence $8+4=\_+5$ some students notice that both expressions include addition and that one of the addends on the left side, 4, is one less than the addend on the other side, 5. Noticing this relation and having an (implicit or explicit) understanding of addition properties enable students to solve this problem without having to perform the computations 8 plus 4 and 12 minus 5.

Some previous studies have provided evidence that elementary students are capable of using relational thinking when solving number sentences, overcoming some issues such as the "lack of closure" as well as an operational understanding of the equal sign (Carpenter *et al*, 2003; Koehler, 2004; Molina and Ambrose, in press, Molina, Castro and Ambrose, 2006).

In this paper we focus on analyzing the range of different ways in which this type of thinking was used by a group of eight-year old students when solving true/false number sentences. Relational thinking allows us making a finer distinction within the duality procedural-conceptual meta-strategies (Molina, 2007).

METHODOLOGY

We applied the "conjecture-driven research design", which Confrey and Lachance (2000) propose for teaching experiments aiming to investigate new instructional strategies in classroom conditions and to analyze different approaches to the content and the pedagogy of a set of mathematical topics. Our research method shared the features of design experimentation identified by Cobb and his colleagues (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003).

We worked with a group of 26 eight-year old Spanish students during six sessions over a period of one year. In this paper we will focus on the data gathered on the last four sessions as the first two were directed to exploring and extending students' understanding of the equal sign. The general aim of this research work was to study students' thinking involved in solving number sentences, in the context of whole class activities and discussion. We analyzed the strategies that students used to solve the sentences, focusing on detecting evidences of use of relational thinking.

The tasks used were number sentences, mostly true/false number sentences[3] (e.g., $72 = 56 - 14$, $7 + 7 + 9 = 14 + 9$, $10 + 4 = 4 + 10$) which were proposed to the students in written activities, in whole-class discussions and in interviews. All the sentences used were based on some arithmetic property or principle (e.g., commutative property, inverse relation of addition and subtraction, compensation relation) and, therefore, could be solved by using relational thinking.

We did not promote the learning of specific relational strategies but the development of a habit of looking for relations, trying to help students to make explicit and apply the knowledge of structural properties which they had from their previous experience with arithmetic. Students' use of relational thinking was favoured by encouragement of looking for different ways of solving the same sentence and special appreciation of explanations based on relations.

The data here presented provide evidence of diversity of approaches without paying attention to how frequently the students evidence each one[4]. The study of the use of this type of thinking and of students' structure of attention allow us to make distinctions between strategies which are more and less conceptual and to provide a description of the variety of approaches available.

STUDENTS' BEHAVIOURS WHEN SOLVING T/F NUMBER SENTENCES

We identified six different behaviours when attending at students' ways of solving the considered true/false sentences[5]. In all of them, except for the first one, we identify some use of relational thinking.

Non-RT Behaviour. Students who display this behaviour solve each sentence by obtaining the numeric values of each side and comparing them. They do not provide any evidence of having noticed any relation or characteristic in the sentence apart from the numbers in it, the operations which combines them and the presence of the equal sign. For example, Irene displays non-RT behaviour when solving the sentences gathered on Figure 1. She computed the numeric values of each side by using the standard algorithms for addition and subtraction[6].

18 – 7 = 7 – 18	**75 – 14 = 340**	**17 – 12 = 16 – 11**	**6 + 4 + 18 = 10 + 18**
False because 18 – 7 = 11 and 7 – 18 = 19 *(She computes 7 – 18 = 19 and 18 – 7 = 11 by using the standard algorithm)*	False because 75 – 14 = 61 and 14 – 75 = 49 not 340 *(She computes 75 – 14 = 61 and 14 – 75 = 49 by using the standard algorithm)*	False because 16 – 12 = 05 *(She computes 17 – 12 = 05 by using the standard algorithm)*	True because 6 + 4 + 18 = 28 and 10 + 18 = 28. *(She computes 6 + 4 + 18 = 28 and 10 + 18 = 28 by using the standard algorithm)*

Figure 1: Irene's answers to some true/false sentences. Italics are descriptions of her computations.

Simple-RT behaviour. Students displaying this behaviour solve some of the sentences by directly applying a known fact (an arithmetic law or principle) after having noticed a particular relation or characteristic in the sentence which led them to recall that fact. Students recognize in the sentence a particular case of a general fact that they know in an implicit or explicit way. Some students recall and use the properties of zero as identity element and the property "$a-a=0$". We see this strategy as a basic use of relational thinking. In other sentences they proceed as in behaviour non-RT.

For example, Jose Luis seems to show this behaviour when solving the sentences $325+0=326$ and $24-24=0$: *"It is false* [Why do you think is false?] *Because three hundreds and twenty-five plus zero is three hundreds and twenty-five, and three hundreds and twenty-six… is nothing"; "It is true because…because twenty-four minus twenty four is zero"*. We infer that he didn't do any computation as the numbers involved in the sentences don't allow easy mental computation and he provided his answers very quickly without having done any writing. In other sentences he computed the numeric values of each side and compared them.

RT-Sameness behaviour. The students displaying this behaviour solve some sentences, without making any computations, by noticing some sameness between the numbers in the sentence. They apply the reflexive property of the equality relation, the commutative property of addition, an over-generalization of the commutative property to subtraction or an overgeneralization of the reflexive property (i.e.

assuming that a sentence is true if and only if it contains repeated numbers, no matter their position). The relations that students appreciate are based on sameness or not-sameness. In other cases they may proceed as in the non-RT or simple-RT behaviours.

This behaviour shows a more elaborated use of relational thinking as it is not based on simply applying a known fact but on making flexible use of observed relations to get an answer.

Miguel shows this behaviour when solving the sentences $18-7=7-18$ and $75+23=23+75$, while solving other sentences by computing the numeric values of each side.

18 – 7 = 7 – 18	**75 – 14 = 340**	**6 + 4 + 18 = 10 + 18**	**75 + 23 = 23 + 75**
True because eighteen minus 7 and the other is the same, and if it is the same they are equal.	False because seventy-five minus 14 is not three hundreds and forty. *(He computes 75 – 14 = 51 by using the subtraction standard algorithm)*	True because 6 plus four plus eighteen is 28 and 10 plus eighteen is 28 *(He computes 6 + 4 + 18 = 28 and 10 + 18 = 28 by using the standard algorithm)*	True because they are the same and then it is the same

Figure 2: Miguel's answers to some true/false sentences. Italics are descriptions of his computations.

<u>One-shot-RT, frequent-RT and all-RT behaviours.</u> These three behaviours correspond to the cases when students solve some sentences by using relational thinking, making use of a pair of distinctions or relations such as sameness between the numbers in the sentence, difference of magnitude between those numbers, a number fact contained in the sentence or some numeric relations between the numbers. Sometimes they also apply some knowledge of the effect of operations on numbers. Table 1 shows some student's explanations which evidence the use of relational thinking based on pairs of the mentioned elements.

