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## DIFFERENT THEORETICAL PERSPECTIVES IN RESEARCH FROM TEACHING PROBLEMS TO RESEARCH PROBLEMS

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### INTRODUCING THE CHALLENGE OF DIVERSITY

One of the characteristics of the European Community in Mathematics Education is that we operate with a large diversity of different theories, research paradigms and theoretical frameworks. Like in a nutshell, this heterogeneity was present in the Working Group 11: 30 researchers, coming from 15 countries, discussed 18 different contributions in which in sum 16 different theoretical frameworks were explicitly stated (cf. Figure 1).

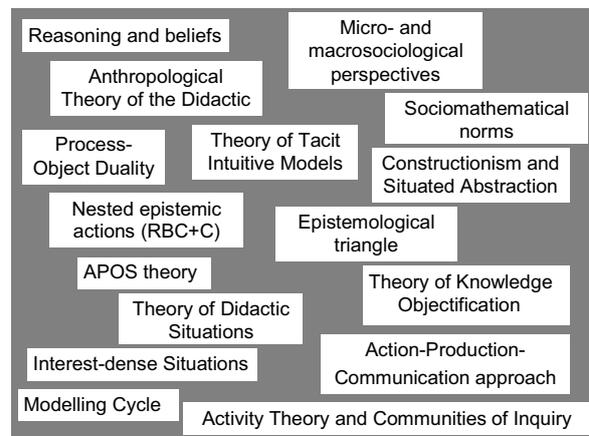


Figure 1. Research frameworks present in WG11

The large diversity of theoretical frameworks already starts with the heterogeneity of what *is called* a theoretical framework or a theory by different researchers of different traditions. In the list of explicitly stated theories, there are research paradigms, basic and comprehensive general theories as well as local conceptual tools, with different scopes and backgrounds. It affects also the way each theory conceptualizes and questions mathematical activities and educational processes, as well as the type of results it can provide. This diversity represents a challenge for the community for different reasons:

- Problems of communication: researchers from different theoretical frameworks sometimes have difficulties to understand each other in depth because of their different backgrounds, languages and implicit assumptions.
- Problems of integration of empirical results: as in the cartoon shown below, researchers with different theoretical perspectives consider empirical phenomena (see Figure 2) from different perspectives and hence come to very different results in their empirical studies. How can the results from different studies be integrated or at least understood in their difference?

- Problems of scientific progress: in the long run, improving mathematics classrooms depends in part on the possibility of a joint long term progress in mathematics education research in which studies and conceptions for school *successively build upon empirical research*. But how to do that when each study uses a different theoretical framework that cannot be linked to others?

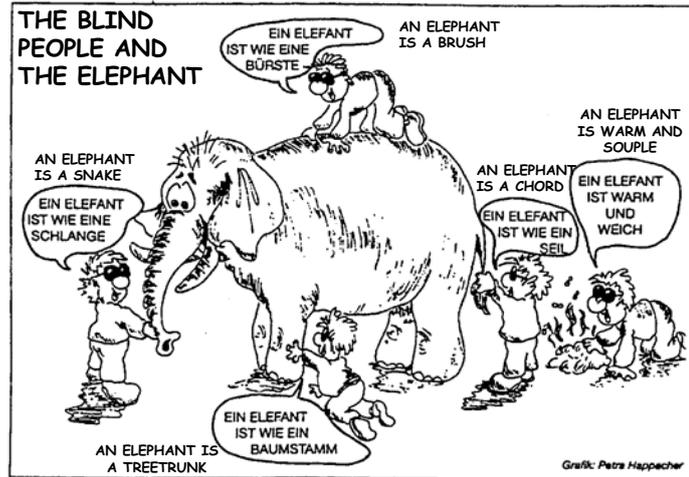


Figure 2. A metaphor for research from different perspectives

The incommensurability of perspectives produces sometimes incompatible and even contradictory results which not only impedes the improvement of teaching and learning practices, but can even discredit a research field that may appear as being unable of discussing, contrasting and evaluating its own productions.

Although these aspects clearly show that the diversity of theoretical frameworks is a challenge for a community which intends to have communication and progress between researchers of different theoretical frameworks, we started the Working Group from the assumption that we do not aim at a progressive unification of all frameworks since we consider the *variety of frameworks to be a rich resource* that is absolutely necessary in order to handle the complexity of mathematics teaching and learning (see Bikner-Ahsbahs & Prediger 2006). This is also emphasized by one conference contribution:

“Simultaneous exploitation of [...] approaches is especially valid [...] when the didactical phenomena occurring in the mathematics classrooms [appear] so complicated with respect to personal, social and epistemological aspects. A multiple approach seems necessary and the researchers have to build up the concrete connections that would make these different tools compatibles.” (Kaldrimidou *et al.*<sup>1</sup>)

## LEARNING TO COPE WITH DIVERSITY

Hence, the proposal of learning how to deal with the diversity, complexity and richness of European theoretical perspectives in mathematics education is a task our community cannot postpone much longer. This necessity was at the origin of Working Group 11 at CERME 4, where a group of researchers lead by Tommy Dreyfus, Michèle Artigue, Mariolina Bartolini Bussi, Eddie Gray and Susanne

<sup>1</sup> All papers cited without reference nor year follow as contributions to these Working Group proceedings.

Prediger met around the discussion on research paradigms within the context of their effect on empirical research. One of the most important directions that emerged was the idea of *networking*:

“If we can develop and maintain a certain degree of networking between some of the advocates of the different theoretical stances that are currently evident within mathematics education, this will constitute an important step on the path towards establishing mathematics education as a scientific discipline.” (Dreyfus *et al.* 2006, p. 1242)

Anyway, bringing to the fore the project of comparing, networking or (partially) integrating theories as an inevitable task does not make it less difficult. As it was pointed out:

“It is crucial to have an awareness of the underlying assumptions of each theory. Only on the basis of such awareness, can a discussion on the possible coherence of underlying assumptions begin to take place so that a common language supporting such networking can be developed.” (Dreyfus *et al.*, p. 1243)

Taking into account this discussion at CERME 4 in year 2005, the succeeding Working Group 11 at CERME 5 proposed to follow these orientations by focusing on two axes:

**Theme 1:** Deepening our insight into the underlying assumptions, relationships and differences of theories or approaches in mathematics education;

**Theme 2:** Handling the diversity of theories in our field of research in order to better grasp the complexity of learning and teaching processes.

More concretely, papers were asked to provide a “piece of answer” to the following starting questions:

**Theme 1: Functions of a given theoretical framework**

- 1.1. How do specific theories allow (re)formulating problems about the considered reality?
- 1.2. What specific methods, methodologies and heuristics are developed because of the use of a theory?
- 1.3. What are the consequences of the use of a specific theoretical framework on the interpretation and formulation of results of an empirical research?
- 1.4. How do empirical studies contribute to the development and evolution of theories?

**Theme 2: Interactions of two or more theories**

- 2.1. How do specific theories allow (re)formulating problems about the considered reality?
- 2.2. What methods, methodologies and heuristics are used to compare, develop, combine, integrate or complement different theories?
- 2.3. What consequences the interaction of theories has on the research, for instance in the formulation and approaching of problems?
- 2.4. What can be said about the issue of mutual consistency of different theories?

Some of the approaches are specifically treated in the papers related to theme 1: the theory of knowledge objectification (Radford), social practice theory and communities of inquiry (Goodchild, Jaworski), the nested epistemic actions (RBC+C) model for abstraction in context (Dooley) and the anthropological theory of the didactic (Rodríguez *et al.*, Wozniak).

The other appearing theoretical frameworks were compared, contrasted or combined with different purposes. Thus, Arzarello *et al.* combine and try to complement the Anthropological Theory of the Didactic with the Action-Production-Communication approach. Bergsten contrasts the way APOS theory, the anthropologic theory of the didactic and research on reasoning and beliefs approach to the problem of teaching limits of functions. Gellert compares the micro- and macro-sociological perspectives in mathematics education research through the analysis of a short transcript of 6<sup>th</sup> graders' collaborative problem solving. Kaldrimidou *et al.* use the sociomathematical norms and the epistemological triangle to analyse the mathematical knowledge under construction in two secondary school lessons. Kidron *et al.* focus on how the theory of didactic situations, the nested epistemic actions and the theory of interest-dense situations take into account social interactions in learning processes. Maracci approaches students' difficulties when solving vector space problems comparing two frameworks: the theory of tacit intuitive models and the theory of process-object duality. Finally, Cerulli *et al.* present a methodology for integrating research teams based on different theoretical perspectives: theory of didactics situations, anthropological theory of the didactic, socio-constructivism and activity theory, constructionism and situated abstraction. Petrou approached the connecting task by combining different research methods for her empirical questions.

Although we cannot summarize the whole process in the working group, we want to give insights in some important aspects and questions that arose in the discussion.

## **SOME ARISING ASPECTS AND QUESTIONS**

### **The problem of the hidden assumptions**

One of the most important obstacles for communication between different theoretical frameworks is the fact that each theory is connected to more or less explicit assumptions on epistemological, methodological, philosophical and sometimes psychological questions.

#### *Nature of mathematical knowledge*

By reminding of the work of Hans Georg Steiner (1987), Günter Törner and Barath Sriraman emphasized that among all the grounded assumptions of any theoretical approach, those related to the nature of mathematical knowledge may

appear as most fundamental (see Törner/Sriraman 2007 which is the elaboration of their preconference contribution). Even if the chosen mathematical epistemology is a crucial element of any approach, it is also important, as Steiner pointed out, to conceive them not as a “credo” or “norm” to follow, but as scientific models that are to evolve and be modified according to their productivity in explaining didactic phenomena. The degree of elaboration of specific epistemological models (or mathematical philosophies) also seems to differentiate mathematics educational approaches from those, more general, coming from psychological or social perspectives. Not only Törner and Sriraman pleaded for this level of comparison between theories, also other studies (Bergsten, Kaldrimidou *et al.*, Rodríguez *et al.*) showed that the assumptions on the nature of mathematical knowledge appears to be one key point for the analysis of similarities and differences between approaches.

### *The individual/social interplay and the challenge to constructivism*

Another important level of basic assumptions concerns the nature of knowledge and learning, is it individually or socially constituted? At CERME 4, this difference was discussed as an important aspect which might even make integration of theories impossible (see Artigue *et al.* 2006). This year, we saw attempts of new conceptualizations of the individual-social interplay: Doodley presented a study on distributed knowledge construction without social learning theories, Kidron *et al.* presented a joint work with the comparison of different roles that the social interaction can play within three different theories, and Radford challenged constructivism by designing a “cultural theory of learning”.

### **Extending theories to higher levels**

Jaworski’s and Goodchild’s papers are good examples of how to extend the scope of a given theory to embrace a wider and more complex phenomenon such as the relationships between teaching/learning practices and research. While Goodchild focuses on the use of an activity theory perspective to analyse the learning process of a didactic team in a developmental research project, Jaworski extends the social practice theory to analyse the practice of teaching as learning in practice.

In another framework that also considers teaching and learning mathematics as social practices of communities, Wozniak illustrates how the analysis proposed by the anthropological theory of the didactic extends the scope and nature of the studied phenomenon. Considering the teaching of statistics in France and the difficulty of its diffusion in compulsory education, the analysis highlights general restrictions coming from different levels of determination like the status of statistics within mathematics, the “reclusion” of statistics in professional education, and, more generally, the negative consideration of statistics in past times in the French society.

### **The problem of non-isomorphism between research questions in different frames**

The anthropological theory of the didactic is also used in the paper by Rodríguez, Bosch and Gascón to contrast how the classical problem of teaching metacognitive strategies in mathematics can be formulated in terms of, on the one side, the passage from point-levels to local or regional levels of mathematical praxeologies and, on the other side, the different division of responsibilities between teacher and students as it is classically stated by the current didactic contract. Thus, a problem that has been historically approached from cognitive perspectives can be converted into a problem of the conditions and obstacles for didactic and mathematical praxeologies at school. This conversion changes the problem into a non-isomorphic one.

The “incommensurability” of perspectives can be made clearer when several perspectives upon a same practical question are taken into consideration. This is also shown by Bergsten who compared three different perspectives considering the questions approached, the methods and empirical evidence used, and the conclusions and implications stated in each case.

### **How different theories approach similar data**

Studies considering different approaches to similar questions or data bring more evidence to the considerations above. In the case of sociological perspectives in mathematics education, Gellert’s work consists in comparing the validity and relevance of analyses coming from a macro and a micro level when applied to a unique set of data. Different interpretations and understanding of the same account rise immediately: the micro-sociological analysis describing the emergence of a convincing argument while, at the macro-sociological level, differences between situations and their recognition by the students (then using everyday knowledge or a logic-mathematical thought) have a prominent role.

## **NETWORKING STRATEGIES**

Theoretical approaches can be connected in multiple ways and degrees. From the extreme of mutual ignorance (or a relativist “laissez-faire”) to the extreme of complete integration, almost all positions can be considered in between. It seems to exist a general agreement in the European community that, even if theoretical diversity is more a richness than a nuisance, we should advance towards a more consequent coordination of frameworks. But at the same time, we plead for respecting the pluralism of autonomous theories as a rich resource.

The working group started to discuss the wide spectrum of strategies for connecting theoretical approaches between the two extreme strategies – the laissez faire on the one hand and the unification on the other hand. We considered the

contributions as first attempts of connecting theories with different strategies like comparing, contrasting, coordinating or combining.

*Understanding each other* is the first strategy on which we spend a lot of time and the first part of the call for papers (see above).

The most modest but already ongoing strategy is *comparing* theoretical approaches and their impact on research processes. A comparison can start from the theoretical base, but also from a piece of data or a problem that has to be conceptualized as a research problem in different theoretical perspectives. A comparison can also lead to competing between different theories with respect to specified research interests.

Other contributions had their focus on a specific empirical research question and use different theoretical lenses for a deeper understanding of a concrete phenomenon, in these cases, *combining or coordinating* strategies can be discussed.

*Integrating or synthesizing strategies* aim at a further development of theories by putting together a small number of theoretical approaches into a new framework. In order to avoid building inconsistent theoretical parts without a coherent base, these strategies can only be applied to approaches with compatible (but not necessarily equal) backgrounds.

### **A shared experience for comparing:**

#### **“From a teaching problem to a research design”**

Prediger and Ruthven designed one common experience for comparing theoretical approaches by considering their meaning for practically oriented empirical research. The main question was “How does a theoretical basis chosen for a study influence the nature of the purpose, questions, methods, evidence, conclusions, and implications of the study?” Whereas most presented comparisons started with a given piece of data and analysed it through different lenses, Prediger and Ruthven wanted to start earlier in the research process by focussing on the way an ordinary teaching problem is conceptualized in terms of the theory and how a research design is made out of it. The proposal was to follow a set of frameworks during the whole process of conceptualising a practical problem, transforming it into a more focused research question, developing a research design and forecast the kind of results that could come up. The answers given by eight teams of researchers were analysed through two main axes: the major intention of research (improve understanding of phenomena *versus* improve teaching and learning practices) and the level of analysis: micro, meso or macro (cf. Prediger/Ruthven).

### **Different networking strategies for connecting theoretical approaches**

Some papers offer first suggestions of how to combine theoretical approaches. Maracci’s combined the local conceptual frameworks for explaining students’

difficulties with linear algebra, namely Fischbein's theory of tacit intuitive models and Sfard's theory of process-object duality. Arzarello et al. combine bigger frameworks, the anthropological approach and the APC-space theory, to analyse the same subject, the "ostensives" or "semiotic tools" (oral words, written symbols, graphical objects, gestures, etc.) used in mathematical activities. Each approach produces different complementary insights on the same phenomenon and the resulting analysis is so enriched by the combined approach.

A paradigm of the necessity of networking theories for the needs of the research is the case described – and analysed – by the TELMA network project (Cerulli *et al.*). A group of six European research teams interested in the Technology Enhanced Learning in Mathematics (TELMA) had to develop a methodology for integrating their research approaches to favour the construction of a shared scientific vision, the development of common project and the building of complementarities and priorities in the considered research area. Their experience based in a cross-experimentation brings some important guidelines to the "dealing with diversity", some of which has been considered before: the "making clear and communicating the implicit" – related to the "hidden assumptions" commented below – and the differences in the conception of the experiments – as the collective work leaded by Prediger and Ruthven also illustrates. Furthermore, what the TELMA project shows is how the interaction of approaches in the design of teaching experiences and their putting into practice in classrooms often reveals the limitations of the theoretical frameworks – "what they do not say" – and appears as an excellent way for their future developing.

Another basic work on connection theoretical approaches is offered by Kidron *et al.* who make a reflected first attempt of networking theories. Starting from three different theoretical approaches (represented of the Theory of Didactical situations, the theory of interest-dense situations and the theory of abstraction in context with its RBC-model) they focus on one crucial aspect, namely the role of social interaction as the core for comparing and contrasting each pair of theories. Especially instructive is the question of what each of the theories has to offer for the others.

### **Networking in Different Profiles**

By comparing these different first networking strategies, the *aim of networking* turned out to make a big difference. Whereas for example Kidron *et al.* search for a general development of their theories, Cerulli *et al.*, Kaldrimidou *et al.*, and Maracci start from an empirical phenomenon or a practical question with the aim of developing or understanding it better connecting different perspectives (Cerulli *et al.*) or local conceptual tools.

Bottom-up development	Top-down development
<ul style="list-style-type: none"> <li>• Aim: understanding a concrete empirical phenomenon</li> <li>• Ongoing growing of the bulk of theories</li> <li>• Focus on the research question for a concrete empirical phenomenon</li> <li>• Trial &amp; Errors</li> <li>• Dialectic among theories</li> <li>• Endogenous development (i.e. driven by the concrete study)</li> </ul>	<ul style="list-style-type: none"> <li>• Aim: use or development of a given set of theories</li> <li>• Different theories on the table from the beginning</li> <li>• Focus on the relationships among theories</li> <li>• Deductive approach</li> <li>• Networking / Combining / Integrating</li> <li>• Exogenous development (i.e. driven by the general interest)</li> </ul>

Figure 3: Comparing different profiles

As a consequence of these different aims, the development of networking follows completely different profiles as sketched in Figure 3. Whereas the top-down profile starts with different theoretical frameworks from the beginning, the bottom-up profile searches for new theoretical tools only if the others turn out to be insufficient. With the focus on the research question, the process of combining theories follows the logic of trial and error with output for understanding the empirical phenomenon as the measure for suitability. In contrast, the top-down profile with its focus on relationships among theories follows a deductive approach. In this sense, the TELMA project offers a new profile while starting from a given set of theories but aiming at the development of a concrete empirical research. Although we only met these different profiles so far, it is still an open question whether others profiles are possible and fruitful which switch between the different columns.

## OUTLOOK

Although we started a very interesting process, the discussion is far from being finished. Three big questions turned out to be crucial for the further work:

For the concrete networking, first appears the problem of how to link theories without getting contradictions and not destroying their internal coherence. A common supporting frame is necessary, at least to offer a location for the linking. Another possibility is to start the networking being located in a given “strong” perspective and developing it in order to “incorporate” the new approaches. Anyway, it seems clear that networking cannot be done in a “theoretical no-man’s land”.

The second question follows. How do the differences or similarities between theories influence the networking strategies? What shared backgrounds are necessary in order to ensure networking without losing the rationale of each approach?

Finally, related to what we can call a “theory of networking theories”, the question arises of what categories are needed to deal with different theories. Is it necessary to build up a common or shared background that may appear as a “neutral land” for the networking or, on the contrary, it is important to maintain a multiple reference system where each theory may appear as a potential chief supporter of the whole construction?

The ambitiousness of these questioning obviously exceeds the scope of a working group that only meets a few days in a two years period. The CERME Working Group on Theoretical Perspectives intends to start a long term project that might work also between the conferences, possibly by splitting into small groups. In each case, the aim is to establish a “reference place” for the study of theories, their differences, commonalities and connections, with the vision of a vivid and theoretically diverse but connected European community of research in Mathematics Education.

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## OSTENSIVES THROUGH THE LENSES OF TWO THEORETICAL FRAMEWORKS

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*The paper develops an analysis of the same subject (the ostensives) combining two different theories: the APC-space and the ATD frame. The philosophy underpinning the research comes from the idea of networking theories, namely of comparing, contrasting and possibly combining different theories in order to get new insights about learning processes.*

### 1. INTRODUCTION

It is well known that there is a strong contrast in mathematical activities between the abstract nature of mathematical objects, which have no perceptual existence, and their representations, which are tangible and upon which subjects' activities can develop in a very concrete way. Such a duality is basic in all learning processes. It is important therefore to develop suitable frameworks to analyse it and to clarify its role in the teaching and learning of mathematics.

The mentioned duality been afforded by Bosch and Chevallard (1999), introducing the dialectic between what they call “*ostensives*” and “*non ostensives*”. They observed that there is a variety of palpable registers through which mathematical activities can develop:

“...the **oral** register, the **trace** register (which includes all graphic stuff and writing products), the **gesture** register, and lastly the register of what we can call the **generic materiality**, for lack of a better word, namely the register where those ostensive objects that do not belong to any of the registers above reside” (Bosch & Chevallard, 1999, p. 96, emphasis in the original, translation from the French by the authors)

According to the *Anthropological Theory of the Didactic* (ATD), it is important to study the “instrumental value” of the ostensives, considering the practice systems (*praxeologies*), within which they are treated (García et al., 2006). A very similar problem has been afforded by Arzarello (Arzarello & Olivero, 2005; Arzarello, 2006). He studies the “semiotic value” of the ostensives through a broadened notion of semiotic system (the so called *Semiotic Bundle*) and frames them within the so called *Space of Action-Production-Communication* (APC-s) model.

The major goal of the paper consists in starting a comparison of the two frames and in drawing some new insights from the “combined” use of the two theories for approaching the ostensives. The idea of comparing, contrasting, combining different theories has been discussed in CERME 4 within the Working Group on Theoretical Perspectives in Mathematics Education. The purpose is developing a networking of different theoretical approaches (Bikner-Ashbash & Prediger, 2006). The general philosophy of the paper as well as some ideas expressed in it are influenced by the discussion in those events but the responsibility of what said here is only of the authors.

The next section of the paper sketchily exposes how the two frames approach ostensives. The subsequent section contains a (tentative) analysis of the ostensives through the “combined” use of the two frames (the main result of the paper). The last section consists of some general considerations about the idea of networking theories.

### **THE APC-s and ATD APPROACHES**

Looking at the phenomenology of learning processes in the mathematics classes, a variety of *ostensives* are observable. They may be produced or used with great flexibility: the same subject generally exploits simultaneously more than one of them (e.g. speech and gesture). Sometimes these resources are shared by the students (and possibly by the teacher) and used as communication tools, other times they reveal as crucial thinking tools. All such (ostensive) resources, with the actions and productions they enhance, appear important in the building of mathematical ideas. In fact they reveal crucial to bridge the gap between the time-less and context-less sentences of formal mathematics and the worldly experience that many times allows people to grasp the meaning of mathematical concepts (non-ostensive objects). These general observations suggest that in order to scientifically describe the learning processes in the classroom, it is necessary to consider all such *resources*, the *practices* they are treated with, and how they *evolve*.

In next subsections we expose sketchily the theoretical frames of the APC-s and of the ATD, underlining their different approaches to ostensives. We thank M. Bosch for the fruitful e-mail discussions she had with us on ATD and ostensives.

### **Embodiment, multimodality and APC-s**

The notion of *multimodality* has evolved within the paradigm of *embodiment*, which has been developed in these last years (for a synthetic overview, see Wilson, 2002). Embodiment is a movement afoot in cognitive science that grants the body a central role in shaping the mind. It concerns different disciplines, e.g. cognitive science and neuroscience, interested with how the body is involved in thinking and learning. The new stance emphasizes sensory and motor functions, as well as their importance for successful interaction with the environment. A major consequence is that the boundaries among perception, action and cognition become *porous* (Seitz, 2000). Concepts are so analysed not on the basis of “formal abstract models, totally unrelated to the life of the body, and of the brain regions governing the body’s functioning in the world” (Gallese & Lakoff, 2005, p. 455), but considering the *multimodality* of our cognitive performances. Verbal language itself (e.g. metaphorical productions) is “part of these cognitive multimodal activities” (*ibid.*). In the more extreme version, the frame of multimodality appear to suggest that “the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuomotor activities, which become more or less active depending of the context.” (Nemirovsky, 2003; p. 108).

The *Space of Action, Production and Communication* (in short, APC-s) allows to frame suitably mathematical learning processes according the multimodal paradigm. Namely it allows to consider how action and perception determine the processes of

learning and to describe them so that doing, touching, moving and seeing appear as their important multimodal ingredients. Specifically, the APC-s is meant to be a model for framing the processes that develop and are possibly shared in the classroom among students (and the teacher) while working together (Arzarello, in press; Arzarello & Olivero, 2005). It analyses them considering their different components and a variety of mutually dependent relationships among them. The components are the body, the physical world, the cultural environment: in a word, the students themselves and the teacher along with the context where they are acting and learning. When students learn mathematics, these and other components (e.g. the emotional ones) take an active part in the learning processes, interacting together. The interaction comes from the students' work, the teacher's mediation and possibly from the use of artefacts. The three letters A, P, C illustrate the main dynamic relationships among such components, namely students' actions and interactions (e.g. in a situation at stake, with their mates, with the teacher, with themselves, with tools), their productions (e.g. answering a question, posing other questions, making a conjecture, introducing a new sign to represent a situation, and so on) and communication aspects (e.g. when the discovered solution is communicated to a mate or to the teacher orally or in written form, using suitable representations). The APC-s is a typical complex system, which cannot be described in a linear manner as resulting by the simple superposition of its ingredients. It particularly models how the relationships among its components develop in the classroom through the specific actions of the teacher. The APC-s analysis also allows to picture how the multimodal aspects of learning processes come to be related to cultural and institutional aspects. In fact, as pointed out by L. Radford:

“an account of the embodied nature of thinking must come to terms with the problem of the relationship between the body as a locus for the constitution of an individual's subjective meanings and the historically constituted cultural system of meanings and concepts that exists prior to that particular individual's actions.” (Radford et al. 2005).

Ostensives can be framed suitably through the APC-s frame: we see them as constituting a palpable aspect of multimodality. To focus their nature and mutual relationships it is convenient to use a semiotics lens, which in any case is an excellent tool to enter into APC-s. Semiotics is a powerful tool for observing the didactical phenomena in their multimodal complexity (e.g. see Ernest, 2006). However, the classical semiotic approaches (for an updated survey see Sáenz-Ludlow and Presmeg, 2006) put strong limitations upon the structure of the semiotic systems they consider and therefore in our view they reveal too narrow to describe the complexity of APC-s ingredients, particularly of the ostensives. This happens for two reasons: (i) Students and teachers use a variety of semiotic resources in the classroom: words (orally or in written form); extra-linguistic modes of expression (gestures, glances, actions, ...); different types of inscriptions (drawings, sketches, graphs, ...); instruments (from the pencil to the most sophisticated ICT devices), and so on. Analysing such resources, we find that some of them do not satisfy the requirements of the classical definitions for semiotic systems as discussed in the literature (e.g. see the semiotic registers by Duval, 2006). (ii) The way in which such different resources are activated is

multimodal, as pointed out above. It is necessary to carefully study the relationships within and between those, which are active at the same moment, and their dynamics developing in time.

To overcome such difficulties, Arzarello (2006) has introduced a broader semiotic tool: the *Semiotic Bundle* (Arzarello, 2006). The Semiotic Bundle is the semiotic tool suitable to analyse the variety of resources and their relationships within the APC-s frame. Encompassing all the classical semiotic systems or registers as particular cases, it does not contradict the semiotic analysis developed using such tools but broaden it with the double aim of getting new results and framing the old ones within a unitary wider picture. To define the Semiotic Bundle, we first need the notion of *Semiotic Set*, which broadens that of semiotic system<sup>1</sup>. A Semiotic Set is:

- a) A set of signs which may possibly be produced with different actions that have an intentional character, such as uttering, speaking, writing, drawing, gesticulating, handling an artefact, and so on.
- b) A set of modes for producing such signs and possibly transforming them; these modes can possibly be rules or algorithms but can also be more flexible action or production modes used by the subject (e.g. in gesturing, in drawing, etc.).
- c) A set of relationships among these signs and their meanings, e.g. between the sign ‘=’ and its meanings, or between a gesture and its meaning (e.g. see the classification in Goldin-Meadow, 2006, p. 6: iconic, metaphoric, deictic, beat gestures).

Examples of semiotic sets are on the one hand all the usual semiotic systems (speech, written languages, the algebraic register, etc.) and on the other hand ‘new entries’ like, gestures, drawings, sketches, etc. In fact, the three components above (signs, modes of production/transformation and relationships) may characterize a variety of resources, spanning from the compositional systems, usually studied in traditional semiotics (e.g. formal languages), to the open sets of signs (e.g. sketches, drawings, gestures). The former are made of elementary constituents and their rules of production involve both atomic (single) and molecular (compound) signs. The latter have holistic features, cannot be split into atomic components, and the modes of production and transformation are often idiosyncratic to the subject, who produces them. The word “set” must be interpreted in a very wide sense, e.g. as a variable collection. Now we can define a *Semiotic Bundle* as the couple formed by:

- A collection of semiotic sets.
- A set of relationships between the sets of the bundle.

A semiotic bundle is a *dynamic structure*, which changes in time because of the semiotic activities of the subject: for example, the collection of semiotic sets that constitute it may change; as well, the relationships between its components may vary in time; sometimes the conversion rules have a genetic nature, namely, one semiotic set is generated by another one, enlarging the bundle itself (we speak of *genetic conversions*: see below). Semiotic bundles are the semiotic lenses, through which one can observe the nature and the dynamics of the ostensives in the APC-s. As such,

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<sup>1</sup> It is a generalisation of the definition of Semiotic System, as it is given in Ernest (2006, pp. 69-70).

they reveal as *semiotic systems of cultural meanings* (Radford, 2006), that is, those systems which make available various sources for meaning-making through specific social signifying practices (e.g. through the actions, productions and communications pictured by the APC-s). Such practices are not to be considered strictly within the school environment but within the larger environment of the society as a whole, embedded in the stream of its history.

An example of semiotic bundle is represented by the unity speech-gesture. It has been a recent discovery that gestures are so closely linked with speech that “we should regard the gesture and the spoken utterance as different sides of a single underlying mental process” (McNeill, 1992, p.1), namely “gesture and language are one system” (*ibid.*, p.2). In our terminology, gesture and language are a semiotic bundle, made of two deeply intertwined semiotic sets (only one, speech, is also a semiotic system). Research on gestures has uncovered some important relationships between the two (e.g. match and mismatch, see Goldin-Meadow, 2003). Another example, made of gazes, speech, gestures and inscriptions has been studied by F. Ferrara in her PhD Dissertation (Ferrara, 2006). A semiotic bundle is not to be considered as a juxtaposition of semiotic sets; on the contrary, it is a unitary system and it is only for the sake of analysis that we distinguish its components as semiotic sets.

Arzarello and his team have used the APC-s frame and the Semiotic Bundle tool to analyse different classroom stories (Arzarello et al., 2006; Arzarello, in press ). They have revealed particularly useful for studying several didactic phenomena that happen in the classroom.

### **The ATD and the ostensives**

The ATD assumes an institutional conception of the mathematical activity:

Mathematics, like any other human activity, is something that is produced, taught, learned, practised and diffused in social institutions. It can be modelled in terms of praxeologies called mathematical praxeologies or mathematical organizations. (García et al., 2006, p.226)

As an example of mathematical praxeology, García and his colleagues give the ‘proportion problems’, that is

“a set of problematic tasks (the classic proportional problems where three measures are given and a fourth one is to be found), techniques to deal with these problems (commonly known as *rule of three*) and a technological-theoretical discourse that explains and justifies the mathematical activity performed (defining what are proportional magnitudes and how to determine if two magnitudes are directly or inversely proportional)” (García et al., *ibid.*).

Mathematical praxeologies are the object of learning and teaching in the schools: “The mathematical knowledge is produced, taught, learned, practised and diffused in social institutions. It is thus not possible to separate it from its process of construction in a specific institution” (García et al., *ibid.*). Hence the learning processes which happen in the classroom are considered by ATD from an institutional point of view.

Any praxeology is always activated through the manipulation of ostensives and the evocation of non-ostensives (e.g. the ostensive  $y = k \cdot x$  and the mathematical abstract concept ‘linear function’), which are like the two sides of the same coin. According to the ATD, in the mathematical activities generally the focus is on the non-

ostensives (the concepts), while the ostensives are underestimated. The ostensives are considered usually according to their *semiotic* function, namely as perceivable objects, which represent other objects (i.e. as signs). But ATD points out another important, usually neglected function of ostensives: the *instrumental* function. In fact, the ostensives are not simple *working media* but genuine *instruments* for the mathematical activity: their careful manipulation does not only allow performing a mathematical task but is essential for its accomplishment, e.g. for solving an equation. The instrumental and the semiotic value of the ostensive objects depend on the practises of the institutional system, where they are activated. Consequently the non-ostensive objects exist because of the manipulation of the ostensive ones within specific praxeological organisations.

Both the frames (ATD and APC-s) focus on the ostensives as a relevant part of mathematics learning in a dialectic relationship with the non-ostensives. Hence both approaches do not tackle the learning of mathematics as a pure learning of concepts. This point is supported through an institutional analysis in the ATD theory and through the paradigm of multimodality in the APC-s frame. In the next section we shall see how both frames can contribute to more complete analysis of the ostensives.

### **OSTENSIVES: A COMPARISON COMBINING ATD AND APC-s**

The two approaches allow to consider the dynamics of learning processes as a result of cognitive, cultural and institutional facts, namely because the teacher and the students are human beings, cognitively reacting in certain ways, living in specific societies, attending certain schools, where specific praxeologies have historically and socially developed in the years, and so on. The APC-s focuses some of these, while the ATD approach is more suitable for focusing others. The ATD approach is particularly apt to focussing the evolution from the use of specific techniques to the use of generic ostensives and to the elaboration of suitable technologies and theories in a variable historical space-time-scale (see García et al., 2006). It is a frame which describes the processes of learning as a cultural and social appropriation phenomenon: the emphasis is on the cultural and institutional aspects in learning processes. The APC-s approach is particularly apt to focussing the dynamics among the different semiotic resources (ostensives) used by the students in the short time-scales of the classroom story. It allows studying classroom events that take place in few minutes or even seconds and that are considered crucial according to that theoretical frame, namely the actions made by pupils, their productions (in different languages: verbal, gestural, written), the interactions (between the students, between students and the teacher, with the artefacts). The analysis is particularly attentive to the *multimodality* of the studied phenomena (because of its relevance in the learning processes of pupils), and to the cultural aspects that they reveal. The emphasis is on the psychological (and possibly neural) and social aspects in learning processes. The cultural aspects are present in the mediating action of the teacher. In this sense, the APC-s frame allows to embrace both the psychological and the cultural dimension of learning. Specifically, it can give reason both of the biological and of the cultural roots of the ostensives produced and acted on in the classroom.

The comments above show that there are complementarities between the two frames. For example, didactical phenomena can be analysed according to different time and space scales, which span from the small-scale flying moment of a learning process in a specific classroom as described in the APC-s to the long term and wide events, which produce the praxeologies at regional level described in the ATD. To grasp properly didactical phenomena, we argue it is fruitful to integrate theoretical frameworks based on complementary scales of analysis. Of course it is a difficult problem to coordinate the fine grain analysis of short-term processes with the analysis of long term processes. On the one hand, APC-s allows to develop a fine-grained cognitive analysis (where the semiotic facts are interpreted within the APC components, which heavily refer to subjects actions, productions and communications from a biological and cultural point of view); on the other hand, the ATD frame allows to develop an analysis from a cultural and institutional point of view (praxeology with techniques, technologies, theories, didactic transposition etc.). What happens in the classroom concerns both dimensions and each analysis can give us useful information and interesting interpretations within the respective frames. In this sense the two approaches can benefit each other if we can merge the different scales. The task is not easy, but it appears intriguing to study the ostensives combining the two frames, e.g. investigating all their roots, from the historical and institutional to the psychological and biological ones.

To make very concrete this tentative of combining the two frames we shall apply it to two related didactical phenomena concerning ostensives, namely the chirographic reduction (in Greek ‘χειρ’ means ‘hand’), studied in the ATD frame, and the genetic conversion, analysed in the APC-s frame. The notion of *chirographic reduction* has been studied in Bosch and Chevillard (1999): they point out the “individual micro genesis of techniques for solving specific problems” (p. 104), that is a process that starting from ostensive objects (in discursive, gesture, graphic, written form) ends with stable techniques, generally developed on the sheet of paper. The chirographic reduction consists exactly in the “transfer of gesture and material objects to the oral and graphic registers” (*ibid.*, p. 105). For example they analyse gesture and speech, which accompany the accomplishment of matrix product. In the end, these ostensive objects are integrated in new mathematical objects, represented through the algebraic formalism, where each trace of gesture and oral activity is eliminated. This is at the root of a didactical paradox. On the one hand, the genuine mathematical job seems to consist in “pure computation and pure syntax”. The other ostensive aspects, which are embedded in the stream of time do not seem to acquire a clear mathematical status. On the other hand, it is exactly the combination of this private component with the official one –i.e. the semiotic sets active simultaneously in a semiotic bundle– that seems able to give meaning to the official mathematical formalism. The notion of *genetic conversion* is apt for analysing some relationships among the semiotic sets of a semiotic bundle that develop in time. The notion is similar, but not identical with that of conversion (or of transformation) studied by Duval (2006). For example, students sometimes describe a function that they must produce starting from a

numerical table, before through gestures and words and only later through a graph or a formula. They produce a genetic conversion from a semiotic set (that of gestures) to a semiotic system (that of Cartesian graphs). This genetic aspect of the process is not encompassed in the standard notion of conversion between semiotic systems (Duval, 2006), which is meant for describing transformations between semiotic systems, e.g. from the sign  $y=x^2$  to the graph of a curve. In other words, conversions presuppose to act between two already existing systems. In our example, on the contrary, there is a genesis of signs from a semiotic set to a semiotic system. The signs introduced in the new set (system) are often built preserving some features of the previous signs (e.g. like the icon of a house preserves some of the features of a house, according to some cultural stereotype). The preservation generally concerns some of the extra-linguistic (e.g. iconic) features of the previous signs, which are generating new signs within the new semiotic set (or system). Usually the genesis continues with successive (genetic or standard) conversions from the new sets (systems) into already codified systems.

The chirographic reduction has been studied within the ATD frame from an institutional point of view; using the APC-s language, this reduction can be described as the tendency in mathematics to convert some semiotic set (e.g. gestures) into some other semiotic system (e.g. the arithmetic language), which can be represented with written inscriptions that can be treated through precise algorithms (as an example think to the conversion from the abacus praxeologies to the techniques of arithmetic). We have observed similar phenomena in the classroom and have studied them within the APC-s frame (Arzarello et al., 2006). Precisely, in students (of different ages) who solve problems we see such genetic conversions from a more material semiotic set to a more structured semiotic system, which can be represented on the paper and treated according to precise rules. For example in Arzarello et al. (2006) some pupils use gestures to model an arithmetic problem, then the gestures are transformed in written inscriptions that freeze their gestures in the air into a written icon; subsequently the pupils operate on the inscriptions to model properly the situation and in the end activate an arithmetic system to interpret it. The APC-s analysis shows that in doing that the students are embedded in the stream of culture because of the multimodal way of their learning processes: gestures and idiosyncratic inscriptions are deeply blended with arithmetic within an evolving semiotic bundle. Sometimes this conversion happens spontaneously, sometimes not; in these cases the semiotic mediation of the teacher is crucial. A further observation that is drawn from the APC-s analysis is that the conversion (reduction) does not mean a definitive pruning of the old semiotic sets (ostensives); namely the semiotic set from which the genetic conversion has been generated can become active later (or produce new stable ostensives). For example, when the students tackle some difficult task the old semiotic set may appear again in the semiotic bundle and can be useful to support their processes. In this sense, the symbols so produced or used within the semiotic bundle often maintain an indexical function with respect to the older semiotic set (we use the dialectic index  $V_s$ / symbol like in Peirce: see Arzarello, 2006, for more details). More than a pruning, the genetic conversion is a flexible enlargement of the semiotic bundle, which can vary in time according to the didactic situations tackled

by the students but often contains more or less explicitly the old semiotic sets that have generated the more formal new ones. Sometimes all these resources of the bundle are active and interacting very closely each other.