Elements at the base of the RT used	**Examples : Student's explanations**
Sameness between numbers in the sentence and knowledge about the effect of operations	In 122 + 35 – 35 = 122: "True because if we add 122 to 35 and we take it away, it is as if we don't add anything"
Number fact contained in the sentence and sameness	In 7 + 7 + 9 = 14 + 9: "True. I did it by adding seven and seven…. which is fourteen. The same than there

between numbers	[right side]". Nine, the same than there [right side] too.
Numeric relations between numbers in the sentence and sameness between numbers	In 13 + 11 = 12 + 12: "True because you subtract one to the twelve and you give it to the other twelve, and you get what it is there [left side]"
Differences of magnitude between numbers and knowledge about the effect of operations	In 75 – 14 = 340: "False because 75 minus 14 is less, it can not be a bigger number"
Numeric relations between numbers in the sentence and knowledge about the effect of operations	In 11 – 6 = 10 – 5: "True because if eleven is higher than ten and you subtract one more than five, you get the same"

Table 1: Examples of students' explanations which provide evidence of use of relational thinking based on various observations and arithmetic knowledge

We distinguish between the behaviours "One-shot-RT", "frequent-RT" and "all-RT", depending on the variety of ways in which students use relational thinking, according to the elements at the base of their thinking from those indicated in Table 1. These behaviours do not differ in the way students proceed when solving the sentences but in the diversity of ways of using relational thinking. In behaviour "One-shot-RT", students provide evidence of having use relational thinking based on just one of the above referred elements. "Frequent-RT" behaviour corresponds to those cases in which the students solve a variety of sentences by using relational thinking based on several but not all the elements in Table 1. The behaviour "all-RT" refers to the cases in which the child provide evidence of solving different sentences by using relational thinking based on all those elements.

Therefore, each one of these six behaviours includes the previous ones as shown in Figure 3, but it is characterized by ways of solving number sentences not included in previous behaviours or by the variety of ways in which relational thinking is used. However, this doesn't mean that, for example, all students displaying RT-sameness behaviour do also provide answers as those from non-RT or single-RT behaviour. The inclusion relation that we are highlighting expresses that it happens in some cases.

In all except the "Non-RT behaviour", we distinguish two different ways in which students invoke relational thinking. In some cases they start by looking at the sentence, making distinctions and noticing some relationship which they used to solve the sentence, following what Hejny *et al.* called a "conceptual meta-strategy". In other cases they start by computing some of the operations involved in the sentences, and during the computation process, they notice a special characteristic of the sentence or a relation between its elements which leads them to change their approach and to solve the sentence without computing the numeric values of both sides. This change of approach can be recognize in the following student's answer to the sentence 51 + 51 = 50 + 52: *"True because as fifty-one plus fifty-one is one hundreds and two, but if you subtract* [one] *from fifty-one, fifty, you can add to fifty-one from the other, one more, and you get fifty-two…and you get fifty plus fifty-two".* In other cases we identify this change of approach by comparing the students´ notes with their explanation.

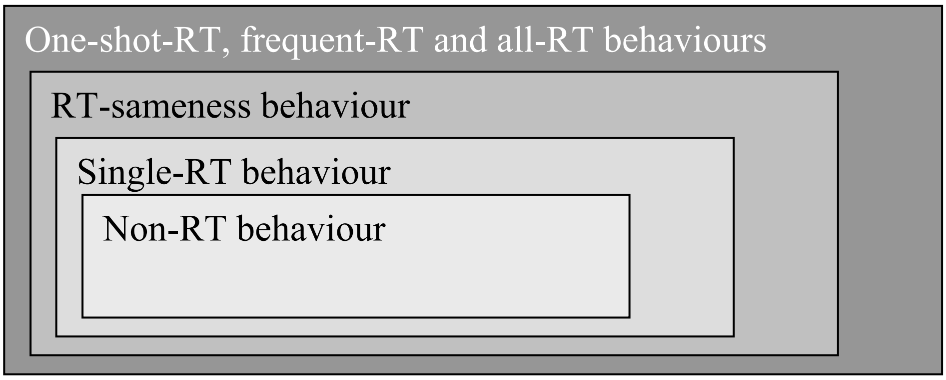

Figure 3: Relationship between the identified students' behaviors when true/false solving number sentences.

DISCUSSION

The different behaviours suggest a variety of ways in which students approach solving true/false number sentences whose design is based on arithmetic properties and relations, as well as a variety of different students' structures of attention when working with the sentences (Mason and Johnston-Wilder, 2004). The behaviours differ in the way students' pay attention to the sentences (initially and during the solving process), the distinctions they make within them, the relations that they display evidence of noticing and the arithmetic knowledge that those distinctions and relations trigger in the students' mind.

Some students consider sentences and expressions with more than two terms in a global way (as a whole) and look across the equal sign as well as within each side to make distinctions and to establish relationships between elements. The way they use these relations is influenced by their awareness of the structure of the sentence (e.g., the equal sign differentiate two sides in the sentence) as well as their knowledge of arithmetic structure (e.g., inverse relation of addition and subtraction, commutative property of addition). Other students proceed to do the computations, apparently paying attention only to the numbers involved and the operations to perform on them, considering each side or even each operation separately. Other students' attention, while initially being placed on doing the computations, fluctuates between numbers, partial results and elements of the sentence. This fluctuation cues to become aware of characteristics or relations within the sentence not previously noticed.

Looking for evidence of the use of relational thinking in the students' solving strategies promotes appreciation of a range of strategies between procedural and conceptual meta-strategies. Considering these distinctions, the appreciated differences between the structures of students' attention provide a point of entrance to allow teachers to help students to develop more conceptual approaches. Some ways could be encouraging students to look at the sentence before computing and formulating questions which draw children's attention to the sameness or differences of some elements in the sentence as well as to look for relations between the terms. Discussions of students' various approaches in which the use of relational thinking is encouraged seem also to be effective.

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NOTES

1 This study has been developed within a Spanish national project of Research, Development and Innovation, identified by the code SEJ2006-09056, financed by the Spanish Ministry of Sciences and Technology and FEDER funds.

[2] Within the context of problem solving, the use of conceptual meta-strategies may be considered to be related to an element of heuristic competence referred as "internal monitor" (Mason, 1985) or "instructed manager" (Puig, 1996). This element include various capacities such as examining possible ways of approaching a problem before addressing its resolution in order to make an informed choice of a solving strategy or keeping an eye on calculations to make sure that they remain relevant to the question. However, the distinction we want to make when distinguishing conceptual and procedural meta-strategies, it is not that the person does a informed choice of a strategy but that he/she uses mathematical relationships (or the mathematical structure of the situation) when constructing the strategy.

[3] In a previous study (Molina and Ambrose, 2006) we appreciated that true/false number sentences, unlike open number sentences, contribute to break the students´ computational mindset as the students don't need to provide a numerical answer. Therefore, this type of sentences eases the consideration of number sentences as wholes and the use of relational thinking in its resolution.

[4] For further description and information about this teaching experiment see Molina (2007), available at http://cumbia.ath.cx:591/pna/Archivos/MolinaM07-2822.PDF.

[5] The examples provided have been translated from Spanish to English.

[6] This student appreciates some structure in the sentence 18 – 7 = 7 – 18 which she applies when solving the sentence 75 – 14 = 340. However, she does not provide any evidence of having use relational thinking for solving any of the sentences in Figure 1.

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STUDENT DIFFICULTIES IN UNDERSTANDING THE DIFFERENCE BETWEEN ALGEBRAIC EXPRESSIONS AND THE CONCEPT OF LINEAR EQUATION

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Algebraic expressions and equations have been an essential part of the history of mathematics and of the secondary school mathematics curriculum. In this study, four ninth grade students were interviewed in an attempt to examine students' understanding of the difference between algebraic expressions and the concept of linear equation. Although these students were able to simplify algebraic expressions and solve linear equations with ease, the data show that they still had difficulties indicating a relation between algebraic expressions and linear equations.

HISTORICAL AND CURRICULAR IMPORTANCE

Expressions and equations have been a vital part of the history of mathematics. Starting with the ancient Egyptians and Babylonians about 3000 years ago, rhetorical algebra was used in the form of words to solve linear equations. In the 3rd century AD the Greek mathematician Diophantus was the first to use abbreviated words for algebraic expressions, giving rise to "syncopated algebra" (Sfard, 1995, p. 18). In 830AD, the Persian mathematician Al-Khwarizmi wrote *al-Kitab al-mukhtasar fi hisab al-jabr wa'l-muqabala* or *The Compendious Book on Calculation by Completion and Balancing*. In this book, the term al-muqabala, which translates to *balancing*, refers to the process of combining like terms when simplifying an equation. Centuries later, and specifically in 1591, Francois Viete wrote an algebra book which formally gave rise to symbolic algebra (Sfard, 1995). Following, in 1830, the British mathematician George Peacock proposed that in algebra letters replace numbers and that, in general, algebra was arithmetic using symbols (Sfard, 1995).

Algebraic expressions and equations have also been a significant part of secondary school mathematics curricula. Particular to the United States mathematics school curriculum, the National Council of Teachers of Mathematics (NCTM) published the *Principles and Standards for School Mathematics,* in which it is recommended that a curriculum be "focused on important mathematics" (2000, p. 14). The Algebra Standard for grades 6-8 states that students "use symbolic algebra to represent situations and to solve problems … that involve linear relationships" and that they "recognize and generate equivalent forms for simple algebraic expressions and solve linear equations" (2000, p. 222). Moving on to high school, the Algebra Standard for grades 9-12 suggests that students "understand the meaning of equivalent forms of expressions, equations, …" and that they "write equivalent forms of equations" (2000, p. 296).

LITERATURE REVIEW

Student difficulties with algebraic expressions

A substantial amount of research has been carried out indicating student difficulties with algebraic expressions. The research shows that students should gain experience with modelling situations using expressions and producing equivalent expressions before they are introduced to the concept of equation. Otherwise, they will view the equal sign as merely a signal to "do something" and not as a symbol of equivalence and balance (Kieran, 1990).

The available literature also indicates students' difficulties with recognizing and using the structure of algebraic expressions (Kieran, 1989a; 1990). According to Kieran (1989a), the *surface structure* of an algebraic expression refers to "the given form or arrangement of the terms and operations, subject … to the constraints of the order of operations" (p. 34). That is, the surface structure of the expression 6+2(x+5) consists of taking 6 and to that adding the multiplication of 2 by x+5. On the other hand, the *systemic structure* of an algebraic expression refers to the operational properties of commutativity, associativity, and distributivity which would permit one to express 6+2(x+5) as 2(x+5)+6.

A study by Tall and Thomas (1991) identified four obstacles to making sense of algebraic expressions. First, many students tend to interpret expressions such as 3+4k as 7k. This, which the authors call the *parsing obstacle*, arises because of the way we read from left to right. From prior experiences with arithmetic students expect to perform some calculation when they encounter an operation sign such as +. So, when faced with algebraic expressions such as 3+4k they expect to produce an answer. This is what the authors call the *expected answer obstacle*. Related to this issue, is the *lack of closure obstacle* which refers to students' view of 3+4k as an incomplete answer. Last, the authors identified the *process-product obstacle* which refers to the inability of students to view algebraic expressions as having a dual nature; that of a *process* and of a *product*. For example, an expression such as 3+4k indicates both the instructions to perform a calculation (process) and it is also the result of such a calculation when a value is not assigned to the variable (product). If an algebraic expression is only viewed as a process then "the powerful way in which it can be manipulated and linked to other expressions makes little sense and failure with algebra becomes inevitable" (French, 2002, p. 16).

Student difficulties with linear equations

Various studies have also been conducted that looked at student difficulties when dealing with the concept of equation. Equations are defined as open number sentences consisting of two expressions which are set equal to one another. This is what Kieran (1989a) calls the *surface structure* of an equation and it is an aspect that students find challenging to recognize. In addition, students have trouble recognizing the *systemic structure* of an equation which includes the equivalent forms of the two expressions given in the equation (p. 34). Kieran (1989a) claims that students who "view the right-hand side of an equation as the answer and who prefer to solve equations by transposing," lack an understanding of the balance between the right and left hand sides of the equation (p. 52). Moreover, Kieran (1990) found that many algebra students "could not assign meaning to a in the expression a+3 because the expression lacked an equal sign and right-hand member" (p. 104). Relating to this, a 1984 study by Wagner, Rachlin, and Jensen found that students added "=0" to any expression they were asked to simplify (as cited in Kieran, 1990).

Students also face difficulties when working with equivalent equations. In a study by Steinberg et al. (1991) participants were given pairs of equations and were asked to identify whether the equations in each pair were equivalent. The data showed that many students could not distinguish between expressions such as 3x and 3+x and some thought that subtracting a number from both sides of an equation would alter the answer because "-4 on each side is subtracting 4 twice" (1991, p. 117).