## CONCLUSIONS

In this paper we tried to use two different theoretical frames –the ATD frame and the APC-s frame– to interpret a specific aspect of teaching/learning processes in mathematics: the ostensives. The philosophy underpinning this research comes from the idea of networking theories, namely of comparing, contrasting and possibly combining different theories in order to get new insights about learning processes. In fact, no theory can describe all the aspects of learning processes. Each one emphasizes some aspects and neglects some others; hence combining them can be fruitful for research. Of course a problem of compatibility and coherence emerges immediately. For this reason we have chosen a specific topic, the ostensives, for starting a tentative joint analysis, that we have developed according to a complementary approach. The general idea of networking is not of constraining theories within a unique frame –an impossible task indeed– but of showing the different insights that different approaches can give to specific learning problems. Hence the issue of coherence is not on the table here. Each theory saves its specificity; the commonality is in the problem to analyse and in the richness that the different approaches can give to our understanding of teaching-learning phenomena. ATD and APC-s are very different: the one considers more the historical-institutional aspects of learning processes; the other is more interested in a social-psychological approach to learning; however, both are interested in the cultural aspect of the teaching-learning processes. Moreover the ATD is a complex and sophisticated theory, which has developed in the last decades and is very well known by many researchers, while the APC-s is a very recent one, used only by a few researchers. Our effort has been to compare and contrast the two approaches with the aim of getting some new insights into the problem of ostensives. Doing that, we have tried to enter into the other's culture and have learnt something new not only about the other theory but also about our own frame. Of course, we are conscious of the limits of our efforts, in particular of the problem of mutual coherence. At the moment, we remain at a more modest level, namely at the issue of the complementarities of approaches.

**Acknowledgments.** Research program supported by MIUR and by the Università di Torino and the Università di Modena e Reggio Emilia (PRIN Contract n. 2005019721).

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# HOW DO THEORIES INFLUENCE THE RESEARCH ON TEACHING AND LEARNING LIMITS OF FUNCTIONS?

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*After an introduction on approaches, research frameworks and theories in mathematics education research, theoretical aspects of didactical research on limits of functions are investigated. In particular, three studies with different research frameworks are analysed and compared with respect to their theoretical perspectives. It is shown how a chosen research framework defines the world in which the research lives, pointing to difficulties to compare research results within a common field of study but conducted within different frameworks.*

## INTRODUCTION

It is generally acknowledged that results from didactical research, as any other research on human behaviour in social settings, depends heavily on the underlying basic assumptions, general approach, and theories and methods used. One may also ask, for a particular study, what factors influence the choice of a specific research framework, and what consequences this choice entails. After a general introduction on research frameworks and the concept of theory, I will go into more detail looking at didactical research on a specific mathematical notion, limits of functions, often referred to as “difficult” for students to learn or understand (Mamona-Downs, 2001). I will give a short overview of some approaches and perspectives used in educational research on limits, and then compare more closely three studies, representing different research frameworks, with a focus on their theoretical underpinnings and claims. In doing this, I will consider the following question: *How does a theoretical basis chosen for a study influence the nature of the purpose, questions, methods, evidence, conclusions, and implications of the study?* This question will be studied using the theoretical notions presented in the next section.

## RESEARCH FRAMEWORKS AND THEORIES

In Lester (2005) reasons are given for why educational research needs to be pursued within a scaffolding framework. A framework is here seen as “a basic structure of the ideas (i.e. abstractions and relationships) that serve as the basis for a phenomenon that is to be investigated” (p. 458), representing its relevant features as determined by the adopted research perspective, and serving as a viewpoint to conceptualise and guide the research. A research framework thus “provides a structure for conceptualising and designing research studies”, including the nature of research questions and concepts used and how to make sense of data, allowing to “transcend common sense” (p. 458).

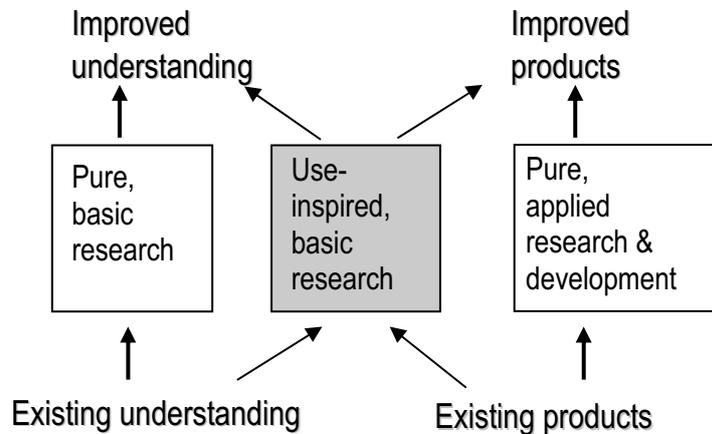
According to Eisenhart (1991) three kinds of research frameworks can be identified, that is a *theoretical*, a *practical*, and a *conceptual* framework. Lester (2005) argues that although making the choice of conforming to a particular theory has the advantages of “facilitating communication, encouraging systematic research programs, and demonstrating progress” (p. 459), it also has serious shortcomings, such as prompting explanations by decree rather than evidence, making data “travel” to serve the theory, offering weak links to everyday practice, and limiting validation by triangulation. Also practical frameworks, based on accumulated experiences and ‘what works’, may suffer from limitations caused by norms and narrow insider perspectives. The focus of a conceptual framework is more on justification than on explanation but still based on previous research. Instead of relying on one particular overarching theory as in the case of a theoretical framework, it is “built from an array of current and possibly far-ranging sources”, and can be “based on different theories and various aspects of practitioner knowledge, depending on what the researcher can argue will be relevant and important to address about a research problem” (Lester, 2005, p. 460). The validity for the chosen framework is context dependent, which is its strength considering the implications of the research. Lester thus pragmatically argues with a focus on justification, the purpose of research to answer the *why* questions, that “we should focus our efforts on using smaller, more focused theories and models of teaching, learning and development” (p. 460). The notion of a conceptual research framework relates to the idea of a *networking* strategy for dealing with the diversity of theories within mathematics education (Bikner-Ahsbabs and Prediger, 2006).

Niss (2007) notes that although the notion of theory is essential for mathematics education research, and often used, a definition of the key term *theory* is seldom or never explicitly given. He goes on to offer such a description of this notion, stating that a theory is an organised network of concepts and claims about a domain, where the concepts are linked in a connected hierarchy and claims are either basic hypothesis taken as fundamental, or obtained from these by means of formal or material derivation. To be a theory this network is also required to be stable, coherent, and consistent.

Niss (2007) also separates the purpose of using theory and its role in research. In the former category he lists explanation, prediction, guidance for action or behaviour, a structured set of lenses, a safeguard against unscientific approaches, and protection against attacks from sceptics in other disciplines. Concerning the role of theory he mentions providing an overarching framework, organising observations/ interpretations of related phenomena into a coherent whole, terminology, and research methodology. He also adds that the inclusion of theory in general is needed for publication.

Mathematics education is characterised by its *double nature* (Niss, 1999), with both a descriptive purpose, aimed at increased understanding of the phenomena studied, and a normative purpose, aimed at developing instructional design. In discussing the role of theory in research, the dynamic model presented in Lester (2005) takes this double

nature into account (see figure 1). The primary outcome of research may be to increase understanding of a specific phenomenon or to improve practice, a goal pursued along different possible pathways of pure, basic, applied, or developmental research.



**Figure 1. A dynamic model of educational research (Lester, 2005, p. 465)**

From a broad perspective, one may identify at least three different general approaches used in research on mathematics education, a cognitive, a social, and an epistemological approach. Within the *cognitive* approach the research interest is focused on the mental structures and thinking processes involved in learning, understanding and doing mathematics, including meta-cognitive dimensions. Taking a classroom perspective, or involving more broad social factors on mathematics education, a *social* approach is used. In an *epistemological* approach, focus is on the structure and use of mathematical knowledge and its diffusion in educational institutions. While acknowledging the fact that, for example, a study with an epistemological approach can use a cognitive as well as a social theoretical framework, or that an epistemological analysis of the object of learning may be used within a cognitive approach, this distinction is made here to identify the *main approach* or focus/interest of the study.

## RESEARCH ON THE MATHEMATICAL NOTION OF LIMIT

Overviews of research on limits are found in Cornu (1991) and in Harel and Trgalova (1996). Cognitive approaches have dominated this research, identifying the critical role played by conceptions of infinity, quantification, epistemological obstacles, visualization, concept images, the dialectic between processes and objects, and between intuition and formalism, conceptual metaphors and image schemata, and students' beliefs about mathematics and their role as learners. Epistemological approaches have discussed historical-philosophical aspects of the mathematical ideas involved in the limit concept (Burn, 2005), epistemological obstacles (Cornu, 1991), or contrasted mathematical and didactical organisations observed in classrooms (Barbé *et al.*, 2005).

Juter (2006) applies a cognitive approach, using a conceptual framework with a focus on concept images and the “three worlds” of Tall (2004) to investigate Swedish uni-

versity students' understanding of limits. Her study confirms the image of limits as a problematic area, but that students often tend to overestimate their own abilities as compared to their achievements. Przenioslo (2005) outlines an instructional design based on a “didactical tool” to enable students “to develop conceptions that are closer and closer to the meaning of the concept of limit of a sequence” (p. 90). Mamona-Downs (2001) also aims at developing a teaching/ learning practice by making tacit intuitive views visible and conscious. Bergsten (2006) applies an epistemological approach to analyse university students' work on limit tasks. In the next sections, three studies are described in more detail in order to discuss the consequences of using particular approaches and frameworks. Two of these studies use the same approach but refer to different kinds of research frameworks, while two differ in main approach but are both conducted within a theoretical framework.

### **APOS theory**

An example of a cognitive approach is found in Cottrill *et al.* (1996), where the theoretical framework used is explicitly stated in the paper as the APOS theory, based on Piaget's constructivism. The focus is on students' understanding of the limit concept, and after acknowledging student difficulties to understand this concept, the stated purpose is to “apply our theoretical perspective, our own mathematical knowledge, and our analyses of observations of students studying limits” to develop a “genetic decomposition of how the limit concept can be learned” (p. 167). This tool is based on the APOS theory, in particular how it treats the reconciliation of the dichotomy between “dynamic or process conceptions of limits and static or formal conceptions” (pp. 167-168). The perspective is based on the following statement about mathematical knowledge (p. 171):

Mathematical knowledge is an individual's tendency to respond, in a social context, to a perceived problem situation by constructing, re-constructing, and organising, in her or his mind, mathematical processes and objects with which to deal with the situation.

The chosen theoretical basis is mirrored in the terminology used, such as the frequent terms *construct* and *schema*, as in “the coordinated process schema is difficult in itself and not every student can construct it immediately” (p. 174). The ‘conclusion’ is an instructional design focusing on getting students to make “specific mental constructions” (p.169) of importance for understanding the limit concept. The research method is a cyclic process, where a genetic decomposition of the topic is developed by an epistemological analysis. This way the research approach also has a strong epistemological component interacting with the cognitive approach. The genetic decomposition is then forming the basis of an instructional design that is implemented. After extensive observation, and interviews of students, a renewed cycle is performed, which may cause changes in the decomposition and the design, and ultimately also in the theory.

The final genetic decomposition described consists of seven steps (see pp. 177-178), which were materialised in the instructional design. Evidence for students' construc-

tions targeted in the different steps of the decomposition is provided by analyses of interview protocols. Some conclusions about concept development are made, indicating that a “dynamic conception of limit is much more complicated than a process that is captured by the interiorization of an action” (p. 190), and that a strong such conception is needed to move to a formal conception of limit, which is not static “but instead is a very complex schema with important dynamic aspects and requires students to have constructed strong conceptions of quantification” (p. 190).

### **Reasoning and beliefs**

In a study by Alcock and Simpson (2004, 2005), the interaction between students’ modes of reasoning (i.e. visual or non-visual) and their beliefs about their own role as learners is investigated. The research is a “naturalistic inquiry into learners’ thinking about introductory real analysis” (Alcock and Simpson, 2004, p. 2), with the goal of the study being to “develop a theory of the interactions between various aspects of students’ thinking” (p. 7). The approach is thus cognitive and the research framework conceptual, since the study uses theoretical concepts from various sources rather than one overarching theory. Examples of such theoretical concepts used are on visualisation, concept image, spontaneous conceptions (Cornu, 1991), perceptual proof scheme (Harel and Sowder, 1998), semiotic control (Ferrari, 2002), and, for the method, grounded theory, and the distinction account of/account for (Mason, 2002).

The empirical data consist of protocols from interviews with pairs of students, engaged in first-year analysis courses, discussing general issues on university studies, working on given limit problems on sequences and series, and a review of the task session discussing proof and definitions. From the data the observed group of students could be classified either as ‘visual’ or ‘non-visual’ depending on their tendencies to introduce diagrams or not during tasks, to use gestures/qualitative terms or algebraic representations when offering explanations, explicitly state their preference or disinclination for pictures or diagrams in reasoning, and to base their sense making to non-algebraic or algebraic reasoning.

The visualizers generally set focus “on the mathematical objects as constructs”, draw “quick initial conclusions”, and show “Conviction in their own assertions” (Alcock and Simpson, 2004, p. 10). However, a further analysis revealed three “bands” of behaviour of the visualizers, depending on the consistency of the way the mathematical objects were displayed with the formal definitions, and on the ability to use those definitions as a basis for argumentation. These behaviours were found to interact with the students’ beliefs about the learner’s role. Students that “expect to see consistency and structure” and use “flexible links between visual and formal representations” in mathematics, show an “internal sense of authority”, setting value to their own judgement (p. 18). Students using images that are not of sufficient generality to justify their reasoning exhibit a belief that “mathematics will be provided by an external authority” (p. 24). In a similar way, the non-visual students could be divided into three “bands” of behaviour, depending on the accurate use of the mathematical

definitions, and on the degree of “semiotic control” connecting algebraic representations with underlying concepts. Also the mathematical behaviour of these students revealed an interaction with their beliefs regarding internal or external authority. The way the course was conducted could not explain the different preferences, and both groups showed a wide range of success and failure, indicating that “there is no “perfect presentation” that will be available to all students” and successful (Alcock and Simpson, 2005, p. 98).

### **The algebra and the topology of limits**

The research presented in Barbé *et al.* (2005) is located in the framework of the Anthropological Theory of Didactics (ATD) and uses the general model of mathematical and didactical activities provided by this theory in terms of mathematical and didactical praxeologies (*ibid.*). One of the main methodological principles of this research is taking into account how the mathematical knowledge as it is *proposed to be taught* constraints the students’ (and the teacher’s) mathematical practices. In the case of limits of functions, due to a complex historical process of didactic transposition, the mathematical knowledge *to be taught* appears to be a disconnected union of two mathematical organisations originated by different fundamental questions in the “scholar” mathematical institution: “the algebra of limits” that starts from the supposition of the existence of the limit of a function and poses the problem of how to calculate it for a given family of functions; and “the topology of limits” approaching the question of the nature of the mathematical object “limit of a function” and responding to the problem of the *existence of the limit* of different kinds of functions. Due to traditional tasks and techniques in textbooks and syllabi, the algebra of limits becomes the practical block of the mathematical organisation to be taught, while at the same the theoretical block remains closer to the topology of limits. This mismatch of the two parts of the taught praxeology causes problems for the teacher, as well as the students, to explain, justify, and give meaning to the work on limits. The available theoretical discourse is not appropriate to justify the techniques students learn to use and thus appears to be unmotivated, without any rationale and unable to justify the practice of the algebra of limits – which, for this reason, tends to be considered as a “mechanical” practice difficult to develop. According to the ATD, the main reason for this phenomenon has to be found, not in the human cognition of teachers and students, but in the severe constraints imposed by the process of didactic transposition on the kind of mathematics that can be taught (and learned) at school. Without taking into account these institutional constraints, it seems difficult to understand what teachers and students do (and cannot do) when facing a problem involving limits of functions.

The “split” mathematical praxeology about limits of functions explains some important “distortions” on the teacher’s and the students’ practice that are due to constraints coming from the first steps of the process of didactic transposition. For instance, the difficulties for the teacher to “give meaning” to the mathematical praxeologies to be

taught, because the rationale of limits of functions (why we need to consider and calculate them) cannot be integrated in the mathematical practice that is actually developed at this level. The empirical data for analysing these issues in the particular case reported, were taken from syllabi, textbooks, and classroom observations.

### An analysis of influences of theory

An overview of the influence of theory on the three studies discussed above is shown in table 1, structured by the research question stated in the introduction, and by the descriptions, terms and models discussed above in the general section on theory.

The two studies using a cognitive approach both investigate the influence of learning environments on the development of students' understanding of the mathematical concept of limit. The chosen frameworks, however, may be characterized as *closed* and *open*, respectively.

<b>Study</b>	<b>Cottrill et al.</b>	<b>Alcock &amp; Simpson</b>	<b>Barbé et al.</b>
<i>Main purpose (see figure 1)</i>	Improved understanding and products	Improved understanding	Improved understanding
<i>Research framework</i>	Theoretical: APOS theory	Conceptual: A set of 'local' theories and concepts	Theoretical: ATD
<i>Approach</i>	Cognitive	Cognitive	Epistemological
<i>Questions</i>	How does a 'genetic decomposition' of how the limit concept can be learned look like?	How do various aspects of students' thinking interact?	How are teachers' practices restricted by mathematical and didactical phenomena?
<i>Methods</i>	Research cycle: analysis – design – implementation – observation – analysis	Open and task based interviews	Epistemological analysis and observations of mathematical and didactical organisations
<i>Evidence</i>	Interview protocols	Interview protocols	Syllabi, textbooks, classroom observations
<i>Conclusions</i>	Dynamic conception of limit complicated  Formal concept of limit not static  Refined genetic decomposition of limit	A theory about the interactions between students' tendency to visualize and beliefs about their own role as learners	The internal dynamic of the didactic process is affected by mathematical and didactical constraints that determine teachers' practice and the mathematics taught
<i>Implications</i>	Further research on quantification needed, along with the genetic decomposition, to design effective instruction	At least in small group teaching situations, different students' tendencies to visualize should be taken into account	Problems of motivation, meaning, atomisation of curricula, etc., need a deeper understanding of institutional restrictions regulating teaching

**Table 1. The influence of theory on the research process**

Cottrill *et al.* (1996) start with, and stay within, a specific theory focusing, along with an epistemological analysis of the limit concept, on the cognitive development of the individual student, forcing interview data to be interpreted in terms of the basic notions of the theory only, that is actions, processes, objects, and schemas: “In trying to fit our observations with the APOS theory, we felt the need to pay more attention to the idea of schema than in our previous work with this theory” (p. 190). The clinical interview is chosen, in line with the Piaget tradition, as the method for collecting evidence on the state of a student’s mental schema. This is a closed framework, and the conclusions may be called a progressive confirmation.

As a contrast, the study by Alcock and Simpson (2004) began as “a qualitative investigation of the way different learning environments influence students’ developing understanding of real analysis” (p. 1), and the centrality of the distinction between visualizers and non-visualizers, and the interacting role of beliefs, did only emerge by “inductive analysis of the data” (p. 1). This is an indicator of a kind of openness of the conceptual framework chosen. Here the aim was not to develop an instructional design by using a specific theory-based tool, but to increase understanding of the influence of learning environments on students’ conceptual understanding. Thus, possibly not to force students’ thinking to fit a specific line of development, the data collection method chosen was task solving in pairs, in addition to open questions on general views on mathematics and of proof and definitions. Based on the conceptual framework, which can be seen as emerging from the research problem and the interpretation of data, the conclusion of the research is the development of “a theory which accounts for the students’ behaviour” based on the interactions between degrees of visualization and beliefs on authority (Alcock and Simpson, 2004, p. 2).

The study by Barbé *et al.* (2005) shares with Cottrill *et al.* (1996) a questioning of the mathematical content in use but outlines a very different kind of questioning of this object. While Alcock and Simpson (2004, 2005) take the “scholar” point of view on limits of functions for granted, the theory of didactic transposition allows this questioning. The fact that institutional constraints rarely are taken into account in didactic research makes it difficult to compare results. In Bosch, Chevallard and Gascon (2006) such a comparison between two studies on the concept of continuity is found, focusing on consequences of considering several dimensions of a mathematical practice instead of only one, concluding that “students’ difficulties in the learning of a “piece of knowledge” that is praxeologically ‘out of meaning’ can be taken as a positive symptom of the educational system, instead of a problem in itself” (*ibid.*).

## CONCLUSIONS

The three studies highlighted in this paper all originate from common observations of student ‘difficulties’ in the mathematical content area of limits of functions, but display, by their different choices of approaches and frameworks, different kinds of research questions and ‘answers’, based on different kinds of methods and evidence.

The conclusions from the research, in particular, differ considerably at a qualitative level: within the APOS theory, claims are made at a local conceptual and instructional level; within the conceptual framework, a local theory to account for the data is postulated; and within the ATD framework, explanations are found at a systemic level. In addition, the implications listed in table 1 stay for the cognitive approach at a local level of understandings and instruction, while the epistemological approach takes another perspective and considers the level of institutional restrictions as necessary to account for teachers' practice and students' behaviour.

It is evident from these examples how a chosen research framework defines the world in which the research lives, and grows, a fact that also has implications on how to interpret research, and points to the difficult task to compare research results within a common field of study taking into account the different approaches and research frameworks used. This is in itself a research task, and, as a consequence, requires a theoretical stance within which to work. As an example, in this paper specific theoretical tools, based mainly on Lester (2005) and Niss (2007), were chosen as a framework to structure the study of the three studies. But how does this contribute to compare and integrate the contributions of these studies, and others, to a deepened progression of our didactical knowledge of limits of functions?

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## INTEGRATING RESEARCH TEAMS: THE TELMA APPROACH

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*In the context of the Kaleidoscope Network of Excellence, six European research teams developed a methodology for integrating their research approaches. In this paper we present the methodology, based on a cross experiment, showing how it gave insight to the understanding of each team's research, and on the relationship between theoretical frameworks and experimental research.*

### INTRODUCTION

This contribution is about a research activity that is jointly carried out by six teams belonging to Kaleidoscope, a European Network of Excellence [1] that brings together many research teams in technology-enhanced learning. The aims are, on the one hand, to develop a rich and coherent theoretical and practical research foundation and, on the other hand, to develop new tools and methodologies for an interdisciplinary approach to research on learning with digital technologies at a European level (TELMA ERT 2006).

Within the activities of Kaleidoscope, a European Research Team (ERT) TELMA – Technology Enhanced Learning in Mathematics – has been established to focus on the improvements and changes that technology can bring to teaching and learning activities in Mathematics. TELMA ERT includes six teams [2] with a strong tradition in the field, and most of which have also been engaged in designing, developing, testing and integrating Interactive Learning Environments (ILE) for use in mathematics learning. TELMA first aim is to promote integration among such teams and to favour (a) the construction of a shared scientific vision, (b) the development of common projects and (c) the building of complementarities and common priorities in the area of digital technologies and mathematics education.

TELMA teams have brought with them different research questions, theoretical frameworks, work methodologies, cultural perspectives and views of the use of digital technologies for the teaching and learning of mathematics. So the teams started sharing knowledge, developing a common language and common topics of interest. This demanding task was addressed by analysing documents and some of the most significant papers provided by each team, focusing on topics considered as important for mutual knowledge and comparison among teams, such as digital technologies developed and used by the teams, theoretical frameworks and work methodologies, and contexts of digital technologies use. This work allowed identifying some common concerns (e.g., contextual, social and cultural dimensions of learning, instrumental issues, etc.), but it also put forward a diversity of ways to

deal with these common concerns which is due mainly to the variety of theoretical frameworks used by the teams (ibid.). For the sake of developing an integrated approach to the research on technology enhanced learning of mathematics, the need emerged to get a deeper insight on the role played by the theoretical frameworks each team uses in its own research. Aiming at finding some common perspectives, the teams decided to prepare a joint short-term project based on a cross-experimentation under which to look at the different teams' approaches concerning three interrelated topics: theoretical frameworks within which the teams face research in learning mathematics with technology, the role assigned to representations provided by technological tools, and the way in which each team plans and analyses the context in which the technology is employed.

This paper focuses on the teams' collaborative work aiming at *highlighting how specific theories may influence empirical research* as well as to *exhibit joint methodologies which can be used to compare, combine, integrate and complement different theoretical approaches*.

## METHODOLOGY

TELMA teams' collaborative work is based on a cross-experimentation whose aims (among others) were to provide a better understanding of the ways theoretical frameworks influence (a) the analysis of a given educational software and of the potential it offers for the mathematics learning, (b) how this potential is exploited in a particular learning context, and (c) how the results of this exploitation are analysed and interpreted. Two main methodological tools were developed and used for achieving these goals: 1) the construct of *didactical functionality* of a tool; 2) a *cross-experimentation* framed by and developed together with collaboratively-produced guidelines.

### The construct of Didactical Functionality

The construct of Didactical Functionality (DF) (Cerulli et al. 2005) was built with the aim of providing a common perspective, independent from specific theoretical frameworks, to address the variety of approaches (possibly depending on theoretical references) to the use of ILEs (as ICT tools) in mathematics education, and to link theoretical reflections and actual uses of ILEs in given contexts.

*'With didactical functionalities we mean those properties (or characteristics) of a given ICT, and/or its (or their) modalities of employment, which may favor or enhance teaching/learning processes according to a specific educational goal.*

The three key elements of the definition of the *didactical functionalities* of an ICT tool are: (1) a set of *features/characteristics* of the tool; (2) a *specific educational goal*; and (3) a set of *modalities of employing* the tool in a teaching/learning process referred to the chosen educational goal.' (ibidem, p.2)

These three dimensions are inter-related: although characteristics and features of the ILE itself can be identified through a priori inspection, these features only become functionally meaningful when understood in relation to the educational goal for which the ILE is being used and the modalities of its use. We would also point out that, when designing an ILE, designers necessarily have in mind some specific DFs, but these are not necessarily those which emerge when the tool is used. This may be especially the case when an ILE is used outside the control of its designers, according to different epistemological or educational perspectives, or in contexts different from those envisaged by the designers.

The notion of DF took a central and unifying role in the design and development of the cross-experimentation:

- on the one hand, the cross-experimentation aimed at exploring the DFs that the different teams would associate with an ILE they did not design;
- on the other hand, this notion was also used to structure the methodology for exploring the role played by theoretical frames in designing empirical research.

In fact, the three dimensions constituting the notion of DF are supposed to be always addressable, no matter what the theoretical assumptions of the research which is being analysed are.

### **The cross-experimentation**

The cross-experimentation was intended to enhance integration among the teams, by addressing a shared set of research questions derived from the three key themes of interest of the project: contexts, representations, and theoretical frameworks. On the one hand the investigation of these themes constitutes a first level of integration among TELMA teams, at least in terms of addressing shared issues. On the other hand such themes are wide and open the space for a huge number of possible research questions: the need emerged to restrict a feasible smaller number of questions. In general, the choice of specific questions to address may depend on one's interests, on possible theoretical frameworks of reference, or on other constrains. This potentially constituted a sort of centrifugal force among the teams which could contrast with the aims of the cross-experimentation itself. Thus, common questions were chosen according to a specific methodology, as detailed in the next paragraph.

One principal characteristic of the cross-experimentation was the request for each experimenting team to design and implement a teaching experiment making use of an ILE developed by another TELMA team. This decision was expected to induce deeper exchanges between the teams, and to make the influence of theoretical frames more visible through comparison of the DFs envisaged by the ILE designers and those identified by the experimenting teams. Table 1 summarises the ILEs chosen, the teams who developed the ILEs and the teams conducting the experimentation.

ILE	Developer's team	Experimenting team(s)
<i>Aplusix</i>	MeTAH	CNR-ITD, UNISI
<i>E-Slate</i>	ETL-NKUA	UNILON
<i>ARI-LAB 2</i>	CNR-ITD	MeTAH, DIDIREM, ETL-NKUA

**Table 1: The tools employed by TELMA teams in the cross experiment**

Finally, in order to allow as much comparability as possible between the research settings, it was also agreed to address common mathematical knowledge domains (fractions and algebra), with students between years 7 and 11 of schooling in experiments lasting approximately one month.

### The Guidelines

The *Guidelines* is a document collaboratively produced during the cross-experimentation which includes the research questions to be answered by each designing and experimenting team in order to frame the process of cross-team communication, as well as the answers provided by the teams before, during and after the experiments. This document was meant to draw a framework of common questions providing a methodological tool for *comparing the theoretical basis of the individual studies, their methodologies and outcomes*. Thus the questions had to reflect on the one hand the shared objectives of the cross experiment and its constraints, and on the other hand, the specificities of each research team. Thus the Guidelines were jointly built according to the following procedure:

- Three researchers of the TELMA group, experts in the subjects, developed three documents (one for each of the three key themes addressed by TELMA) each consisting of a set of possible research questions to focus on.
- The teams reviewed such documents and jointly chose a small set of questions to be addressed. The choice followed the criteria of (a) relevance to teams' interests and (b) feasibility within the cross-experimentation constraints.
- *A priori*, *a posteriori* and *a priori/a posteriori* sets of questions were developed to be answered by the experimenting teams respectively before, after and both before and after the experiments.
- In addition, each team that produced a tool employed in the experiment was required to provide a description of the educational principles underlying the design of the tool, and to indicate possible DFs of the tool.

Two examples of questions concerning theoretical frameworks are the following:

*Example 1 (theoretical frameworks - a priori):*

What theoretical frame(s) do you use and what motivated your choice? How do you see their potential and eventually limitations for this project?

*Example 2 (theoretical frameworks - a posteriori):*

In your opinion, in which ways do your theoretical choices have influenced:

- the analysis of the software and the identification of its didactic functionalities?
- the conception of the experiment?
- the choices of the data and their analysis?
- the results you obtain and the conclusions you draw from these?

### **The cross-experimentation and the Guidelines**

After the production of the first version of the Guidelines document containing a set of key questions to be addressed and identifying basic information to be provided by each team, the Guidelines became a key element around which the main phases of the cross-experimentation were developed:

1. Production of a pre-classroom experiment version, containing plans for each experiment and answers to some questions (a-priori questions).
2. Implementation of the classroom experiments.
3. Analysis of the experiments.
4. Production of the final version of the Guidelines containing answers to all the addressed questions (including a-posteriori questions).

The Guidelines may be considered both as a product and as a tool supporting TELMA collaborative work: a product in the sense that the final version contains questions and answers as well as plans, descriptions of the experiments and results, and a tool in the sense that the Guidelines structured each team's work by:

- providing research questions concerning contexts, representations, and theoretical frameworks;
- establishing the time when to address each question (e.g., before or after the classroom experiment, etc.);
- establishing common concerns to focus on when describing classroom experiments, on the basis of the definition of DFs;
- gathering under the same document, the answers provided by each team to the chosen questions, in a format which could possibly help comparisons.

In a sense, the Guidelines go both in the direction of investigating how to employ a given ILE in maths education and in the direction of integrating the work conducted by teams.

The Guidelines became also a tool for analyzing the role played by theoretical frameworks in the design, implementation and analysis of the experiments themselves and for comparing and possibly integrating different research approaches of the teams. In fact the process of building the Guidelines, and at the same time of using them as a reference for comparing teams' researches, contributed to:

- the investigation of the relationships between teams' assumed theoretical frameworks and the employed/defined DFs (and questioning the effectiveness of such DFs).
- the analysis of teams' classroom experiments design processes, and the explanation of the key choices characterising such processes, could they be depending on theoretical assumptions, institutional, cultural or other constrains.

Such objectives were addressed on the one hand, by comparing and questioning teams' answers to the questions contained in the guidelines, and on the other hand, by addressing extra questions, like the one of *example 3*, a preliminary question for preparing the terrain for answering the a posteriori question of the guidelines reported in *example 2*:

*Example 3 (DF – extra question):*

If you were to design a new experiment aiming at the same mathematical educational goal and employing the same ICT tool, which characteristics of the experiment would you keep unchanged? Which of these characteristics do you think, according to the theoretical framework you chose, are necessary conditions for the experiment to be successful?

This kind of questions bridges the DFs employed/defined by teams for their experiments, and the theoretical frameworks they assumed.

## RESULTS

As specified in the previous paragraphs, different issues concerning the role of theoretical frameworks in the design of teaching experiments were explicitly addressed by the cross-experimentation. In what follows, we outline the most significant elements emerging from the compared analysis and discussion of many aspects of the experiments carried on by TELMA teams. We start with TELMA researchers' retrospective reflections on the methodological tool itself.

### **Making clear and communicating the implicit**

The relationship between theoretical reflection and cases of practice is certainly one of the main issues that characterised the effectiveness of the cross-experiment either as a tool for comparing/integrating research approaches, or as a tool for investigating how to employ ILEs in mathematics education. In particular, researchers involved in the cross-experiment witnessed the importance of the request of conducting an explicit reflection on issues such as “research questions”, “theoretical frameworks”, “educational goals”, “analysis of the ILE”, and the relationships between them, which influence each other, and which remain often implicit. The request to communicate to the other teams how these issues influenced each other and how they influenced/determined the design, implementation and analysis of classroom

experiments, forced each team to address them explicitly, and to leave as less unexplained choices as possible.

The effort of making explicit the possible implicit factors when designing teaching experiments may not be new, however even when a researcher autonomously faces this task, s/he often deals with her/his own concerns, addresses self-posed questions. On the contrary, the reflection brought forward during the TELMA cross-experimentation required researchers to address (in practice, not only at a hypothetical level) also questions/issues raised and formulated by other researchers. As a consequence, each researcher was asked to cope with theoretical frameworks and with approaches to research in mathematics education that could possibly be not compatible with her/his own ones.

TELMA researchers share the common feeling that though highly demanding *the request of making clear and communicating resulted in a very useful effort both in terms of refining each team's investigation concerning ILE in maths education, and in terms of making the descriptions of the single classroom experiments as comparable as possible.*

### **The interaction between theoretical reflection and cases of practice**

The cross-experiment gave insights on how cultures and theoretical frameworks influence deeply how researchers conceive, conduct and analyse experiments. In what follows, we report on some interesting results with this respect.

*On the conception of the experiment.* Contextual and representational issues were central aspects of the study developed within TELMA project together with issues related to the role of teacher, social interaction and so on; consequently these were central issues of the cross-experimentation as well. Nevertheless the research teams did not address such aspects in the same way: rather, the cross-experimentation shows that though addressing the same main issues, *different teams had different priorities when designing their experiments.*

Such priorities (and differences among teams' approaches) may be determined by cultural backgrounds, theoretical frameworks and ways of approaching and conceiving research in maths education. For instance, in the experiment carried out by the DIDIREM team, the main theoretical references were the *Theory of Didactic Situations* (Brousseau 1997) and the *Anthropological Theory of Didactics* (Chevallard 1992). As a result, major attention was paid to (a) a detailed organization of a (potentially) cognitively rich 'a-didactic milieu' and (b) a distance between the experimental and the usual institutional contexts, as well as the necessity to keep this distance manageable by the teacher. Consequently, other aspects, even if considered interesting, were less emphasized (e.g., students' collaborative work, teacher's role beyond the management of the devolution and institutionalization processes).

On the contrary, the CNR-ITD team mainly referring to *Socio-constructivism* and *Activity Theory* (Cole and Engeström 1993; Engeström 1991; Vygotsky 1978)

assigned a high priority to social construction of knowledge and to the role of the teacher. Therefore, the experiment was mainly focused on these issues and minor attention was paid to other aspects (e.g., detailed organization of the milieu), many choices were not set up by the experimenting team but left to teachers (e.g., specific tasks and orchestration of the work).

Finally, let us quote ETL-NKUA team's theory-driven choice of not defining a 'strictu sensu' didactical goal for its experiment. Mainly referring to theories on 'the generation of mathematical meanings' such as *Constructionism* (Harel & Papert 1991) and *Situated Abstraction* (Noss & Hoyles 1996), ETL-NKUA researchers paid emphasis not on 'closed didactical goals' but on pupils' active construction of meanings as they operationalize the use of the available tools while making judgments and taking decisions in the process of solving a problem.

We hypothesize that such priorities may remain implicit and act as hidden variables – out of one's control – when designing experiments. The request of making clear and communicating allows/makes these variables revealed.

*What theoretical frameworks do not say.* In the previous paragraph we cited a few examples of how theoretical frameworks may – implicitly or explicitly – drive the design of a teaching experiment. This is but a part of the story; in fact the cross-experimentation revealed that though a theoretical framework may influence/inspire an experiment at a global level, it may not address/define many specific relevant aspects for the actual set up of the experiment itself. There seems to be a sort of a gap between what a theoretical framework offers, and what is needed to put into practice (within a classroom experiment). Such a gap is at the core of the relationship between theoretical reflections and cases of practice, and it remains often implicit. In the case of the TELMA cross-experimentation, the gap was revealed through comparisons among the different teams' experiments.

With this respect, the comparison results inspiring between UNISI and ITD-CNR experiments and between MeTAH and DIDIREM ones.

UNISI and ITD-CNR teams referred to compatible theoretical frameworks – respectively the *Vygotsky's Theory* (as for the construction of higher psychological functions) and the *Activity Theory* – and centered their experiments on the use of the same ILE, namely Aplusix. Nevertheless, from the ILE analysis, they identified different educational aims for their experiments. This resulted in two teaching experiments, both consistent with the respective theoretical frames, but deeply contrasting between them for the role of the teacher, the kind of tasks given to pupils, the validation of pupils' work, the use and set up of the tool.

Similarly, MeTAH and DIDIREM teams shared the same theoretical background - *Theory of Didactical Situations*, *Anthropological Theory of Didactics* - and experimented the same ILE: AriLab2. But their experiments still differed (though less dramatically than UNISI and ITD-CNR experiments) for important aspects such

as: who/what is responsible for validating pupils' work? Does validation emerge as a social product? Does it rest with the teacher? Or the opposite, does it rest with the ILE? Are pupils allowed/obliged/forbidden to use systems of representations other than those provided by AriLab2 (e.g., paper and pencil)?

## CONCLUSIONS

In this paper we exhibited the specific methodology followed by TELMA teams to address the question of investigating how specific theories may influence empirical research. We have reported on four main facets of the TELMA work: (a) the use of the construct of DF as a means to link theoretical reflections and actual uses of ILEs in given contexts; (b) the collaborative design and realisation of a cross-experimentation approach as a joint methodology to help different developing and experimenting teams to make explicit their assumptions and the set up of their experimental investigations; (c) the development of a methodological tool (i.e., the Guidelines) for comparing the theoretical basis of the individual studies, their methodologies and outcomes, and (d) the preliminary analysis of the experiments.

This preliminary analysis evidences two essential facts that contribute to the existence of a gap between the theoretical and the practical facets of an experiment:

- theoretical frames do not fully determine the design of situations aiming at an efficient use of an ILE. Many decisions taken in the design and the implementation of such situations engage other forms of rationality or are shaped by cultural and institutional habits and constraints.
- theoretical frames themselves often act as implicit and naturalized theories, more in terms of general underlying principles than of explicit operational constructs.

These issues certainly contribute to explain why the first step of the TELMA work based on the reading of published papers was only moderately productive. Making the role played by theoretical frames visible and not just invoked needed specific methodologies. With this respect, the results sketched above comfort the efficiency of the methodology developed within the TELMA project, but the exportability of the presented methodology cannot be taken for granted. Is it applicable to other research projects? What are the conditions for its applicability? Moreover, given that different forms of rationality are implicitly engaged in the design and implementation of teaching experiences, to what extent may such implicit factors be accessible to an explicit study? Finally, we believe that this kind of research is of particular importance in the European context where more and more teams are involved in cross-country projects. With this respect, the TELMA experience rises the question of what level of integration among different research teams is actually possible and what level of integration is desirable if one wants to preserve the richness of the teams' differences. Some of these questions are being addressed in ongoing work of TELMA, and in other projects involving TELMA teams.

## NOTES

1. Kaleidoscope is an initiative founded by the European Community (IST-507838) under the VI Framework Programme. See [www.noe-kaleidoscope.org](http://www.noe-kaleidoscope.org).
2. The teams (whose acronym is indicated in brackets) belong to the following Institutions: Consiglio Nazionale delle Ricerche – Istituto Tecnologie Didattiche – Italy (CNR-ITD); Università di Siena – Dipartimento di Scienze Matematiche ed Informatiche – Italy (UNISI); University of Paris VII – France (DIDIREM); Grenoble University and CNRS – Leibniz Laboratory – France (MeTAH); University of London – Institute of Education – UK (UNILON); National Kapodistrian University of Athens – Educational Technology laboratory – Greece (ETL-NKUA).

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# CONSTRUCTION OF KNOWLEDGE BY PRIMARY PUPILS: THE ROLE OF WHOLE-CLASS INTERACTION

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*The effect of peer interaction is a focus of attention in research on the RBC model of abstraction. This model has heretofore been used to analyse interactions of dyads or small groups of students rather than whole class discussions. In this paper, the RBC model is used to explore the ways in which whole class interaction facilitated construction of primary pupils' knowledge about decimal fraction expansion and non-terminating decimals. Whilst it cannot be claimed that all individuals constructed new knowledge, it will be shown that that one pupil's 'recognising' led to 'building with' by another and to 'construction' of new ideas and strategies by others. In particular, the interaction that took place during a plenary session in the final phase of the lesson seemed to facilitate rich and sophisticated construction.*

## INTRODUCTION

A major mathematical idea that young children are capable of understanding is that of infinity. There is evidence to suggest that children as young as seven and eight understand that the sequence of natural numbers is non-terminating (Fischbein, Tirosh and Hess, 1979; Hartnett and Gelman, 1998). In other research, the ability of primary school children to understand the infinite divisibility of number has been reported (Falk, Gassner, Ben-Zoor and Ben Simon, 1986; Smith, Solomon and Carey, 2005). In this paper, a lesson with primary pupils on decimal fraction expansion, non-terminating decimals and convergence towards a whole number is described. A model of abstraction proposed by Hershkowitz, Schwarz and Dreyfus (2001) is used to analyse the way pupils constructed some of these ideas. Although this model has been validated in a variety of contexts and across a range of ages, it is generally used to analyze the behaviour of individuals or small groups of students. The contribution of this paper to Working Group 11 is to show how this model might be expanded to describe the behaviour of individuals in a whole-class setting.