Furthermore, a study by Hall (2002) examined the errors that secondary school students make when attempting to solve simple linear equations. The results showed that many students "find the process of collecting "like" terms so difficult that they cannot confidently simplify an expression such as 3x+2x" (p. 46). Moreover, Hall reports that some students have difficulties combining "like" terms in expressions such as "3x+2y+4x" which involve "unlike" terms within the expression (p. 46).

PURPOSE OF THIS STUDY

This study attempts to address students' understanding of the differences between algebraic expressions and the concept of linear equation. The reason for carrying out this study arose after a close reading of the available literature relating to the two concepts. Since algebraic expressions and linear equations are essential in the mathematics curriculum and since the literature has shown that students have difficulties recognizing the structure of both of these concepts, as well as making sense of them and working with them, this led to question whether the problem lies in that students might not see a relation or any differences between the two concepts. Furthermore, since mathematics curricula use a specific vocabulary associated with each of these concepts, it seemed possible that this vocabulary might have an effect on students' understanding of the difference between algebraic expressions and linear equations. Thus, the need arose to study the way that students make sense of this vocabulary.

Since school algebra in the United States curriculum formally begins either in the eighth or ninth grade, ninth grade students were asked to participate in this study in the attempt to make sure that the participants had instruction in algebraic expressions and linear equations prior to this study.

RESEARCH QUESTIONS

Specifically, the following questions are addressed in this study:

1) What difficulties do 9^{th} grade students have with identifying differences between algebraic expressions and linear equations even when they have the ability to simplify expressions and solve linear equations?

2) What sense do 9^{th} grade students make of the various verbs (such as 'simplify' and 'solve') and nouns (such as 'solution', 'value', and 'variable') associated with algebraic expressions and linear equations?

DATA SOURCES

Participants

Four fifteen-year old ninth grade students participated in the study. All students received classroom instruction on simplifying algebraic expressions and solving linear equations prior to the study. Two of them, Ellen and Tim, were enrolled in an algebra class during the time of the study whereas the other two, Kathy and Susan, took algebra in grade eight and were at the time enrolled in a geometry class. Ellen and Kathy attend the same high school in a middle-sized city in the state of Michigan, Tim attends a different high-school in that same city, and Susan attends high school in a metropolitan area in Michigan. These students were chosen because they represent a variety of high schools, had different teachers, used different curricula in their algebra classrooms, and they were all of high mathematical ability.

Data Collection

The students were observed as they worked with various mathematical tasks relating to algebraic expressions and linear equations. The interviews were audio taped and later transcribed. The students were asked to provide written responses to the items on the instrument. In addition, they were asked to talk about their thinking and the processes they used to respond to the items. In some cases, field notes were also taken. In particular, Activity 1 (Instrument section) required the students to sort cards on which either an expression or linear equation was written. Since this task did not involve providing a written solution, field notes helped in keeping record of the student actions while performing the task and the result of their actions.

Instrument

Overall, the students worked with five types of tasks, organized under four activities: Task 1) sorting cards on which an algebraic expression or linear equation was written (Activity 1); Task 2) observing common characteristics of and identifying differences between algebraic expressions and linear equations. (Activities 2 and 4); Task 3) combining like terms/simplifying expressions (Activity 3); Task 4) solving linear equations (Activity 3); Task 5) producing algebraic expressions and linear equations (Activity 3).

Activity 1: Sorting cards

Twelve index cards are provided. On each index card I have written something. I will place down the first two cards (cards 1 and 2). Sort the remaining ten cards according to what you see as common among them. If a phrase is written on a card, sort according to what the phrase would result in if written mathematically.

$3x + 2$ 1	$x + 10 = 5$ 2	add 3 to 5a 3	$6 + n$ 4
$z + 5 = 19$ 5	y is 10 more than x 6	$n + \bigcirc\bigcirc = \bigcirc\bigcirc\bigcirc\bigcirc$ 7	$25 + \square = 142 + \square + \square$ 8
$\frac{1}{2}x + \frac{1}{4}y = 26 + 22$ 9	$\frac{1}{2}p + \frac{1}{2}q + \frac{1}{8}r$ 10	$4(6r - 3s) + 5r$ 11	$x + y + z = x + p + z$ 12

Activity 2: Characteristics of algebraic expressions and equations

Why did you sort the cards the way you did? What do you observe that the cards in each of the piles you sorted them into have in common? What do they have different?

Activity 3:

1) Simplify $2x - 5 + 11x - 41$

2) Solve $2x - 5 = 11x - 41$

3) Simplify $2x - 5 + 9y + 11x + 41 + 6y$

4) Solve $2x - 5 + 9y = 11x + 41 + 6y$

5) Combine like terms: $14z + 134 + 3z + 3(12z + 22 + 9z)$

6) Solve $14z + 134 = 3z + 3(12z + 22 + 9z)$

7) Give an equation that has 2 as a solution

8) Give an expression such that when x is 3 the value of the expression is 0

9) Give an expression such that when x is 0 the value of the expression is 3

Activity 4:

1) What is/are the differences between expressions and equations?

Mathematics curricula usually provide students with problems such as those found in Activity 3, asking students to either solve an equation or simplify an expression. Such problems are used in this study in order to identify whether the participants have the ability to simplify algebraic expressions and solve linear equations, thus accommodating for the condition set in the first research question. In addition, the tasks in Activity 3 will provide insight into what sense students make of words and phrases such as 'solve', 'simplify', 'combine like terms', 'expression', 'equation', 'solution', 'value is', and 'x' thus helping in gaining insight relating to the second research question. Activities 1, 2 and 4 are not commonly asked of students. These activities are used in the study in order to help address the first research question. In particular, Activity 1 is meant to indicate whether students can distinguish between expressions and equations. Activity 2 is used as a follow-up to Activity 1 in order to gain insight into whether students identify any differences or similarities between expressions and equations thus directly addressing the first research question.

METHOD OF ANALYSIS

In attempting to address the research questions, the focus was on how and to what extent the students used the terms *expression* and *equation* during the interview. Attention was also paid as to whether the students made a distinction between the two terms. In relation to the literature, attention was given as to whether the students recognized and used the *surface structure* and *systemic structure* of an expression and/or an equation as defined by Kieran while attempting to respond to the mathematics problems they were given. Moreover, the actions that the students took when encountering the terms *expression* and *equation* were observed. That is, did they immediately perform any particular procedure when they viewed an expression or an equation? Did the students associate the verb *simplify* with expressions? Does the word *solution* lead students to think that they must be asked to deal with an equation? Furthermore, attention was paid as to whether the students were able to correctly simplify algebraic expressions and solve linear equations given to them in symbolic form.

FINDINGS AND DISCUSSION

Overall, all four of the students simplified algebraic expressions and solved linear equations with one unknown given to them in symbolic form with ease, arriving at the correct result. When faced with the instruction *simplify* all students combined like terms. During the interview, all students mentioned that *simplifying* and *combining like terms* is one and the same thing. However, they all mentioned the term *simplify* in relation to algebraic expressions and none of them related the term to equations. When the students were faced with the instruction *solve* (Activity 3 tasks 2, 4, 6) three of them (Ellen, Kathy, and Tim) performed a *do the same thing to both sides* procedure indicating that they viewed the equal sign as a balance sign. Susan was the only student who followed the procedure *move the unknowns on one side and the numbers on the other*.

The table on the next page shows the way that each student sorted the cards in Activity 1. The *Intended Sorting* column indicated in the table is the sorting of cards that, as judged by the author, would have indicated that students observe a difference between expressions and equations and that would, furthermore, specify that the students view the equal sign as part of the structure of equations.

Intended Sorting		Ellen		Tim			Kathy		Susan	
1	2	1	2		1	3	1	2	1	2
3	5	3	5	9	2	6	3	5	3	5
4	6	4	7	10	4	7	4	7	4	7
10	7	6	8		5	8	6	8	6	8
11	8	10	9		11	12	10	9	10	9
	9	11	12				11	12	11	12
	12									

Table 1: Results of the task on sorting cards (Activity 1)

Notice that the sorting performed by Ellen, Kathy, and Susan is identical whereas Tim chose to sort the cards in a way that he thought was more appropriate. As Tim said, he placed cards 9 and 10 on the same pile by themselves because "they were the only ones that contained fractions". He then placed cards 1, 2, 4, 5, and 11 on the same pile because "they contained either one or in the case of card 11, two variables. So, they had a small number of variables." As he indicated, he chose to place cards 3, 6, 7, 8, and 12 together because they "have more words or shapes like circles or they have many variables". It is interesting to note Tim's conception of v*ariable*. He seems to associate *variable* with *letter* denoting an unknown whether that unknown is a constant (cards 2 and 5) or can vary in value (card 1). So, according to Tim, the number of letters in an expression or equation denotes the number of variables. However, Tim's conception of variable falls short when considering cards 7 and 8. Tim seems not to view shapes (circles and squares) as place holders for an unknown. If he did, then he would likely have realised that a single letter could be used to denote the unknown on cards 7 and 8 and would have placed cards 7 and 8 in the same pile as cards 1, 2, 4, 5, and 11.

When Ellen, Kathy, and Susan were asked about the differences in the two piles in which they sorted the cards (Activity 2), they all indicated that the cards on the right pile had an equal sign whereas those on the left did not. Also, these three students identified the cards on the right pile as "equations". However, they could not remember the term "expressions" when asked what name they would give to the cards on the left pile. This however, does not indicate a lack of understanding from the part of the students of the differences between expressions and equations. None of the students though indicated that equations are a pair of expressions set equal to one another by the equal sign. Thus, none of the students identified the surface structure of an equation.

When students were asked to produce expressions and linear equations in Activity 2 tasks 7-9 their responses were as follows:

Student	7)Give an equation that has 2 as a solution	8)Give an expression such that when x is 3 the value of the expression is 0	9)Give an expression such that when x is 0 the value of the expression is 3
Ellen	1 + x = 2	0 x = 0	x + 3 = 3
Tim	1 + 1 = 2	x = -3	x = 3
Kathy	12 – x = 2	3 – x = 0	3x = 3
Susan	10 ÷ x = 2	x – 3	x + 3

Table 2: Results of Activity 2 Items 7-9

The only student who made a distinction between expressions and equations in the above set of tasks was Susan who provided correct algebraic expressions in tasks 8 and 9. During the interview Susan was asked about her choice of equation she gave as a response to task 7. At that time she corrected her response to task 7 to be 10÷x=5. Ellen on the other hand did not distinguish between the terms *equation* and *expression* in the above items and instead provided equations in all three cases. During the interview, Ellen specified that it was because of the words ‘value is’ that she associated the last two tasks with equations.

When students were asked in Activity 4 to describe what, according to them, are the differences between expressions and equations they gave the following responses:

Ellen: Equations have an equal sign and variables. Something equals something else. Expressions don’t have an equal sign.

Tim: Expression is like giving a certain number to a variable, I think. And then equation is a whole bunch of numbers added together kind of a thing.

Kathy: I think equations don’t have variables and expressions do. When you have an expression and you simplify you don’t find the exact answer, but you just kind of add like terms together. But when you solve an equation you find the exact answer.

Susan: Expressions don’t have the equal sign. They have a variable like x.

From the above comments, one is able to see that Ellen identifies the surface structure of equations (“something equals something else” and “equations have an equal sign”). However, Ellen’s response to activity 4 makes her responses to Activity 3 tasks 7-9 problematic. Ellen recognizes that expressions do not have an equal sign.

Yet, when asked to provide expressions in tasks 8 and 9 she gives an algebraic sentence that indeed contains an equal sign. Susan also identifies that expressions as opposed to equations do not have an equal sign. However, Susan, Tim, and Kathy were unable to view equations as being made up of two expressions. Kathy's comments are particularly interesting. She seems to be viewing variables as obstacles to arriving to the "exact answer" but does not seem to reject expressions as answers to problems. Kathy seems to be more of an 'action person' since she refers to the actions or procedures she would use when faced with expressions or equations. Kathy clearly identifies that the noun *expression* is related to the verb *simplify* and the noun *equation* is related to the verb *solve*. None of these students seemed to face any of the obstacles suggested by Tall and Thomas. They all accepted expressions as answers to problems and did not view algebraic expressions as incomplete answers.

CONCLUSION

Ellen's association of "value is" with equations makes one realize that language plays a vital role in student performance and understanding. Student confusion may not arise because of student misconceptions or difficulties with using mathematical concepts but perhaps because of the language they face. Teachers should be extremely careful with interchangeably using mathematical terms in the classroom such as *expression* and *equation*. If they are not careful with language use then students might come to believe that the two concepts are actually the same and thus, not be able to distinguish between them. Also, the inability of the students to recall the term *expression* accompanied by their ability to spontaneously recall the term *equation* when asked to provide a general name for the piles in which they sorted their cards in Activity 1, seems to suggest that perhaps the term *equation* is more frequently used in the mathematics classroom. Teachers should use the term *expression* more frequently even after having finished instruction on expressions. Students should be reminded of the term *expression* while they receive classroom instruction on equations so that they acquire the correct mathematical vocabulary.

In general, the data collected in this study has shown that students associate the term *simplify* with algebraic expressions and the term *solve* with equations. Although students may not have difficulties manipulating, and simplifying algebraic expressions and solving linear equations, they may still not be able to identify the difference between the two and may find it very challenging to generate expressions or equations. Thus, instruction on linear equations should include identifying the difference between algebraic expressions and linear equations so that students can make connections between concepts they learn in their mathematics classrooms.