## THEORETICAL BACKGROUND

Hershkowitz *et al.* (2001) suggest that the genesis of abstraction passes through three stages: (a) the need for a new structure, which may arise from an intrinsic motivation to overcome obstacles or to deal with uncertainty (b) the construction of a new abstract entity within which recognizing and building with available structures are dialectically nested and (c) the consolidation of the entity so that it can be used with ease in the future. The three epistemic actions that they identify as giving a strong indication that abstraction is taking place are *recognizing*, *building with* and *constructing*. Recognition of a familiar structure occurs when a student realizes that the structure is a component of a given mathematical situation. This is not the first

time that the student has met the structure. When building with, the student is not enriched with new more complex structural knowledge but is using available structural knowledge to deal with the problem at hand. This stage is evident when he or she is involved in an application task or making an hypothesis or justifying a statement. Constructing, the most significant of the epistemic actions that are constituent of abstraction, is a process of building more complex structures from simpler structures. It involves the reorganization of mathematical elements so that a more refined structure emerges. In order to distinguish between building with and constructing, it helps if the goals of the particular activities are considered. In constructing, students use a new mathematical structure to attain their goal. In building with, a goal is attained by combining existing structures. Generally speaking, typical textbook problems are heavily based on recognition and building with whereas the solution of non-standard problems might require construction. These three epistemic actions are not linear but nested. In other words, the process of constructing does not follow recognizing and building with but simultaneously requires these other epistemic actions. Recognition is nested within building with and both of these actions are nested within constructing. A constructing action might also be nested within a more global constructing action. This model of abstraction has thus been termed ‘the dynamically nested RBC model of abstraction’ (Dreyfus, Hershkowitz and Schwarz, 2001a: p. 378). Once construction has taken place, it will progressively become more consolidated and has recently been termed the RBC + C model where the second ‘C’ stands for Consolidation (Hershkowitz, Hadas and Dreyfus, 2006)

The RBC model of abstraction described by Hershkowitz *et al.* (2001) was based on data derived from a teaching interview with one student who had a computerized tool at her disposal. However, they suggested that epistemic actions might be distributed among participants, that is, a method might be recognized by one member leading to building with by another and then to the collective construction of a new structure by the other members of the group. They later investigated RBC in the context of peer interaction in a dyad (Dreyfus, Hershkowitz and Schwarz, 2001a, 2001b) and demonstrated that the flow of RBC and flow of interaction patterns develop in parallel with each other. They found that abstraction can result under varied and numerous patterns of interaction, but that collective construction was richer than might have been the case for the two individuals within the dyad. Hershkowitz *et al.* (2006) were also interested in the relationship between construction of knowledge by individuals and the ‘shared knowledge’ of a group that is constituted by these individuals. They showed that, in the case of a triad of students working on an elementary probability problem, knowledge flowed from one student to another, that questions, explanations and self-explanation in the course of constructing played a crucial role and that shared knowledge became the basis of further constructing and/or consolidation tasks. However, each individual had a unique way of constructing a method for solving the problem and constructs of individuals within the group varied at different points in time.

The difficulties of studying ways in which a large group of students might construct knowledge are well recognized. Data will probably be complicated and messy, some members might seem passive although they are attentive to the interaction and there is ambiguity around knowledge that is shared and the personal constructs of individuals (Hershkowitz *et al.*, 2006). Yet, pupils are most often situated within the large group that constitutes the ‘class’ for their formal learning of mathematics. In this paper, I will focus on a lesson on the square roots of square and non-square numbers that took place in a senior primary class. I will endeavour to show how interactions that took place in the plenary session at the end of the lesson facilitated individuals’ constructions of mathematical knowledge. The ways in which other phases of the lesson contributed to these constructions will also be discussed

## RESEARCH

The aim of my research is to investigate the factors that contribute to the construction of advanced mathematical ideas by primary school pupils. The methodology is that of ‘teaching experiment’ in which students’ mathematical development is analysed in the social context of the classroom (Cobb, 2000). Between October 2005 and June 2006, I worked with a class of thirty pupils, aged 10 -11 years, for fifteen one hour periods. There were eighteen girls and twelve boys in this class. In a lesson that took place almost four months prior to the one described in this paper, the children had engaged in a calculator activity called ‘bullseye’ whereby they had to reach numbers using the ‘multiply’ key. e.g., use the ‘multiply’ key to reach 40 from 32 or to reach 100 from 23. In that session, most children had used numbers with two places of decimals to endeavour to reach the target number. In the plenary session at the end of the ‘bullseye’ lesson, some pupils began to experiment with the use of numbers containing more than two decimal places to reach target numbers. As the calculator seemed to facilitate children’s use of decimal numbers in the ‘bullseye’ activity, I was interested in seeing if it could assist pupils’ construction of understanding of decimal fraction expansion. I chose a context which was already familiar to the pupils, that is, to find the length of one side of a square given the area. Other ideas that pupils might be expected to construct in this context include convergence and the impossibility of expressing the square root of a number such as the number eight as a terminating decimal. The ‘bullseye’ activity was designed so that children could be eased into increasingly more difficult examples. I anticipated that the pupils would find this lesson more challenging as no natural number has a square root with only one or two decimal places, that is, the square root of any natural number is either a natural number or an irrational number. I endeavoured to offset possible frustration that this might cause by asking pupils to find their ‘nearest answer’.

The lesson was divided into three parts. In the first part, I worked with the whole class to explore how the length of the side of a square could be found if the area were given. In the second part, the pupils worked in self-selecting groups of two, three, or four on similar problems. In the third part, the whole class discussed findings with me and their teacher. All phases of the lesson were audiotaped. When children were

working in groups, audio tape recorders were distributed around the room. Groups were also encouraged to show their thinking by using ‘ways of thinking’ sheets (Lesh and Clarke, 2000). In order to demonstrate how children constructed ideas about non-terminating decimals and square roots of ‘non-square’ numbers, I will use the RBC model to analyze parts one and three of the lesson. Some of the written work produced by the children will also be presented to show that the whole class discussion during part three of the lesson facilitated the construction of new ideas. In the transcripts below, the following codes will be used, I: The researcher (myself); T: the (regular) class teacher; Chn: simultaneous contributions from two or more children - these different contributions are separated by the symbol >; Ch: a single child whose name could not be identified, otherwise pseudonyms are used. Three dots (...) represent a pause and comments are inserted in brackets.

### Part One

Initially, children discussed the properties of the square. Among the properties mentioned were those of equal sides, four right angles, parallel sides and its two-dimensional aspect. They had very little difficulty determining the length of a side of a square if the area were a square number, such as  $4 \text{ cm}^2$ ,  $9 \text{ cm}^2$  and  $36 \text{ cm}^2$ . We then went on to discuss the length of a side if the area were  $10 \text{ cm}^2$ .

I: I am going to show you a square now. (*draws on blackboard*) and [the area of] this square is ten centimetres squared... What do you think the length of the side is?...(*quiet sounds from children*)

Ch: Ten by one.

I: Would it be ten by one?

Chn: No.

Ch: Cos it's a square.

I: And what, if it were ten by one, what would it be?

Chn: A rectangle.

I: A rectangle. ...Tara, what do you think?

Tara: Two by five.

Chn: Ah!

I: Would it be two by five?

Chn: A square, that would be a rectangle.

It seems that the majority of the pupils were recognizing the link between multiplication and area or the difference between a square and a rectangle<sup>1</sup>. After some conversation of this type, one of the pupils began to conjecture that it could be a decimal:

Oran: 3.5

I: Where are you getting 3.5 from?

Oran: Cos 3.5 multiplied by two is ten.

Ch: Ohh

I: Three point...

Ch: No, it has different sides.

I: Would it be three point five, do you think?

Chn: No> different sides> then that would 70> it would be 3.5 by 2 > 70 > it would be 7 as well > ohh

On the basis of results he achieved in standardized mathematics tests, it is unlikely that Oran would have difficulty with a procedure such as 3.5 multiplied by 2 and it is possible that he meant 3.5 multiplied by itself. However, he did not choose to elaborate on this after the response from other class members. Dan now conjectures that there is no answer.

I: Dan, what do you think?

Dan: It can't be done.

I: It can't be done. Why can't it be done?

Dan: Cos it's, there's only two multiples of ten and that's ten and one and five and two, all rectangles.

I: Do you all agree with that? That it can't be done?

Chn: No.

Lisa: Unless you go into points.

Dan: Unless you go into decimals.

I: Unless you go into points!

Lisa: That's more sensible, decimals.

I: Lisa, do you think if you went into points, you might get something? What do you think?

Chn: Yeah> then it would all be the same.

When Lisa suggests that decimals could be used, other children 'build with' as they latch on to her idea. Rian now says that the answer might be a non-terminating decimal:

I: So what would you be looking for?... Rian?

Rian: Three point three three et cetera multiplied by three point three et cetera.

I: How do you know it's three point three three et cetera?

Rian: Cos, eh, one third is, it would be one third, it would be three point three multiplied by three point three is nine point nine nine and it's...

Ch: Close to...

As Rian appears quite comfortable with the idea of 3.333... it seems that he is not constructing a new idea but applying his understanding of the decimal representation of  $\frac{1}{3}$  to this situation. In other words, he too is building with, albeit assuming that 3.333 multiplies by itself to give 9.999. As the conversation progressed, other children in the class began to build on the decimal idea, e.g.,

Oran: Three point three three five

Ruth: Multiply three point three three by three point three three.

Tom: Around or about three

However, other children were thinking in terms of the perimeter:

I: What do you think?

Jack: Two and a half.

I: Why do you think two and a half?

Jack: Cos half of ten is five and then half of five is two and a half.

### Part Two

Space does not permit an analysis of the transcripts of audiotapes of conversations that took place among groups of pupils during this part of the lesson. However, what is apparent both from the audiotapes and the 'ways of thinking sheets' is that most children consolidated the idea of the link between the length of a side and the area of a square, particularly when the area was a square number. For the problem involving a square of area  $8 \text{ cm}^2$ , some groups of children gave answers such as 2.8, 2.82 or 2.83 while others suggested  $2 \times 4$ ,  $2+2+ 2+ 2$  or 'impossible'. Dan and Harry, the only pupils to extend their answers beyond two places of decimals, systematically arrived at an answer of 2.82835 from 2.82.

### Part 3

This part of the lesson centred on finding the length of a side when the area was given as eight  $\text{cm}^2$ . Finn expressed his puzzlement as follows:

Finn: You can't really do it because you'll only get it if you, if the numbers are odd, they are not the same.

However, based on her work completed during part two of the lesson, Emer suggested an answer of 2.82, and used her calculator to show that this would give an area of 7.9524. A few minutes later, she got a closer answer:

Emer: I got 8.0089.

I: ...How did you get 8.0089?

Emer: 2.83 by 2.83.

It seems that she is building with as she applies her understanding of the magnitude of decimal numbers to get an answer close to eight.

I: Now, so 2.82 by 2.82 gives you 7.9524 and 2.83 by 2.83 gives you 8.0089, so what is that telling you, is that telling you anything?

Chn: You can't>it's in between them.

I: It's in between, who is saying that it is in between?

Cara: It's kind of in between.

I: Can you give me a suggestion of what it might be?

Cara: Em 2 point...8, ... 25.

The hesitations in the last line suggest that Cara is constructing the name of the number between 2.82 and 2.83 rather than using an idea with which she is already familiar. She uses this new construction to 'build with' and a few minutes later says:

- Cara: I got 7.991.  
I: How did you get 7.991?  
Cara: I multiplied 2.827 and multiplied by 2.827.  
I: So 2.827 by 2.827.  
T: Oh, right, so...  
I: 2.825 and now changed it to...  
T: Gone up to...  
I: 2.827.

At this stage the pupils were working with their calculators to find a closer answer. There were sounds of excitement as they got closer answers. Laura built on Nicole's idea:

- Laura: Oh, I got closer (*excitedly*)  
I: What did you get, Laura?  
Laura: I got 7.997584.  
T: Ah  
...  
I: 7584 and what did you do to get that?  
Laura: I went 2.82...yeah 2 828 multiplied by 2.828.

Dan continued the work he had started during phase two.

- Dan: We got 7.9999992.  
I: Using what?  
Dan: Eh, 2.828457...multiplied by 2.828457.

Pupils were busily engaged using their calculators to find closer answers. There was a hum of excitement around the room and bursts of 'Oh, I got closer'. The teacher then asked them how long this process could take:

- Chn: Years>infinity  
T: How many years?  
Chn: Hundreds> millions> you could keep getting closer>hundreds  
I: Closer, right?  
Ch: You could use smaller and smaller and smaller fractions  
I: Do you think you could ever actually get to it?  
Chn: Maybe>never

It now seems that several children have gained significant insights into decimal fraction expansion. The chorus of responses shows that this knowledge is shared by many members of the group. The above interaction also indicates that some pupils are constructing an understanding of limits ('you could keep getting closer') and of non-terminating decimals ('never'). A short while later, I explained that a special button on the calculator could give an answer:

- I: Right, there's a special button on the calculator

- T: Where?  
I: and what you have got to do, I will give you a hint, press 8, say for example we wanted to find it for 8, you press 8 and this particular button  
...  
Chn: Em> we got it>we got it>we got it (*voices raised*)  
T: Show me  
Chn: I got it, no , oh, what is it, 2.82824271> Yes> Got it (*excitedly*)  
Chn.: We got it> we got it> we got it> I got it> Teacher> Where> I got it> child singing (*voices raised*)

Although it is difficult to gauge the extent of children's comprehension of square root, it was apparent that, at this point, the children were acting in unison and shared some appreciation of the significance of the square root operator. Later some children found the square root of five and multiplied it by itself to check its validity

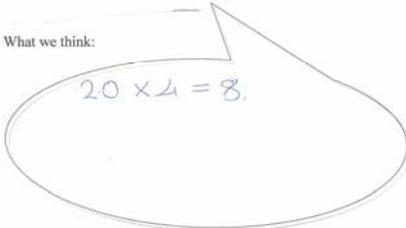
- Lisa: Oh no, that isn't our answer. Our answer is 4.99999996.  
I: Using what?  
Lisa: Using the square root button.  
I: Did you press in 5 and the square root?  
Lisa: Yes.  
I: And what did you get?  
Lisa: Well we got 2.2360679.  
I: And then when you multiplied that by itself, you get 4.99999. We explained, didn't we why it's 4.9999, do you understand why it doesn't give you exactly 5?  
Lisa: Yeah.  
I: Why?  
Dan: There's not enough...  
Jack: I know.  
Lisa: It's impossible.  
I: It's impossible. Why is it impossible?  
Lisa: It's not fitting any numbers on the calculator.  
I: And if you could fit all the numbers on the calculator?  
Jack: It's an uneven fraction.  
Lisa: Well it's not the square root of it.  
Dylan: It's an improper fraction.  
Lisa: It hasn't got a square root.

It seems that Lisa, who in phase one of the lesson suggested the use of decimals to find a square root, has developed a new construction, that is, the impossibility of finding a square root (as a terminating decimal) of certain numbers. Jack and Dylan, although not technically correct in their belief that the number on display is an 'uneven' or 'improper' fraction, are beginning to discern something about its irregularity.

## CONCLUDING REMARKS

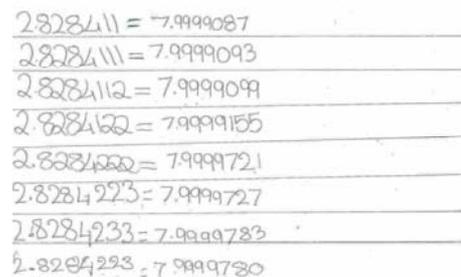
The nested RBC flow was evident in this lesson. This occurred on an individual level where some children's construction of ideas of expanded decimal notation had nested within it recognition (the relationship between area of square and length of its side) and building with (an understanding of decimal notation for tenths and hundredths). Abstraction was also distributed among members of the class. For example, Emer's building with was followed by Cara's construction which was then followed by a similar construction by Laura. In fact, Cara's construction (of using a number between 2.82 and 2.83), sparked off the quest by the group as a whole for closer and closer numbers. This led to the construction, for some individuals, of a more global construction concerning limits and non-terminating decimals. Although further investigation would be required to determine the extent to which all individuals within the class have constructed new ideas, it is the case that phase three of the session produced richer constructions than did phase two. This is true, for example, of Lily and Ellen. In figure a, the solution they presented in phase two is shown while figure b shows work they completed in phase 3 of the lesson:

What we think:



$$2.0 \times 4 = 8.$$

**Figure a**



$$\begin{aligned} 2.828411 &= 7.999087 \\ 2.8284111 &= 7.999093 \\ 2.8284112 &= 7.999099 \\ 2.8284122 &= 7.999105 \\ 2.8284222 &= 7.999121 \\ 2.8284223 &= 7.999127 \\ 2.8284233 &= 7.999133 \\ 2.8284233 &= 7.999130 \end{aligned}$$

**Figure b**

This is not to dismiss the importance of each phase of the lesson. The recognition and building with that occurred in phases one and two facilitated the constructions that occurred in phase three. In phase two, most children consolidated their understanding of the relationship between area and length of side. They also developed an awareness of the complexity of the task and this became an important motivational factor. In phase three, my mediation and that of the teacher, the distribution of ideas among students, the development of a shared goal and the almost immediate confirmation offered by the calculator were important aspects of the construction of complex mathematical ideas and strategies. In particular, what is suggested by this paper is that the RBC model, in combination with a theory that investigates epistemic processes from a more social perspective, is a useful means of analysing construction of advanced mathematical ideas in a whole-class situation.

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<sup>1</sup> In the Irish primary mathematics curriculum, the square and the rectangle are introduced as separate 2-D shapes. As a consequence, primary pupils typically perceive the rectangle as a right angled 2-D shape where length and breadth are not equal.

# **EMERGENCE OR STRUCTURE: A COMPARISON OF TWO SOCIOLOGICAL PERSPECTIVES ON MATHEMATICS CLASSROOM PRACTICE**

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*Abstract. In this paper, I explore the issue of interdependency of theory and research findings. I exemplify how analyses of a short transcript of 6<sup>th</sup>-graders collaborative problem solving – by using two different theoretical perspectives – lead to different interpretations and understanding of the same account. I use this exemplification for discussing the fundamental issues of validity and relevance in sociologically oriented research on mathematics education.*

## **INTRODUCTION**

Theories encompass the aims and goals of research, including what constitutes a problématique. They privilege some research methods over others and they determine what counts as a result. They provide a language for description and discussion (Mason and Waywood, 1996). Any research finding is deeply rooted in the underlying theories that have initiated and framed the research process. In order to understand the ways in which researchers' aims, theoretical frameworks, research methods, and research findings interact, a comparative perspective has proved to be productive (Even and Schwarz, 2003; Sfard, 2002).

The aim of this paper is to discuss the issues of validity and relevance in sociologically oriented research on mathematics education. For this purpose, the paper compares two theoretical perspectives. As an exemplification, one single piece of transcript is analysed both from a micro-sociological and from a macro-sociological perspective. Although these two theoretical perspectives have both been labelled *social* perspectives, there appears to be considerable difference between their problématiques, their methods, their results and the concepts they built on. The discussion of these differences, particularly of the divergent ways of data interpretation, is focused on the question whether micro- and macro-sociological approaches to research in mathematics education, meet the criteria of validity and relevance satisfactorily. Relevance and validity are, of course, not only prevalent within sociologically oriented research in mathematics education.

The first theoretical point of view, which is considered, here, is committed to the immediate interactions that can be observed in mathematics classrooms, for instance, amongst students when doing mathematics (e.g., Cobb, 1995; Krummheuer, 1995, 2000; Voigt, 1995). The research purpose of this approach is, roughly, to examine the relationship between the participation of students in classroom interaction and individual content-related learning. According to this view, learning is socially

constituted in such a way that social interaction is a necessary component for learning to take place. This perspective has been called *micro*-sociological (or interactionist).

In contrast, researchers who concentrate on the social distribution of knowledge, on access and resources of students favour a *macro*-sociological (or structuralist) perspective [1]. From this viewpoint, the interplay of classroom practice and external issues of social order, justice and conflict is crucial for mathematics teaching and learning. How does mathematics instruction deal with the correspondence between the hierarchy of social groups and their differential power external to the school and the hierarchies of knowledge, possibility and value within the school? Although this problématique is not unique to the teaching and learning of *mathematics* (e.g., Anyon, 1981; Bourne, 2003), issues of equity and social justice are well within the scope of research in mathematics education (e.g., Atweh, Bleicher and Cooper, 1998; Cooper and Dunne, 2000; Dowling, 1998; Lerman and Zevenbergen, 2004).

The comparison presented in this paper focuses on differences between, rather than on the common ground (such as a shared emphasis on the importance of language issues) of, micro- and macro-sociological perspectives. The paper consists of two parts. The first part presents a piece of transcript and provides some background information, before exemplifying how the transcript can be analysed from a micro-sociological and a macro-sociological perspective, respectively. The reader may find these interpretations and the presentation of the two theoretical frameworks fragmentary and considerably compressed, though this is due to the restrictions of format. By drawing on the two divergent analyses, the second part discusses whether the two sociological approaches to research in mathematics education meet the criteria of validity and relevance satisfactorily.

### **EXEMPLIFICATION: ANALYSING FROM TWO PERSPECTIVES**

The transcript, which is analysed from two theoretical perspectives, is taken from an ongoing study of language use in secondary mathematics classrooms. 6<sup>th</sup>-graders have been involved in collaborative problem solving activities and their talk has been audio taped. All students, from different types of secondary schools (modern school, comprehensive school and Gymnasium), have been quite unfamiliar with the type of problems presented to them. Problems have not been selected with respect to their mathematical importance; the main criterion has been the assumed stimulus for collective argumentation.

In a mathematics classroom of an inner-city modern school, three boys (B1, B2 and B3) and one girl (G) are starting to solve the following problem:

Hannah, Sabrina and Catherine go on vacation. One of the girls travels to the South of France, another girl travels to the Black Forest, and the third one travels to the North Sea. You know:

- Sabrina borrows snorkel equipment from the girl, who travels to the Black Forest.

- The girl, who travels to the Black Forest, and the girl named Catherine go on vacation together with their parents.
- Sabrina needs more suitcases than the girl, who travels to the South of France.

Which girl travels to which holiday destination?

The four students finish reading the problem and after some seconds their conversation starts.

- 1 B1: May I? Sabrina borrows from the girl, who travels to the South of Germany, the snorkels?
- 2 B2: This is diving gear like, hum, like diving goggles, snorkel, and ...
- 3 B1: Yeah, snorkel.
- 4 B2: Or, or, there is other diving gear. Either snorkels or kind of gas mask, yeah, gas masks.
- 5 B1: Then, I think, this Sabrina travels somehow to the Baltic Sea, or so.
- 6 B2: But it says to the North Sea.
- 7 B1: North Sea, that's what I mean.
- 8 B3: But she can also go to the South of France, 'cause it's certainly warm over there.
- 9 B1: No, not that.
- 10 B2: Okay. There is a Sabrina and ...
- 11 B3: ... she goes either to the South of France or to the North Sea.
- 12 B1: Yes, 'cause she is borrowing the snorkel gear.
- 13 B2: Or to the Black Forest.
- 14 B3: In the Black Forest you definitely cannot ... In the Black Forest there is just a river and in the river you cannot ...
- 15 G: Sabrina is borrowing the diving gear, thus I would say, Sabrina goes to the North Sea, after all.
- 16 B2: Well, but in the South of France there is a sea, too.
- 17 B3: Yeah, there is a sea, too, there you can ...
- 18 B2: The South of France is much warmer.
- 19 B3: Yeah, you can perfectly swim there.
- 20 B2: The North Sea, well, the North Sea is somehow ...
- 21 B3: Well, I would rather dive in the South of France, but you never know.
- 22 B1: The South of France! They do not travel to the South of France just for swimming in the sea! I would rather say, the North Sea.
- 23 G: Me too.
- 24 B2: I would say, the South of France. 'Cause it's warmer over there and the water is better. The water is cleaner.
- 25 B3: And, in addition, who has ever been in the North Sea for swimming roughly knows how the North Sea looks like.
- 26 B1: Okay, but the North Sea, many more people are swimming there.
- 27 B3: Okay, but diving ...

- 28 B2: Diving! Those many people just swim!  
 29 B1: In that case I write it down, now: Sabrina / South of France.

This is not their final solution. They continue by looking at the second condition. When they arrive at the third one they realise the incompatibility of it with their first solution. In the course of their conversation they finally manage to resolve all difficulties. However, they need more than 20 minutes to solve the problem.

### **Micro-Sociological Analysis of Argumentation**

From an interactionist perspective, communication in the classroom is seen as “a process of mutual adaptation wherein individuals negotiate meanings by continually modifying their interpretations” (Cobb and Bauersfeld, 1995, p. 8). A basic assumption of the interactionist viewpoint is that social dimensions are neither peripheral conditions nor societal constraints of learning mathematics but are intrinsic to it. Learning occurs during the “co-creation of interaction” (Krummheuer, 2000, p. 23) and is, thus, essentially social. As the focus of attention of interactionist research is on the negotiation of mathematical meanings in the local events of classroom life, this perspective is termed adequately *micro*-sociological.

An analysis of argumentation is prevalent in interactionist research on the learning of school mathematics. From the micro-sociological perspective, argumentation is seen primarily “as a social phenomenon, when cooperating individuals [try] to adjust their intentions and interpretations by verbally presenting the rationale of their actions” (Krummheuer, 1995, p. 229). The social genesis of argumentation is in the focus of micro-sociological analysis.

Krummheuer (1995) proposes a theoretical framework within which to investigate the social processes of argumentation that occur in mathematics classrooms. He assumes that “the claimed validity of an assertion or statement is established by an argumentation in a way that the questioned assertion appears as the conclusion of other assumed undoubtedly valid statements” (p. 247). Analytically, he distinguishes (a) a conclusion, the validity of which is doubted, (b) data, on which the conclusion is grounded, and (c) warrants, which give reason for the legitimacy of the applied inference from data to conclusion [2]. It is the goal of an appropriate analysis of interaction to identify these categories and, thus, to reconstruct the emerging rationality in the development of a collective argument.

*Analysis.* In the transcript presented above, B1 and B2 clarify the data of the argument: Sabrina borrows snorkel gear (Lines 1 to 4). So, the conclusion B1 draws is that Sabrina travels to the North Sea (Lines 5 to 7). B3 doubts the validity of this conclusion (Line 8), and these doubts are finally permitted by B1 (Line 12). B2 presents additional doubts to the conclusion (Line 13: Sabrina could go to the Black Forest), but these are successfully rejected by B3 (Line 14). There appear to exist two competing conclusions: G and B1 favour the conclusion drawn by B1 (Sabrina travels to the North Sea; Lines 5 and 7), while B2 and B3 follow the doubts of B3

(Sabrina could also go to the South of France; Line 8). Since no substantive reason has been given, the situation is undecided at this point. In the subsequent course of their argumentation the students present various warrants for the inference from the data to the conclusion they draw, respectively (B2 and B3 in Lines 18, 19, 21 and 24; B1 in Lines 22 and 26). They come to a final decision (Line 29), when B2 and B3 successfully reject the validity of one of B1's warrants (Line 26 to 28).

The analysis of this passage from a process of collaborative problem solving explicates the denseness of the course of argument: The students confront two conclusions from the given data and warrant their claims by a wealth of argumentative support; they give attention to each other's positions and they continue argumentising until the majority of them are convinced of the validity of one of the competing conclusions. Characteristically, the micro-sociological perspective focuses on the *emergence* of meaning: Classroom interaction is regarded as contingent upon how the participants of this interaction argue, how they negotiate the meaning of tasks, and what they consider as relevant information. In order to reconstruct this emergence, the micro-sociological analysis is "close" to the transcript. In the case presented, here, the analysis of the process of collaborative problem solving identifies a collective struggle for a convincing argument, in which all students participate.

The arguments, which the students collectively produce, can be termed *substantial* (Krummheuer, 1995, p. 235; 2000, p. 30), in contrast to analytic arguments that are given in formal logical conclusions. Substantial arguments are used for gradual support of statements and decisions. They do not have the logical stringency of formal deduction. From the interactionist point of view, this lack of stringency "is not taken as a weakness, but rather as a sign that fields of problems exist which are not accessible to formal logic. ... Substantial argumentation has a right to exist in itself" (Krummheuer, 2000, p. 30). The micro-sociological position takes argumentation as the art of convincing rather than as formal logical inference. The micro-sociological analysis presented, here, explicates how a convincing argument emerges through the students' collective negotiation of meanings.

### **Macro-Sociological Analysis of Discourse**

Macro-sociological studies of classroom practice often try to reveal how social advantage and disadvantage are reproduced through this practice. Classroom practices are regarded as social representations that are more or less accessible to students, depending on their social backgrounds. External to the school, there exists a hierarchy of social groups and differential power. The fundamental assumption of macro-sociological studies in mathematics education is that this *structure* translates into the hierarchies of knowledge, possibility and value within the classroom; that there is, to put it crudely, a relationship between socio-economic status, cognition and achievement.

As a rule, macro-sociological studies in mathematics education are committed to specialised theoretical tools and languages. Bernstein's general theory of pedagogic

codes and their modalities of practice (Bernstein, 1996) is most frequently referred to, though some researchers use other theoretical tools (Bourdieu, 1991; Halliday, 1978). As Lerman and Zevenbergen (2004, p. 29) remark, “Bernstein (1996) is detailed in explaining how power and control are translated into different pedagogies; the implications are that if students are to be successful they need to recognise the unspoken, or invisible, aspects of some pedagogies”.

Bernstein’s theory describes the learner of mathematics in terms of access to *recognition rules* and *realization rules*. Recognition rules are the means by which “individuals are able to recognise the speciality of the context that they are in” (Bernstein, 1996, p. 31). Realization rules allow the production of “the expected legitimate text” (p. 32). When the mathematics teacher poses a problem, students need to respond in a manner that is seen as appropriate. They must be able to recognise that particular responses are expected, and they must be able to produce a desired response. It has proved empirically that students’ access to these rules is distributed unevenly with respect to their different socio-economic background (e.g., Cooper and Dunne, 2000; Lubienski, 2000). The second analysis of the transcript focuses on the students’ possession of recognition and realization rules.

*Analysis.* The students’ collaborative problem solving occurs in a specific situation. This situation is a mathematics lesson and it is, of course, not classified as a leisure time activity. Accordingly, attached to the problem, which the students try to solve, there is an expected solution and a procedure by which this solution can be generated. Contrary to common practice in many mathematics lessons, the procedure is not given. However, a correct solution must refer to the logical relationship of the three statements about the girls’ holiday destinations. In order to concentrate on these logical relations, the students need to ‘unpack’ the problem. They must decide which part of the textual information is relevant and which not. This decision is mainly informed by the recognition rule. The four students, whose talk is documented, regard the problem as an issue of everyday life. They draw on their everyday knowledge about seawater pollution and travel distances. Consequently, their decision-making is characterised by a presentation of arguments that are intended to convince the other group members, but not by logic-mathematical thought. The four students do not recognise that drawing on their everyday knowledge is inappropriate, here, and, in fact, misleading. The transcript documents an incident of the students’ lack of the recognition rule. Since these students misrecognise the speciality of the context they are in, the production of the expected legitimate text is not possible for them.

It has been observed, that in order to solve mathematical tasks working-class [3] students are more likely to refer to their everyday life [4] and that they more often fail to recognise correctly the context, in which their mathematical activity is embedded, than middle-class students (Cooper and Dunne, 2000; Zevenbergen, 2001) [5]. The macro-sociological analysis of the transcript confirms these observations: This talk was recorded at a modern school, and the students needed more than 20 minutes to

get the – finally correct – solution. Students from other types of school, which participated in the study, were much faster and, generally, did not refer extensively to their everyday knowledge about vacations. It has been emphasised that the issue of pace can be critical for students' differential achievement in school mathematics (e.g., Boaler, 1997, pp. 125-142).

## **DISCUSSION: VALIDITY AND RELEVANCE OF RESEARCH**

It would hardly be an exaggeration to say that, by means of the micro- and the macro-sociological analysis, the students' collective problem solving has been interpreted differently. The results of the analyses, even if preliminary, clearly reflect the differential aims and goals of the research. While the micro-sociological analysis has aimed at reconstructing the *emergence* of meaning in the students' collaborative activity, the macro-sociological analysis has focused on the *structural* differences with respect to students' access to the ruling principles of school mathematics. This contrast can be exemplified by the value that micro- and macro-sociological perspectives give to substantial and analytic arguments. On the one hand, substantial arguments are valued for their sense-making capacity. Substantial argumentation "has a right to exist in itself" (Krummheuer, 2000, p. 30). It is a modality of the emergence of meaning. On the other hand, with respect to achievement in school mathematics, substantial arguments are less valued than analytic arguments. This poses the question of whether all students are equally aware of this differential valorisation. There is disagreement between the two positions, whether, in the context of schooling, lack of mathematical-logical stringency is a weakness of arguments.

The tension between emergence and structure is now used to discuss whether the two sociological approaches to research in mathematics education meet the important criteria of validity and relevance satisfactorily. As the discussion will show, there is reason to advocate the thesis that the micro-sociological approach presented, here, is strong with respect to the validity of its results, but lacks sufficient consideration of relevance – and the macro-sociological approach vice versa.

The interactionist approach to research in mathematics education is based on sociological theories and is particularly influenced by ethnomethodology, conversation analysis, and symbolic interactionism. Out of these theories a meticulously detailed repertoire of methods has been developed. It is critical for interactionists that interpretations drawn from data do not lose their footing. Since in the interactionist's view, the course of a mathematics lesson is contingent upon the actions of students and teachers (as can be seen in the students' collaborative problem solving), the *reflexivity* of these actions is highly important. Reflexivity, here, refers to the fact that in the process of social interaction, participants make their actions understandable. They use linguistic markers to make themselves understood, and these markers may serve as starting points within micro-sociological analyses. Interactionists generally dedicate plenty of time for the reconstruction of the emergence of shared meanings among the students (or teacher and students). As a

consequence of the close relationship between data and interpretation in interactionist research, the *validity* of its results is high. Interactionists avoid any claims that are not soundly supported by the data. Idealistically, theoretical concepts are developed through analyses of empirical data.

Structuralist empirical research, in contrast, is already committed to specified languages of description (e.g., as provided by Bernstein's theory). The theoretical concepts are 'already there'. Metaphorically, the theory is the lenses through which to look at the data. From an interactionist's perspective, there is a risk involved in doing that. There is the danger that empirical data is subsumed under pre-existing theoretical constructs without paying sufficient attention to the fine-grained particularities of conversations. From an interactionist's point of view, macro-sociological research may appear merely as a looking for incidents that can be used to illustrate the soundness of the underlying theory.

The *relevance* of research approaches and results is, on one hand, an issue of consistency with current research paradigms. On the other, mathematics education as a social science is reflexively related to social practices. Relevance may, or may not, be established from outside the community of researchers in mathematics education.

Structuralist research on the teaching and learning of mathematics is concerned with social inequalities of access to knowledge, with bias against lower class children with respect to learning opportunities and forms of assessment, etc. These concerns resonate in the public sphere. From an ethical perspective, research on these issues is regarded as highly relevant. In contrast, the results of research on the micro-sociology of mathematics classrooms have been widely ignored outside the community of researchers in mathematics education. Indeed, interactionist research in mathematics education can be reproached – from the vantage point of various sociologists (e.g., Laclau and Mouffe, 1985; Rorty, 1989) – for being epistemologically naïve. The relationship of intellectual discourse and social practice is disturbed, as the research does not re-act upon the social practice it is intellectually concerned with.

## FINAL REMARK

It is one question whether my critical assessment of validity and relevance is justified. Another is whether micro- and macro-sociological perspectives on mathematics education might learn something to their profit from the other's strengths.

## NOTES

1. The labelling of micro and macro is metaphorical. De Abreu (2000, p. 2) takes as the micro-context „the immediate interactional setting where face-to-face interactions take place. The macro-context is used to refer to non-immediate interactional settings“.
2. This represents a simplified version of Krummheuer's scheme: The schematic representation of argument is expanded, but not completed. For a more fully description, which gives reference to the sociological roots of the scheme, see Krummheuer (1995, pp. 239-249).

3. The categorisation of students as „working class“ or „middle class“ is, of course, an oversimplification of their socio-economic status, at least in countries like Germany.
4. A trip to the South of France is more the bourgeois' than the worker's way of spending the summer holidays. This aspect may complicate the students' decision-making, which is based on their specific everyday knowledge.
5. As Dowling (1998) shows, while most of the tasks mathematics textbooks for upper track secondary students provide fall within the esoteric domain of (abstract) mathematics, the tasks given to lower track secondary students do not introduce these students into abstract-logical thinking.

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# **AN ACTIVITY THEORY PERSPECTIVE OF DIDACTICIANS' LEARNING WITHIN A MATHEMATICS TEACHING DEVELOPMENT RESEARCH PROJECT**

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*'Learning Communities in Mathematics' is a developmental research project established with the purpose of developing teaching and learning mathematics through the creation and nurturing of inquiry communities within and between teachers and didacticians. One area of development explored is that of the team of didacticians managing the project. This report uses an activity theory perspective of 'learning as expansion' as a means of exploring data collected over the course of the first two years of the project to expose evidence of learning. Failure to expose 'expansion' leads to a proposition that the account might reveal the 'creation' of a new activity system rather than the adaptation of an existing activity system.*

## **INTRODUCTION**

'Learning Communities in Mathematics' (LCM)[1] is a development research project in Norway that seeks to improve teaching and learning mathematics in school. The project aims to achieve this goal through the creation of communities of inquiry including teachers working at all levels in school, from 1<sup>st</sup> through to 13<sup>th</sup> grade. Workshops designed to establish the community and promote inquiry in teaching and learning mathematics take a central place within the project. In these, participants work together in small groups on mathematical tasks and didactical issues. There are also plenary sessions that focus on aspects of inquiry in learning and teaching. Additionally, teachers collaborate with each other within their schools to design, implement and evaluate inquiry approaches to learning and teaching mathematics within their regular curriculum. Didacticians visit classes, often making video recordings of mathematics lessons, and follow these up in discussion with the teacher about what has taken place. More detail of the content of workshops and school meetings will emerge later in this paper. Research attention is also given to the development of the team of didacticians[2], based at a university college, who manage the project; it is on this development that this report focuses from the perspective of activity theory. Activity theory arises from the epistemological principles within which the research is framed, it offers a means of analysing development within the sociocultural and historical context within which it occurs.

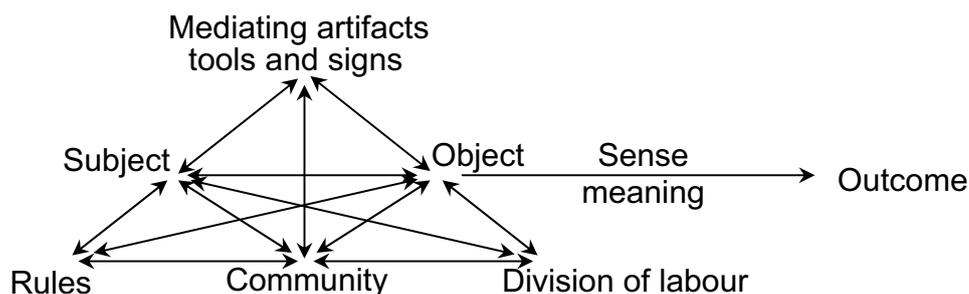
My research question: *'what has the team of didacticians learned from the first two years of the LCM project?'* requires a means of operationalising 'learning' in the context of a group collaborating in a common activity. Engeström's (1999) account of 'learning as expansion,' provides an approach to both defining and exposing learning in this context. I include here a very brief introduction to activity theory and

‘learning as expansion.’ The major part of this report comprises an account of the first two years of the project following a single line of inquiry through the project’s data to expose evidence of expansive learning. This account is necessarily long because the theory leads one to expect that “activity systems move through relatively long cycles of qualitative transformations” (Engeström, 2001, p. 137). Reflecting on this account leads me to suggest that the theoretical perspective might be developed rather than changed to accommodate the creation of a new community with its own activity system, that is, a collaborative community of researcher-practitioners.

## ACTIVITY THEORY AND LEARNING

The following is only intended as brief introduction to activity theory, more detail specifically related to the LCM context can be found in Jaworski and Goodchild (2006). My purpose here is to lay out an account that will support the explanation of ‘learning as expansion’. The development of activity theory can be traced back to Vygotsky (1978) and beyond. Vygotsky identified the use of tool and sign operations that are learnt through social activity as a distinguishing feature of human behaviour. Language is the most important of these tools and signs – as a ‘tool’ which is ‘the conductor of human influence on the object of activity’ (ibid., p. 55) and as a ‘sign’ which ‘is a means of internal activity aimed at mastering oneself’ (ibid., p. 55). Learning is then the internalization of sign operations that first appear external to the person and then reappear ‘*inside the child*’ (ibid., p. 57, italics in original).