With consideration to the above, it is important to analyze the effects of teaching on the students' ability to recognize the differences between algebraic expressions and equations. Due to time constraints, classroom observations were not carried out in this study. However, classroom observations of teachers teaching the units on algebraic expressions and linear equations along with interviews with their respective students as they respond to the instrument in this study would help clarify possible sources of students' difficulties even further.

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TEACHERS' PRACTICES WITH SPREADSHEETS AND THE DEVELOPMENT OF ALGEBRAIC ACTIVITY

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This paper reports on one aspect of a longitudinal study which seeks insight into the ways in which spreadsheet experience and teachers' pedagogic strategies shape pupils' construction of meaning for algebra. It offers a categorisation of what teachers actually do when using spreadsheets in their teaching, and discusses how these practices relate to pupils' construction of meaning for algebra. Five practices are identified: demonstrating; developing socio-mathematical norms; reflecting on activity; focusing attention on meaning; and legitimising algebraic activity. These practices are illustrated with extracts from lessons.

ALGEBRAIC THINKING/ACTIVITY AND SPREADSHEETS

Research suggests that spreadsheets can support the learning and teaching of algebra. In a spreadsheet a cell reference refers to the particular number in a cell, and any number that may be entered into that cell, as well as describing its physical location. Spreadsheets also have the facility for filling down a formula through a range of cells; hence the column can also be seen as representing a variable. In a longitudinal study of two groups of 10-11 year old pupils working on traditional problems, Sutherland and Rojano (1993) conclude that 'a spreadsheet helps pupils explore, express and formalise their informal ideas' (p.380), moving from thinking with specific numbers to symbolising a general rule. The use of algebra-like notation and the activities of writing formulae and graphing are typically cited as offering access to the meaning of symbolic notation and to algebraic activity.

Several researchers point to the important role of the teacher in guiding pupils' construction of meaning when working with such technological tools (for example Dettori et al, 1995). The practice of what teachers actually do, however, has received relatively little attention in the literature. Monaghan (2004), for example, notes that

> 'research focused on teachers' practices in mathematics classes is a relatively recent phenomenon and what there is to date has largely focused on forms of knowledge and on beliefs, with little attention to the whole experience of using technology in the classroom' (p.328).

Hennessy, Deaney and Ruthven (2005) draw on socio-cultural theories of mediation and guided participation to identify pedagogic strategies for mediating pupil interactions with technology in secondary schools. Although mathematics teaching was not a specific focus of the research, the typology of strategies points to the need to maintain focus on subject discourse and to the 'underdeveloped' role of whole class interactive teaching using technology. In terms of technology in mathematics education, teachers' practices have been discussed in relation to the instrumental approach, considered further below.

The instrumental approach (described in Trouche, 2004), is a theoretical framework that offers a way of describing how an artefact such as a spreadsheet becomes an instrument. Trouche describes an instrument, which includes a psychological component related to how an artefact is used, as:

> 'the result of a construction by a subject, in a community of practice, on the basis of a given artefact, through a process, the instrumental genesis' (p.289).

The process of instrumental genesis, in which an artefact is appropriated by a learner, consists of two combined processes: instrumentation and instrumentalisation. The former refers to the way in which an artefact such as a spreadsheet influences the learner, allowing them to engage in mathematical activity using the artefact. The latter is directed towards the artefact, and refers to transforming the artefact to become a mathematical tool. Trouche introduces the term 'instrumental orchestration' to describe the 'external steering of students' instrumental genesis' (p.296). Exploratory research by Haspekian (2005) within this frame indicates the complexity of such processes (see also Artigue, 2002). The ways in which teachers orchestrate, or guide their pupils' construction of meaning is the focus of this paper.

DATA COLLECTION AND ANALYSIS

This study builds upon the Purposeful Algebraic Activity project[1] which aimed to explore the potential of spreadsheets as tools in the introduction to algebra and algebraic thinking. The project involved the design of purposeful spreadsheet-based tasks (see Ainley, Bills and Wilson, 2005a; Ainley, Bills and Wilson, 2005b) which formed the basis of a teaching programme for pupils in the first year of secondary school (aged 11-12). Five classes participated in the project, spending approximately twelve hours of spreadsheet-based activity within their mathematics curriculum over the course of the year. Four experienced teachers, who collaborated with the project team throughout, taught all of the lessons to their usual classes. In the schools involved, pupils were divided into four mathematics sets by attainment. In one school Graham taught set 1 and Judith taught two sets, set 3 and set 4. In another school Anne taught set 1 and Rachel taught set 4.

This research seeks insight into the ways in which spreadsheet experience and teachers' pedagogic strategies shape pupils' construction of meanings for algebra. It draws upon data from the Purposeful Algebraic Activity project, but focuses more closely upon the role of the teacher in order to gain further insights into the social construction of meaning. This paper reports on one aspect of the longitudinal study; the practices of the teachers. It attempts to address the question 'When working with spreadsheets, how (if at all) do teachers guide pupils' construction of meaning for algebra?' Although the tasks had been designed by the research team in collaboration with the participants, the four teachers had the freedom to use the materials as they wished. For example, the teachers made decisions about how to introduce tasks, when to intervene, how to respond to pupils seeking help and how (and whether) to orchestrate plenary discussions.

A range of data was collected from each of the teaching programme lessons, including audio recordings from a radio microphone of teachers' interactions with pupils. These interactions included work with the whole class, often incorporating the use of an interactive whiteboard, and discussions with individuals, pairs and small groups when circulating the room. Field notes were also made with a view to recording what happened; these included chronological jottings of non-verbal activity. In light of the broader aims of the study, other sources of data included video and screen recordings from a targeted pair of pupils, and semi-structured interviews with targeted pairs over a two to three year period.

The audio data from the radio microphone was semi-transcribed, focusing on the development of mathematical meaning rather than general classroom management activities such as organising groupings and managing behaviour. Using a broadly grounded approach (Strauss and Corbin, 1998) NVivo software was used to develop and refine coding of the teacher transcripts. The analysis focused on how the teachers guided pupils' construction of meaning for algebra; it was informed by the ideas of instrumental orchestration (Trouche, 2004) and the role of the teacher in semiotic mediation (Mariotti, 2002). The analysis involved identifying commonalities in the practices of the teachers specifically in relation to constructing meaning for algebra. Essentially, the analysis sought overarching themes beyond the specifics of the task at hand. In total, over five hundred and fifty passages were coded from the sixty-six lessons, and five themes, or practices, emerged.