Activity Theory develops Vygotsky’s account of tools and signs as ‘mediating’ between the person (subject) and the object of their activity to include a number of features that arise in cultural activity: *rules* regulating behaviour and engagement in a task, *community*, and *division of labour*, each with *socio-historical roots*. Figure 1. illustrates the ‘extended activity system’ proposed by Engeström (1987 and 2001).



**Figure 1: The structure of a human activity system (Engeström, 1987 and 2001)**

As a unit of analysis the extended activity system is very accommodating as it can be used at a variety of levels in which the subject might be a single person or could be a group of people. *Tools*, *rules*, *community* and *division of labour* will each relate to the specific activity. In the present account it is the team of didacticians who constitute the subject; the object of the activity is planning and implementing a professional development programme for mathematics teachers; the desired outcome is a functioning community of inquiry; the goal is better learning and teaching

mathematics. The didacticians relate to several communities – the community of researchers locally and internationally, the community of the university college, and the community of mathematics teachers. The tools used within LCM to achieve the goal are the various activities in workshops (e.g. plenary presentations, small group tasks and discussions), classroom visits and other meetings with teachers. A complex set of rules, both implicit and explicit can be identified, notable among these are the curriculum, the textbook, the constraints of time and space, and a fundamental principle upon which the LCM has been established, that of didacticians working on research and development in *co-learning partnership* (Wagner, 1997) with teachers. The division of labour can be readily appreciated by observing that didacticians have the major role in planning workshops – although it is always intended to respond positively to teachers’ needs and wishes. In schools it is the teachers who design lessons and implement these within their own classes. Didacticians might be active in the design stage but during the implementation of the designed lesson they can become, at most, participant observers. Thus it is teachers who take the risks involved in teaching-innovation, recognition of this ‘risk’ introduces an additional moral and ethical *rule* within the didacticians’ activity system.

### **LEARNING AS EXPANSION.**

In addition to the development of the extended activity system depicted in Figure 1, Engeström introduces the idea of ‘learning as expansion’ (1987) and extends the concept of internalization to include the possibility of a group of people in joint activity. He observes that within activity systems contradictions arise from deep seated tensions; these provide creative energy in an activity system. Engeström explains that ‘contradictions are not the same as problems or conflicts. Contradictions are historically accumulating structural tensions within and between activity systems’ (2001, p. 137). Further, Engeström observes that contradictions and tensions might appear both within and between the elements of the activity system. In the first instance something new might be produced by individuals to overcome or remove the contradiction, the ‘innovation’ being used by the individuals but external to the activity system of the group, Engeström refers to this as ‘externalisation’. Over time the ‘innovation’ becomes incorporated within the activity system, he refers to this as ‘internalisation’. Together, externalisation and internalisation form an ‘expansive cycle’, learning will have occurred as each cycle is completed. Engeström explains:

Creative externalisation occurs first in the form of discrete individual innovations. As the disruptions and contradictions of the activity become more demanding, internalization increasingly takes the form of critical self-reflection – and externalization, a search for solutions, increases. Externalisation reaches its peak when a new model for the activity is designed and implemented. As the new model stabilizes itself, internalization of its inherent ways and means again becomes the dominant form of learning and development. (1999, pp. 33,34)

In the present context I interpret ‘individual’ above to refer to separate innovations introduced by single members of the team of didacticians and ‘critical-self reflection’ to refer to explicit processes taking place within the team as a whole. This theory provides the means of exposing learning within the team of didacticians. It requires first to seek and expose contradictions and tensions and then see how these result in innovation, in the form of new or adapted resources, rules, processes or relationships, first by individuals and then adopted by the group as a whole, thus constituting a ‘new model for the activity’.

## **DATA ANALYSIS**

My approach to the data was first to listen to and complete data reductions (factual summaries) of all didacticians’ meetings that took place in the nine month period from April to December 2004, seeking evidence of tensions felt by didacticians. This exposed a number of potentially productive lines of inquiry. The discussions in didacticians’ meetings led to exploring the data arising from other meetings (both within schools and with teachers at the university college) and workshops. In this second source of data I have sought more precise details of events and discussions that subsequently became the background to didacticians’ discussions and decisions. Although my data consists, mostly, of the spoken word I have not engaged in a micro-level analysis of ‘discourse.’ I start with naturally occurring data, the spoken word recorded at regular meetings or workshops (i.e. arising from the developmental activity of the project), or written notes and briefing papers. My intention is to explore the development of the project team, to expose what was decided and implemented and, if possible expose individual innovations that are subsequently adopted into the activity system of the team. Because the data arises from discussions in which the team of didacticians shared, I believe it is possible to infer something about their collective beliefs and concerns, goals and constraints. I have not engaged in analysis that might reveal the ways in which decisions are reached.

## **A FUNDAMENTAL TENSION**

The first workshop was held on September 1<sup>st</sup> 2004, it was the outcome of planning that took place over a period of four and a half months. Planning started with a whole day meeting that took the form of a workshop that comprised four teachers and eight didacticians. This was followed by further meetings which took place concurrently with didacticians’ visits to schools to negotiate teachers’ participation in the project. The planning meetings were thus directly and indirectly informed by teachers.

There was no difficulty in identifying sources of tension, especially one, felt keenly by the didacticians from the outset. This related to the possibility that the intention to develop inquiry as an approach to teaching and learning mathematics might be perceived by teachers as pulling in a different direction to the curriculum, a fundamental *rule* in the teachers’ activity system. The Project Director (PD)[3] expresses this concern in a meeting at the beginning of May 2004:

I would just like to say that I think there is at least one major issue and for me that issue is a tension between choosing activities that are designed to promote inquiry and choosing activities that can be clearly seen to relate to the curriculum[4].

Many mathematics teachers may experience this tension, albeit with some differences, in their practice; they want their students to develop a range of basic mathematical competencies such as: problem solving; communicating mathematically in number, diagrams, symbols and words, and mathematical thinking skills. Their aim is to enable students to understand mathematical concepts and processes and to become competent learners of the subject. Contrary to these goals it often appears that the curriculum operationalises students' mathematical knowledge through tests that favour memorisation of facts and routine skills.

Subsequent discussion in the meeting reveals that PD's sense of this tension was shared and it was recognised that the tension would be realised by teachers in a number of forms. Some teachers work through a text book, as an embodiment of the curriculum, the introduction of additional or different approaches might interfere with their progress through the book. Some teachers, especially in the upper secondary schools, feel the curriculum places them under a lot of pressure given the time available to prepare students for their examinations. They would be rather cautious about adding to their work. Some teachers might come to the workshops hoping for, perhaps expecting, to be given ideas and activities which they could immediately implement with their classes. The project, however, aims to introduce teachers to 'inquiry' so that they are not dependent upon others' ideas but can create new learning opportunities for themselves within their own school teams. The project also aims to bring together teachers working with pupils from grade 1 to grade 13, clearly the curriculum varies across the grades and thus the intention to develop community might be contrary to having a clear curriculum focus.

The issues surrounding this perceived tension continued to be a significant focus of discussion and the programme and content of the first workshop did not emerge until quite late in the planning process. In a meeting just two weeks before the workshop PD expressed her opinion that the first workshop should be concerned primarily with 'community building' and working out 'how we are going to work together, and building up our relationships'. The focus on community building and inquiry are crucial but the perceived tension arising from consideration of the curriculum had not been forgotten. Five days before the workshop PD circulated briefing notes in which she drew attention to the possibility that some teachers might want to 'address questions of curriculum, and relatedness of inquiry to curriculum' and she asked colleagues, other didacticians, to deal sensitively with these but to try to avoid being 'sidetracked' from the intended programme, holding out the possibility that issues relating to the curriculum would be considered in the second workshop.

This must be interpreted in the context of everything else expressed about the tension felt over the issue of the curriculum. The concern about being 'sidetracked' does not

arise because PD is unconcerned about the curriculum (as became evident when the issue did arise in the workshop) but rather that there were project goals to be achieved which could be missed if discussions relating to the curriculum were to take over at this early stage. The issue continued to be of concern to didacticians, in a meeting just two days before the workshop, Leo remarks:

I wonder when the workshop is coming to the end on Wednesday how does the teachers think about this? Will they say that ‘oh this mathematics is nothing for my class!’ It was too low or too high or are they thinking about this as learning as a way to work, what is the purpose for this meeting, will the teachers understand why they are here? Or are we agreeing with them why we have this workshop?

## **THE FIRST WORKSHOP**

The workshop programme included a welcome and introduction to the project by PD. This was followed by very short introductions to 3 inquiry based mathematical investigations that, it was felt, could be tackled at many levels and thus suitable for all participants. The idea was to present three tasks in plenary session, this would be followed by a vote in which one of the tasks was voted out. Then, in small groups, a further choice was to be made in which each group decided to work on one of the remaining two tasks. This would offer participants some choice for the task and facilitate a coherent plenary discussion afterwards. The work in small groups was followed by two presentations by didacticians, one on the characteristics of rich mathematical tasks, the other on the meaning of ‘inquiry’. The workshop ended with school teams planning their project-focused activity and a short plenary discussion.

The perceived tension between ‘inquiry approaches’ and curriculum demands did emerge during the course of the workshop. Osvald, an upper secondary teacher contrasted the tasks presented, two of the tasks he described as fun, the other being useful in the curriculum. It is interesting to note that the ‘useful’ task was the one voted out in the first stage, Osvald being amongst the majority voting for one of the ‘fun’ tasks. However, he later remarked in plenary session:

Yes because we have a textbook we have to get through, and that kind of problem is, it’s, they’re nice problems, but um it’s easy to take quite a lot of time. That’s the problem for, if you have a fixed pensum (curriculum) you have to go through.

It seems that Osvald was expressing the same tension felt by the didacticians and made the same decision – in this first workshop he wanted to work on a ‘fun’ task!

## **THE FIRST YEAR OF WORKSHOPS**

During the course of the first year of the project workshops focused on ‘inquiry’ in mathematics from a number of different perspectives. Of the six workshops, the first two focused on ‘inquiry’ tasks. Workshop 3 took a specific curriculum focus in response to one upper secondary school team requesting that some attention be given to the topic of probability. In workshop 4 a major focus was on taking regular text

book tasks and transforming these into ‘rich mathematical (inquiry) tasks’. Workshop 5 focused on the inquiry arising from a design cycle implemented in one of the upper secondary schools where teachers had designed an ‘inquiry’ approach to linear functions (from the very first workshop this team of teachers had spoken about the need to design some material to complement the textbook presentation which they experienced as ‘a little boring’[5]). Workshop 6 used students’ responses to some questions on number, algebra and graphical representation from tests that had been produced as part of LCM, as the starting point to inquire into the reasons for errors and possible teaching responses. Following the second workshop the evidence in the record of meetings is that these complementary approaches to inquiry, were largely in response to teachers’ comments and they intentionally focused on issues arising from the curriculum, from text books and from students’ activity in the teachers’ own classrooms. The emergent nature of the workshop programmes as a reaction to teachers’ requests and experiences was intended from the outset as it was argued in the initial planning meetings that if too much detail were planned in advance it would leave little opportunity to develop the desired co-learning partnership. Recall, my intention is to expose ‘individual innovations’ in response to the perceived tension between inquiry and the teachers’ interpretation of the curriculum, but what I observe is a number of different approaches to inquiry that are intended to be relevant to the curriculum.

Towards the end of the first year the teachers at the two upper secondary schools met and subsequently invited the didacticians to meet with them to discuss the progress of the project. From the didacticians’ perspective this was a welcome sign because it demonstrated the teachers taking a share in the ownership of the project. The meeting took place in June 2005. Olav, one of the teachers, opened the discussion as follows [The evidence of hesitation and agreement from others has been retained in this quotation because it gives a sense of Olav being very careful over the choice of words as he introduces a line of criticism]:

We, in principle, we are very positive to the project, is that right? [Others express agreement: ‘Mmmm’] Right, and we think that it’s a very good idea and um but we, what we perhaps feel has been (pause) well we feel that there has been a lack of progress perhaps throughout the, during the year. Um we, we understand and accept that to begin with we have to perhaps some um (pause) um when we were working together in groups to begin with I think it was a great idea to to motivate people for the um for inquiries, use of inquiries in mathematics right? To, to um get people to know the method and to motivate and convince people that this is a very good idea. And also to get people to know each other but we felt perhaps that we have been working for too long for too long period with the same kind of inquiry problems that’s what we feel perhaps that [others: ‘Mmmm’] The problems that we have been working on have been um have been too general so to speak. We think especially for (upper secondary school) it’s um um where we are, we have a um more pressure on time and and um curriculum and so on. We feel that we need to um to make the, the themes for the inquiries more specific perhaps so that

they can be applied more directly to our curriculum. [pause; others: ‘Mmmm’] So we feel that we more or less have been doing perhaps the same things over and over again.

It would be interesting to explore reasons for the teachers’ perception of the amount of repetition which they express given the variety that has been outlined above. Whatever the reasons the tensions expressed amongst didacticians before the first workshop were plainly evident amongst the teachers one year later.

## **PLANNING THE SECOND YEAR OF WORKSHOPS**

Following the meeting with the upper secondary teachers, all teachers within the project were invited to a meeting at the university college in August 2005 to discuss progress throughout the year and lay plans for the second year. The meeting resulted in an open and purposeful discussion that set the agenda for workshops throughout the second year. Immediately after the meeting was closed Osvald broke in to say:

Can I be allowed to say one thing more that lies a bit on my heart, and that is that when. ... I have spoken much today, but also said that like as, .. what we have done, isn’t it, we will rather do it a little differently. Then the rumour could easily spread that we are dissatisfied with what is done or that I am dissatisfied. On the contrary, I will gladly praise you for what you have done and I think it has been very interesting what we have done so far. Let that be clear. It is not therefore that rather we will do it differently, but I think you have been clever to get interesting, varied subjects. That you must be praised for. [Original in Norwegian, author’s translation.]

Osvald, an upper secondary school teacher had been present at the previous school meeting, his interjection at this point reveals an additional perspective to the critique. It would appear that the project was valued and the efforts to make it successful were appreciated. Teachers’ readiness to come forward with constructive criticism is seen as evidence of teachers acknowledging the project as being of value to them and their entering into the ‘co-learning agreement’ as intended.

In the meeting, in which Osvald made the above remark, teachers had requested workshops to be much closer to their daily concerns in lessons. Within workshops they wanted time with colleagues working at the same level to begin designing lessons for their classes. They also wanted input, possibly from didacticians, on major curriculum themes. The result was a series of workshops (7, 8, 9 and 10) focusing successively on broad themes suggested by teachers – probability, geometry and algebra. The focus of group work was ‘planlegg et opplegg’ (roughly ‘plan a lesson’), with planning activities in one workshop followed by reporting the implementation of plans in the following workshop. This process fitted very well with the project’s practical foundations in action research, design research, and Japanese lesson study (Stigler and Hiebert, 1999).

The evolution of the workshops does not stop at this point. In workshop 10, focusing on algebra, the upper secondary teachers challenged the didacticians to offer suggestions for addressing some of the common and persistent errors that pupils

make in algebraic manipulation. The challenge resulted in a rich discussion between didacticians in a later planning meeting. The didacticians concluded that the best way to meet the teachers' challenge was to facilitate teachers in having a similarly rich discussion amongst themselves. To achieve this, in workshop 11 teachers were set a task to do a small piece of research inquiring into their pupils' understanding of algebra, or in the case of younger pupils, 'pre-algebra'. Teachers were invited to report the outcome of their research in workshop 12 and pool their ideas to begin to identify a 'red thread' of development of algebraic understanding and see pupils' errors in the context of their developing understanding. This marks the end of the second year of the project and the extent of data considered for this paper.

### **WHERE IS THE EXPANSIVE CYCLE? WHERE IS THE EVIDENCE OF DIDACTICIANS' LEARNING?**

The theory of 'learning as expansion' led to my focus on the tension between 'inquiry' and 'curriculum' and then to seek evidence of innovation that became internalised within the project; internalisation that might be described as learning by the team of didacticians. However, this account has not exposed evidence of innovation and internalisation. It seems that the team of didacticians has been reactive to teachers' requests and suggestions, always conscious of the curriculum demands felt by the teachers – although the evidence also suggests that the responses have not always been entirely successful in convincing the teachers. The development of the project appears to have been evolutionary, with adaptation to the context – as expressed through the rules, community and division of labour. So where is the learning that I set out to reveal?

This question has challenged me increasingly as I have explored the data and prepared this account, I do not think there is evidence here to challenge the theory, hence I seek answers elsewhere. One possible answer is that the complexity of mathematics as a subject, and the processes of teaching and learning mathematics offer means of addressing the 'inquiry-curriculum' tension other than innovation. Another possible answer is to believe my inquiry is misplaced. The account that I have provided is not that of didacticians in an activity system that develops over time as a result of internal contradictions and tensions – in the manner described by Engeström; at least, not starting with the inquiry-curriculum issue as the tension. The account describes the conjunction of two, possibly more, quite separate activity systems: one activity system of the didacticians and the other system or systems of the teachers (possibly a separate system for the team of teachers in each school). The two years of the project have not been about the development a new model of an existing activity system but rather a process of creating a new activity system in which teachers and didacticians come together as a unified subject. As the project enters phase 3 it will be interesting to expose evidence of this new activity system and signs of its development – through a process of expansive learning.

## NOTES

- 1 LCM is supported by the Research Council of Norway (Norges Forskningsråd): Project number 157949/S20
2. This report follows Cestari, Daland, Eriksen and Jaworski (2005) who, at the 4th ERME conference, introduced LCM and reported concerns relating to 'the roles of didacticians in working with teachers to develop an inquiry approach' (p. 1) that were felt at the outset of the project.
- 3 Throughout this paper names have been changed. However, I believe that the Project Director has a specially significant voice within the project and thus I will refer to the Project Director as PD.
- 4 Excerpts are transcriptions taken from recordings made at the time. In most cases pauses and hesitations have been removed to improve readability. Except where indicated transcriptions are in the language of the original.
- 5 An account of this design cycle is available in Fuglestad., Goodchild & Jaworski (2007).

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**THEORY IN DEVELOPMENTAL RESEARCH IN MATHEMATICS  
TEACHING AND LEARNING: SOCIAL PRACTICE THEORY AND  
COMMUNITY OF INQUIRY AS ANALYTICAL TOOLS**

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*In a research project exploring development in mathematics teaching we use theory to illuminate a complex terrain and provide analytical tools. The project is located theoretically within a sociocultural epistemology, which recognizes both community and individual learning. Social practice theory (SPT) offers frames in which to characterize, explain and analyze learning situations. However, treatment of teaching is more problematic. The paper shows that teaching, itself, can be seen as learning. Extending SPT to include a critical dimension allows analysis of teaching situations in an inquiry community model of developmental practice. The paper emphasizes the analytical role of theory in developmental research and the need to rationalize different areas of theory.*

In a research and development project *Learning Communities in Mathematics* (LCM) in Norway, we are concerned with learning in three complexly inter-related layers: students' learning of mathematics, teachers' learning of ways of promoting students' learning of mathematics (commonly called *teaching*) and didacticians; learning of ways of promoting teachers' learning of teaching (we might call this *didacting*). The words used to describe these three ways are only a shorthand for the complexities of knowledge and knowing<sup>1</sup> involved in these layers. Lave (1996) speaks of "teaching as learning in practice". Our learning communities in the project consist of groups of learners within the practices in which we engage: students doing mathematics in classrooms, teachers, planning for teaching mathematics and interpreting plans with students in classrooms, and so on.

The research field is that of mathematics teaching and its development with questions that address how mathematics teaching can develop to provide principled learning of mathematics for students. By this we mean that students should achieve both fluency with mathematics and conceptual understanding of mathematical topics. These questions are being addressed within a field of practice which includes teachers and didacticians engaging in and with teaching activity that promotes or facilitates mathematics learning. Teachers and didacticians are positioned as co-learners within the practice field (Wagner, 1997). Both engage in research to develop knowledge of practice. Thus teachers and didacticians are both practitioners and researchers, with complementary knowledge and practice, so that fields of practice and research are deeply intertwined and mutually constitutive (Jaworski 2003; 2005; 2006). Research studies *processes* and *activity* in both design and interaction in teaching. Here activity addresses what people do, and the associated processes address what is involved in doing it.