CHARACTERISING FIVE PRACTICES

The categorisation offers a broad description of what teachers do which, to varying degrees and in different ways, guides pupils' construction of meaning. The five practices offer a way of characterising the complexity of what teachers actually do when working with technology.

Practice	Definition
Demonstrating	Demonstrating or supporting a local process, such as writing a spreadsheet formula or constructing a graph.
Developing socio-mathematical norms	Establishing expectations or fostering emerging norms through verbal interactions and computer-based activity.
Reflecting on activity	Reflecting on a local process at a global level in terms of the value of the local process and/or the value of the local process in the context of the task at hand.
Focusing attention on	Focusing attention on the meaning of an idea or an image,

meaning	such as the variable cell or the use of notation.
Legitimising algebraic activity	Legitimising or validating mathematical activity, such as using algebraic conventions. Making links to 'algebra.'

Within a particular episode, such as a teacher-pupil interaction or teacher-class interaction, various passages were coded. Most passages fell clearly within one of the five practices; others overlapped two or more. Some episodes consisted of various passages which spanned a range of practices. For example, a teacher may demonstrate a local process, focus pupils' attention on the notation used, reflect upon why the technique is useful in the context of solving the problem, and then develop the sociomathematical norm that she expects the class to use that process. It is acknowledged that the interaction between practices is very powerful, but for the sake of clarity, each of the five practices is outlined below and illustrated with extracts from the lessons. In considering each of the practices, commonalities and differences between the teachers are described.

Demonstrating

This practice involves demonstrating or supporting a local process, such as how to construct a spreadsheet formula, how to drag down a formula or how to construct a scatter graph. It is parallel to the strategy that Anghileri (2002) describes as 'showing and telling.' Teachers showed pupils processes, often using the interactive whiteboard or projected image to demonstrate a series of technical steps, which may involve strong mathematical content. In many cases the technical aspects were interwoven with commentary which aimed to draw pupils' attention to particular aspects, such as the need to use certain notation.

When classes were engaged in spreadsheet activity the teachers responded to pupil-initiated questions and made interventions. Teachers used such opportunities to reinforce processes that had been previously demonstrated and to teach new processes, as they interacted with individual pupils. With her lower attaining set in particular, Judith supported pupils in writing a formula by encouraging them to express a calculation for a particular number or explain in words and then 'tell the computer.'

Ashley I would add um fifteen twenties

Judith Okay, fifteen twenties, so how do we tell the computer to do that? (task 4, Mobile Phones, lesson 1)

In plenaries and mini-plenaries during the lesson, demonstration was also used to consolidate and extend pupils' understanding of how a spreadsheet can be used. Over two hundred passages were coded as instances of teachers demonstrating a process. Given the nature of the practice, it is unsurprising that more demonstration occurred in the earlier tasks. In terms of instrumental genesis, the practice of demonstrating supports the development of instrumentation, whereby techniques are increasingly regarded as an appropriate response to a given task.

Developing sociomathematical norms

Developing sociomathematical norms, a practice described by Yackel and Cobb (1996) and Hershkowitz and Schwarz (1999) includes establishing expectations and fostering emerging norms. The kinds of norms developed included normative routines for using the spreadsheet related to the layout, the process of writing a formula, and the methods used to solve problems. In addition to the procedural aspects of activity, normative ways to engage with and think about notation were also developed during the teaching programme.

Over one hundred and eighty passages were coded as developing sociomathematical norms, and across all of the classes this practice was most prominent in the first task. This practice declined (with one slight exception) over the course of the teaching programme. Whilst all of the teachers actively established norms for spreadsheet activity with their classes, these were reinforced more with the two lower attaining classes where the number of passages coded was approximately double that of each of the other classes. Two common norms were strongly established in relation to formalising rules and generating data: writing a formula and filling down a formula. Normative ways of entering a cell reference in a formula were established by three teachers. Both Judith and Graham encouraged pupils to click on the cell reference rather than type it, whereas Rachel emphasised strongly the idea of telling the computer where to find the number:

> 'You've got to tell the computer "where to find the number to calculate with, and to do that, you give it its "cell reference with the letter first and then the number' (Rachel, task 1, Multiplication Tables, lesson 1)

Thus, whilst there were commonalities in the practice of developing norms, the actual norms and emphasis varied to some degree.

Reflecting on activity

Reflecting on activity involves considering the value of a local process either at a global level or in relation to the context of the task. Whereas the practice of demonstrating introduces pupils to and supports them with carrying out processes, and developing sociomathematical norms encourages pupils to interact with the spreadsheet in certain ways, the practice of reflecting relates to the usefulness of such activity. It includes teachers meta-commenting about why particular processes are useful as well as inviting pupils to interpret data and graphs.

All of the teachers engaged in the practice of reflecting, with over one hundred and eighty passages coded and a number of themes emerging in relation to usefulness. A theme that Judith and Rachel referred to frequently was the value of the spreadsheet in performing a calculation. A second theme relates to the efficiency of using spreadsheet formulae beyond that of performing a calculation. Within this theme are examples of teachers referring to changing the number in a cell and to filling down a formula:

> 'It'll be a lot quicker if you use the formulas … you've just gotta change these numbers and you can see immediately what the total is' (Anne, task 6, Fairground Game, lesson 1)

> 'you don't have to keep typing that in, you can pull it down and it … saves you some time' (Judith, task 3, Sheep Pen, lesson 1)

> 'the computer can do this in the next few seconds. Before the end of the lesson the computer can have worked this out for us' (Rachel, task 3, Sheep Pen, lesson 1)

Teachers tended to consider the efficiency of solution strategies in the tasks that lent themselves to a graphical approach. The interpretation of graphs involved consideration of the task at hand, and seemed to be useful in offering visual support for reflecting upon progress towards solving the problem at hand.

Focusing attention on meaning

The practice of focusing attention on meaning includes amplifying a particular aspect of activity, offering images or metaphors and negotiating meanings. The discussion of this practice is organised around two key themes: the variable cell; and the variable column.

Focusing attention on the meaning of the variable cell was a practice used by all of the teachers. Anne was responsive to the difficulties of individuals in constructing meaning for notation, explaining that 'it'll change everything automatically.' Rachel similarly explained that a calculation with numbers only works once whereas with formulae the numbers change, an idea that Graham also made reference to. Judith's practice, however, involved frequently amplifying the idea of the variable cell. The extract below illustrates Judith's language use in developing the idea of a variable:

> 'I want "any number. I want to be able to change that number there. So I want whatever's in that box. It doesn't have to be forty-three, it's whatever's in that box. So how can I tell it "that box? (.) … Right so, click, to say I want that box there, just click on that box. Right, and it puts B14 look. B14 so that means whatever's in that box' (task 2, Hundred Square, lesson 1)

The phrase 'whatever's in the box' interpreted out of context could refer to the number that happens to be in the cell or to whatever could be entered into the cell. In context, however, Judith's intention is clear in referring to changing the number, which she also demonstrated in different tasks and reinforced by developing the norm of 'doing a sum,' that is writing a formula.

Judith	Right once you've typed in the number five, once you've typed in the number five it stays a five, it won't change, it'll stay a five. How can we make that five "change, if we change the two above?
Pupil	Um by doing a sum

Judith — By doing a sum, good, by doing a sum on the computer. (task 5, Bonus Points, lesson 1)

Judith's emphasis on change was also seen in tasks where the pupils did not actually change the numbers in cells; instead they listed values in a column.

> 'Now you notice he just said well whatever was in that box there, because I want to change that number to a different number, I want to change it to a three or a five or a ten or an eleven. I want to "change that number so it's whatever's in that box, whatever I choose to pick' (Judith, task 3, Sheep Pen, lesson 2)

In this extract Judith was responding to a pupils' use of the phrase 'whatever was in that box there.' Although she said 'I want to change that number,' she did not literally mean that she wanted to replace the number in the particular cell, but that the class would need to consider different widths of sheep pen and hence write a formula that could be filled down. Thus the idea of change referred to the variable, the width, which she also expressed as 'whatever the width' in the Sheep Pen task and 'however many minutes' in the Mobile Phones task.

Focusing attention on the meaning of the variable column was used, but was less common as a practice. Teachers tended to focus pupils' attention on the variable cell and develop the sociomathematical norm of dragging down a formula, but relatively little attention was given to the meaning of this process. Rachel focused attention on the idea of location, which is consistent with the norm that she developed for constructing a formula.

> 'So, that formula tells the computer to find the number that is in cell B2, and add one to it … And then, as you drag that down, it tells it always, to add one to the cell before it … it's telling it where it is and it's telling it what to do with it' (task 1, Multiplication Tables, lesson 2)

Anne similarly used the idea of location to help pupils understand filling down a formula, explaining to a pair of pupils that 'it's taking a cell above it and adding one'. Anne tended to focus attention on the meaning of the variable column more than the variable cell. Graham and Anne tended to refer to dragging down in the context of generating data to solve a problem, encouraging reflective activity, rather than focusing attention on meaning. Judith did on occasions invite pupils to attend to the notation used in a variable column, although little attention was given to the notation used in a variable column in comparison to Judith's focus on the variable cell.

> 'Let's have a time to think. Have a look at it. This one says E2, E2 which is that one, plus one. This one says E4 plus one. This one says E6, which is that blue one, plus one. What have I actually told the computer to do this time? … Right so it's three add one, "four add one, "five add one, "six add one' (task 1, Multiplication Tables, lesson 2)

Legitimising algebraic activity

The practice of legitimising algebraic activity involves legitimising or validating spreadsheet and other activity and ideas as mathematical, and specifically algebraic. It includes the social appropriation and validation of culturally mediated practices and conventions. Examples of the kinds of passages coded are given below.

> 'How on earth can I write in there, not just any particular number, but "any number? … x, that's a great idea' (Graham, task 2, Hundred Square, lesson 2)
>
> 'I've got an a and a b and I've got a b there as well … a plus two lots of b. How do I write two lots of b in algebra? … a plus 2b' (Graham, task 5, Bonus Points, lesson 2)
>
> 'We've had one or two people succeed with the letters, see if you can explain, see if you can use that to explain' (Anne, task 6, Fairground Game, lesson 2)
>
> 'I'd like you to try and "prove to me that definitely putting one in the middle gets used more frequently than putting one at the end. See if you could actually do that in some way, maybe with symbols' (Judith, task 6, Fairground Game, lesson 2)
>
> 'Over the past few weeks you have had quite a bit of practice at doing algebra, using formulas etcetera' (Rachel, task 6, Fairground Game, lesson 1)
>
> 'We're gonna do exactly the same but this time with some letters there so that first box is whatever the number in a is, plus the number in b' (Judith, task 6, Fairground Game, lesson 2)

Three kinds of legitimising emerged from the analysis. The first two examples illustrate teachers legitimising mathematical conventions, in these cases the use of a literal symbol for a variable, and the convention for multiplication. The second two examples, Anne's reference to explaining and Judith's reference to proof are examples of legitimising the significance of practices. In the final two examples the teachers attempt to make links, one global and one local, between spreadsheet activity and algebraic activity, thus legitimising spreadsheet activity as algebraic. The first two kinds of practices were seen less in the lower attaining sets, often a reflection of less work with standard notation.

DISCUSSION

The categorisation of five practices offers a broad description of what teachers do which, to varying degrees and in different ways, guides pupils' construction of meaning. Alongside the design of tasks (see, for example, Ainley, Bills and Wilson, 2005b), the practice of teachers is a key aspect of pedagogy. The five practices were identified in a specific context, with particular sets of pupils. Nonetheless, they offer a way of characterising the complexity of what teachers actually do when working with such technology. They form an interpretive framework, which is primarily grounded in data, but also links with key ideas in the theoretical frameworks outlined earlier in the paper; 'instrumental orchestration' (Trouche, 2004) and the role of the teacher in developing 'instruments of semiotic mediation' (Mariotti, 2002).

In light of the broader aims of the study and the broader range of data collected, the categorisation was used in the analysis of pupils' evolving meanings for algebra. More specifically, the relationships between teachers' practices and pupils' meanings were tentatively explored. It is likely that a number of interwoven factors contributed to the patterns observed at class level, not least of which was the attainment level of the set. A more fine grained level of analysis attempted to embrace this complexity by tracing pupils' evolving meanings through chains of episodes (Cobb and Whitenack, 1996) and developing case studies of pairs of pupils. Within these, data relating to teachers' practices was a particular focus of analysis. Each of the five practices was seen to steer and colour pupils' meanings. Aspects of pupils' activity, such as clicking on spreadsheet cells or making errors, were identified as triggers for mobilising particular meanings.

The categorisation outlined in the paper is grounded in rich, longitudinal data. The five emerging themes offer a framework for characterising and comparing teachers' practices. This offers some insight into how teachers guide the social construction of meanings in a spreadsheet context. In this study, the themes were used in the longitudinal analysis of pupils' evolving meanings for algebra. Future reports on the evolution of meanings will draw upon the categorisation outlined here.

NOTES

1. The Purposeful Algebraic Activity project and this study were both funded by the Economic and Social Research Council

2. The following conventions are used in the transcripts

" precedes emphatically-stressed syllable

… indicates omission of part of transcript for presentation purposes

(.) timed pause (number of dots corresponds approximately to number of seconds)

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