Teachers work with students in classrooms and design activity for classrooms to promote and facilitate mathematics learning. Students in classrooms engage with

mathematics in an interactive setting through tasks created by teachers, using associated materials or tools. Didacticians design workshops and work with teachers in schools to promote learning and teaching and study it. A developmental aim is that students' learning will improve as teachers and didacticians come to know more about learning processes and the tasks and tools that promote learning.

Teachers bring knowledge of mathematics, didactics and pedagogy, of school systems and practice, of curriculum and assessment, of students and the social setting in which classroom interactions take place. This knowledge has been characterized as *craft* knowledge that is largely rooted in the practice of teaching, creating 'normal; desirable states' within which teachers and pupils can work comfortably together (Brown and McIntyre, 1993, p.54). Didacticians bring knowledge also of mathematics, didactics and pedagogy: in contrast with teachers' knowledge, this is rooted in the literature, in theories of learning and teaching, and in principles, processes and practices of teacher education. Although such knowledge might be seen as more theoretical when compared with craft knowledge of teachers, there is nevertheless a craft involved in *didacting* – working to promote teaching development. *Craft* knowledge denotes *knowledge in practice* (Schön, 1987). "Practice", is a term with different levels of meaning: it has been used in a number of ways in the text so far, and in *social practice theory*, discussed further below, it takes on a wider meaning.

Fundamental in the LCM project is the concept of *community* and our meaning in speaking of *learning communities*. In the following sections, I start with a consideration of *community* leading into a discussion of *situated cognition* and *social practice theory* and the associated concept of *community of practice* which allows conceptualization of established practices within schools. To capture practice within the project community, I introduce concepts of *inquiry* and *critical alignment* which allow analysis of relationships between interpretation of project design and established ways of being and doing in schools. I offer two brief examples to illustrate outcomes and issues in analysis of project data according to the theoretical ideas outlined, and end with an attempt to show how progress according to developmental aims can be discerned within the complexity of the project.

### **Community: the individual in the social**

In LCM, a team of 13 didacticians at a university college and teacher teams of 3 or more teachers in 8 schools engage collaboratively to develop learning and teaching of mathematics in schools. Teachers are members of their school communities and didacticians of their university community; all are members of the project community; all are members of communities within societal systems and structures in Norway. The term 'community' designates a group of people identifiable by who they are in terms of how they relate to each other, their common activities and ways of thinking, beliefs and values. Activities are likely to be explicit, whereas ways of thinking, beliefs and values are more implicit. Wenger (1998, p. 5) describes community as "a way of talking about the social configurations in which our

enterprises are defined as worth pursuing and our participation is recognisable as competence". In a learning community, "learning involves transformation of participation in collaborative endeavour" (Rogoff, 1996, p. 388).

Within the communities of our project we *recognize* both individuals and groups: that is we ascribe *identity* to both. For example, the teacher team within a particular school has identity related to their school as a social system and group of people. Any individual teacher or didactician has identity related to their involvement in the project, particularly, but constituted through the many other communities of which the individual is part. Individual identity and group identity are complexly related.

We see knowledge as socially rooted, with individuals forming identity as part of social engagement. *Engagement* is a dynamic concept denoting active participation and mental inclusion. People *think* as they *do* and speech is a mediating force in both doing and thinking, allowing formation, expression and communication of thought and action. This position is fundamentally Vygotskian, "giving analytic primacy to social processes" (Wertsch & Penuel, 1996, p. 417). Nevertheless, individual identity figures strongly in our analyses, and we have to be careful that the theoretical and analytic do not trap us into taking an ontological position in which "What one side says *should* be is thought and spoken of as if it *is*" (Elias, 1991, in Wertsch & Penuel, 1996, p. 421). In other words, we take care to see theory as a means to enable analysis and should beware of acting as if the theory determines what is.

### **Situated cognition and social practice theory (SPT)**

In situated cognition, a situated position sees knowledge as being *in* the practice, with learning as transformation of participation in practice (Lave, 1988; Rogoff, Matusov and White, 1996). For Wenger,

The concept of practice connotes doing, but not just doing in and of itself. It is doing in a historical and social context that gives structure and meaning to what we do. In this sense practice is always social practice (1998, p.47).

According to Wenger, a *community of practice* involves three dimensions: mutual engagement, joint enterprise and a shared repertoire (1998, p. 73). People participate together in activity with an agreed focus and purpose and common ways of acting and being. Learning can be seen as deeply *situated* within the contexts or practices in which we engage, with the *transfer* of this knowledge across contexts as problematic (Lave & Wenger, 1991). For example, within the LCM project, concepts explored between teachers and didacticians in workshops at the university may not be mindful to teachers in their school context. Thus, knowledge can be seen as constituted differently in workshop and school communities and we have to work on revealing differences and relating concepts across contexts.

Wenger (1998) proposes that identity develops through *belonging* to a community of practice involving *engagement*, *imagination* and *alignment*. We *engage* with ideas through communicative practice, develop those ideas through exercising *imagination* and *align* ourselves "with respect to a broad and rich picture of the world" (p. 218).

The terms *participation*, *belonging*, *engagement* and *alignment* all point towards the situatedness of doing and being and the growth of knowledge in practice. In the LCM project teachers and didacticians *engage* in practices in workshops and school settings and *align* themselves with existing or emerging practices. *Imagination* contributes to the emergence of new practices. I will discuss aspects of *emergence* below, and contrast established practice with emergent practice.

### **The place of teaching in SPT**

Seeing learning as *participation in social practice* raises questions about the meaning of teaching and the role of a teacher. Lave (1996) writes

People who have attended school for many years may well assume that teaching is necessary if learning is to occur. Here I take the view that teaching is neither necessary nor sufficient to produce learning, and that the socio-cultural categories that divide teachers from learners in schools mystify the crucial ways in which learning is fundamental to participation and all participants in social practice. (p. 157)

Lave here points to what I see as a key issue in using social practice theory as an analytical frame. This theory is illuminative in offering a means of characterizing and analyzing learning: for example, teachers' learning of mathematics teaching, or students' learning of mathematics. However, the frame is unhelpful in characterizing or analyzing mathematics teaching – indeed, according to Lave, “teaching is neither necessary nor sufficient to produce learning”. Children learn in many contexts outside the classroom. However, where mathematics is concerned, many concepts are not available to children through everyday activity. So something else is needed to make mathematics accessible for children's learning. If we are to call this ‘something’ *teaching*, how do we interpret the term “teaching”? What exactly is taken to be the role of a teacher? Lave (1996, p. 158) writes further

... if teachers teach in order to effect learning, the only way to discover whether they are having effects and if so what those are, is to explore whether, and if so how, there are changes in the participation of learners learning in their various communities of practice. If we intend to be thorough, and we presume teaching has some impact on learners, then such research would include the effects of teaching on teachers as learners as well.

This statement captures well what we are trying to achieve in the LCM project. A key term in both statements is “participation”. According to Lave and Wenger (1991), knowledge is in participation in the practice or activity, and not in the individual consciousness of the participants. “The unit of analysis is thus not the individual, nor the environment, but a relation between the two” (Nardi, 1996, p.71). So, the practice, or activity, in which participants engage is crucial to a situated (social practice theory) perspective.

It is possible to see teachers' engagement in teaching in these terms. This is to see teaching as a practice in which the knowledge of teaching (craft knowledge) is in the practice of teaching. We can study aspects of this practice and provide deep accounts of both the practice and the knowledge within (e.g., Brown & McIntyre, 1993). The

unit of analysis here is *practice*, and the focus of research is on an arena or situation (Lave, 1988) which allows a study of practice – for example one classroom, or a whole school. Such a study might try to capture learning, or growth of knowledge, within teaching – i.e., teachers learning about teaching. This however, says nothing about the learning of students to whom the teaching is presumably directed. We need a link between students’ learning of mathematics and teachers’ learning of teaching mathematics. We can offer similar arguments for *didacting*.

Such a link is provided by Vygotsky’s conceptualization of zone of proximal development (ZPD: Vygotsky, 1978, p. 84), where teaching is seen as a process of enabling students’ *potential* achievement. Seeing teaching as learning in practice can be viewed as an exploration of such a process. In what ways can teaching afford such achievement on the part of the learners with whom it is associated? In a mathematics classroom it is expected (or highly desired) that students will achieve conceptual awareness and competency in mathematics, and such an aim is the focus of teaching. Vygotsky emphasised learning of *scientific concepts* – concepts which cannot be grasped empirically, but require a theoretical mode of thinking for their appropriation (Schmittau, 2003). We might see most mathematical concepts as fitting with this category. Schmittau, drawing on Davidov, speaks of scientific concepts as requiring “pedagogical mediation for their appropriation” (ibid, p. 226). In other words, for the learning of mathematics, teaching is necessary. In our study, it is important that the learning we are talking about is the learning of mathematics (for students) or learning of the teaching of mathematics (for teachers). Despite research which shows mathematical learning in everyday contexts, the arena of mathematics is much less available for participation, unlike tailoring or supermarket shopping, both of which are arenas of practice (used in Lave’s exposition of SPT) with clear tangible dimensions. Even at university level, where learners might be seen as being enculturated into the practices of mathematicians, the field of mathematics for participation has to be created by teachers.

Thus, mathematics teaching, fundamentally, has to create opportunity for engagement with mathematics, and to offer critical guidance as to what mathematical achievement means or can mean. This is not so transparent to participants as the tailor’s cutting of a garment or finding goods in a supermarket. So, an important part of the LCM project concerns a developmental approach to learning about *the creation of opportunity for engagement in mathematics (learning to create learning* – a principle aim of the activity of teaching) leading to principled mathematics learning. Thus, taking our unit of analysis as *practice*, in this wider sense, enables us to go beyond either the narrow separation of teaching/learning processes, or assumptions of a linear progression from teaching to learning.

### **Knowledge in Practice & Community of inquiry**

I turn, now, to the nature of knowledge and learning in LCM. Drawing on Lave and Wenger, we can see *learning* as transformation of participation in social practice and

*knowledge* as being *in* participation. So, for example, teaching in a particular school involves working with pupils within the school system, and learning includes coming to know the system and the pupils for smooth ways of working and achievement of the ‘normal desirable states’ of school practice (Brown and McIntyre, 1993, p. 54). Such knowledge-in-practice might be largely tacit: unrecognized and unarticulated by teachers (Schön, 1987). In LCM, we are centrally concerned with analysis of teaching *development*, a shift from looking at teaching per se to considering what will or can improve teaching. This requires us to make the practice of teaching explicit, revealing the forms of knowledge inherent in the norms of practice in order to create other possibilities. Thus we see the shift from learning as part of engagement in a community of practice to *learning to promote learning in a community of practice*. Whereas SPT offers us ways of conceptualizing knowledge and learning within a field of practice, it is more problematic to see how it offers a conceptualization of *learning to promote learning* (i.e., of developing teaching).

Two theoretical devices enable us to extend the basic idea of CoP to enable the required conceptualization. The first comes from modifying Wenger’s three characteristics of “belonging” to a community of practice (engagement, imagination and alignment) to conceptualize “*critical alignment*” a means of not just aligning with practice as established in the community, but of *looking critically* at that practice while aligning with it. The second, which offers a *means* of looking critically, is to engage in *inquiry* as a mode of practice (Jaworski, 2004, 2006). “Inquiry” brings with it a critically questioning attitude towards practice and knowledge in practice that allows critical reflection on the practice of teaching and hence can lead to development of teaching. Notions of *inquiry* develop from longstanding research in mathematics education, for example, in the problem solving movement (e.g., Schoenfeld, 1985) and in forms of action research into developing mathematics teaching (e.g., Zack, Mousely & Breen, 1997).

As discussed elsewhere (e.g., Jaworski, 2005; 2006), a theoretical construct underpinning activity in the LCM project is that of *community of inquiry*. Community of inquiry extends notions of community of practice to assume a goal of practice that is overtly developmental. It takes practice itself to be developmental, and assumes a developmental role in the study of this practice. We describe our research paradigm as *developmental*. *Inquiry* permeates both practice and the research of practice. Students engage in mathematics in classrooms in inquiry modes; teachers engage with inquiry in designing and critically examining teaching materials and approaches: didacticians engage with inquiry in designing, and critically examining, ways of working with teachers. Teachers’ and didacticians’ insider research of practice leads to new knowledge *in* practice, and outsider research of the entire process in all its complexity provides access to developmental knowledge for sharing with the wider research community. These two theoretical devices are closely linked: I see *community of inquiry* extending Wenger’s exposition of *community of practice* through introducing *inquiry* which adds the critical element to *belonging*, as in *critical alignment* (Jaworski, 2006).

## Manifestations of theory in practice

In LCM, we have communities of teachers and didacticians participating in practices in schools and university and together in the project. School and university practices are well established with historically rooted sociocultural antecedents, and can be described as communities of practice in Wenger's terms. In contrast, the LCM community is under construction, with overt developmental goals and a focus on *inquiry*. Thus we can use SPT as a tool for analyzing learning in the established communities and our extension of CoP to CoI potentially offers tools for analyzing practice(s) in LCM. One of these is our use of a *design* or *inquiry cycle* as seen commonly in action research. This is complex, since we are constructing the analytical tool we want to use to analyse the practices of which the tool is a part. I will give a brief example to illustrate these points.

According to the design of the project, there should be a teacher team in each school with teachers planning together for mathematical activity in classrooms, carrying out the designed activity with observation, subsequent reflection and feedback to the planning stage. This constitutes the design/inquiry cycle of plan→act→observe→reflect→feedback. The intention is that teachers will design activity according to their own school developmental goals, and engage in inquiry to learn more about implementation of goals. However, organization in certain schools limits what is possible. Teachers work together in grade or year teams – a horizontal structure – with diverse subject expertise in any team. It is thus likely that only one person in each year team has mathematics as a specialism. Teachers joining the project are those specializing in mathematics and so come from across the year teams. So the project teacher team in any school is a vertical cross-section. This makes it difficult for teachers in the project team to meet and plan for classroom activity: such planning would cross the horizontal structure, making it difficult to organize planning meetings, and also difficult to plan across year boundaries. A consequence that we have observed is that individual teachers plan innovative mathematical activity for their own class, possibly relating to other teachers in their year team. In doing so they develop a personal inquiry cycle, and use inquiry in the tasks they design for the classroom. However, in some schools, there has been little possibility for vertical collaboration between project teachers in the school environment, which means that development of inquiry processes in planning for mathematics within a school is limited in scope<sup>2</sup>. In analyzing data from interactions in workshops and schools, a CoP perspective allows us to characterize the school setting in which horizontal teams are central to practice; a CoI perspective allows us to analyse the relationship between the project community and any school community to gain insight into the transformative process through which teachers interacting within the project (for example in a workshop setting) bring their perceptions of developmental inquiry to the norms of school practice. Here we can see critical alignment in action. Analysis has to make sense of the ways in which practices and ways of thinking develop as teachers work within their school's practice while bringing new thinking from the

project, designing new forms of activity for their classroom, and rationalizing for themselves and for the project the outcomes and issues that emerge. We see in reality how development is a long term process: it is only after three years of the project that we are able to start to trace emergence of new ways of being and modes of practice.

This illustration relates to one of the more global outcomes of learning within the project. More locally, we can point to developmental patterns that can be traced through practice within the project and which form a part of emergent practice as a whole. Such patterns are starting to form ‘cases’ within the project, analysis of which provides the more detailed, in-depth, understandings of project situations that enable the more global conceptualizations. One such case is the “sum factors” case. At an early workshop, in plenary, one didactician offered a mathematical problem, *the sum factors problem*, for work in small groups. Video data captures his presentation of the problem as well as the subsequent activity of small groups of teachers and didacticians working together on this problem. We can see from the data something of the mathematics that was done, the ways of working on the mathematics and didactical questions that arose related to this work in the small groups and potentially in school classrooms. Further recording shows the plenary gathering in which issues from small groups were discussed in the LCM community as a whole. Analysis of this data endeavours to capture a sense of the growth of awareness, within the project community, of mathematics related to the particular problem, and of the didactical potential of such a problem for the classroom. Subsequent video data from a classroom in one school shows a teacher having transformed the sum factors problem for his class of 10 year old pupils, working with the class on the problem, and the pupils’ mathematical participation in the activity. Thus analysis can be extended to explore relationships between workshop and classroom activity. This is within-project analysis. Harder for the project is to do a detailed analysis of the in-school situation since we do not have the breadth of data to make this possible.

### **The analytical potential of the theoretical perspectives**

The LCM project has developmental goals and a research design for achieving these goals. However, the nature of practice in the project is such that research design can not be simply applied, tested and modified as in a traditional design-research iterative process (e.g., Kelly, 2003). Research is exploring ways in which project design is interpreted in practice and the outcomes and issues that emerge through human interaction in the project. We bring CoP theory to characterize ways in which teachers belong to their school community, engage in and align with its practice and exercise imagination to achieve their own professional goals in working with pupils. We bring CoI theory to try to capture the complex levels of interaction between the established communities in the schools and the emerging relationships in the project community. Inquiry is a key concept in looking at the changes of thinking and practice that become evident. The concept of critical alignment can be seen in terms of ways in which teachers’ alignment with school practices becomes challenged as

teachers think in new ways within the project. Didacticians' activity is no less challenged, as we show in other writing (e.g., Goodchild, 2006).

We see *inquiry* as a way of being in practice that is inherently transformative. The concept of *teaching as learning in practice* positions teachers, analytically, as bringing a critical dimension to their alignment with school practice and exploring possibilities for practice as they engage also in the project. Such a position enables us to address the growth of knowledge in teaching and the complex dimensions of rationalization between project and school communities. These words characterize the more global perspective, while micro analyses of cases allow us to explore particularly ways in which mathematics learning and teaching develops within the project.

While it is easy to write, “the complex dimensions of rationalization between project and school communities”, a characterization of the tensions that contribute to the complexity is a challenge for didacticians. We are using *activity theory* to enable us to point to particular dimensions of complexity such as community, rules, division of labour (e.g., Engeström, 1999) within a system and between systems (see Jaworski & Goodchild, 2006; Goodchild 2006). Such characterization helps us to understand better the complexity of relationships and commitment to existing communities of practice that constrain the developmental processes in which we engage. However, we recognize also that employing a multiplicity of theoretical perspectives begs some overall theoretical rationalization – not a unification, but a clarity on how the theoretical perspectives we are using are juxtaposed and what issues of commensurability arise from juxtapositioning. Space here has limited a further discussion of such issues.

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### Notes

<sup>1</sup> Here “knowing” refers to the dynamic processes of constituting knowledge, whereas “knowledge” relates to the socially accepted *products* of knowing.

<sup>2</sup> Here we see one of the tensions revealed within the project for which we are using an Activity Theory approach to analysis. See the Goodchild (2006) paper presented also in Group 11 at CERME 5.

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## ON THE NATURE OF THE MATHEMATICAL KNOWLEDGE UNDER CONSTRUCTION IN THE CLASSROOM

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*School mathematics and scientific mathematics are two related but epistemologically distinct bodies of knowledge. The present work reports on an attempt to identify the epistemological status of the former as it is interactively constituted in the classroom. To this purpose, two relevant theoretical constructs, that is, the sociomathematical norms and the epistemological triangle are utilized in order to analyze two lessons offered by two secondary school teachers', aiming at characterizing the epistemological status of the knowledge under construction through the lenses provided by them. The results showed that both perspectives allow limited access to the specific epistemological features of this knowledge.*

### INTRODUCTION

Despite the considerable research interest shown in the last two decades for the study of the conditions under which the mathematical meaning is constructed in the classroom, the nature of the mathematical knowledge shaped within this context has attracted little attention. The reason for this surprising limited research activity might be sought in the difficulty of defining the exact epistemological status of the knowledge under consideration in didactical contexts in a coherent manner. What do we mean by the term 'school mathematics'? How does it relate to mathematics as a scientific discipline? Although the latter appears to play a decisive (but ambiguous) role in the determination of the former, the two types of knowledge are epistemologically distinct (Sierpinska & Lerman, 1996).

One of the main features of the difference between the two bodies of knowledge relates to the social contexts within which each develops and which affect their epistemological status substantially (Steinbring, 1998). This suggests that the epistemological status of school mathematics knowledge cannot be deduced from the scientific mathematical knowledge, but needs to be studied in relation to the social contexts of teaching and learning processes. To this direction, the focus of the present work is on the nature of the meaning emerging in the classroom characterized as 'mathematics' in connection with the classroom phenomena which determine this construction. In particular, in an attempt to identify the nature of the mathematical knowledge interactively constructed in the classroom contexts, we adopt two well known relevant theoretical constructs, i.e., the concept of socio-mathematical norms and the notion of the epistemological triangle. The comparative reading of the same lessons, through the lenses offered by these two approaches, allows us to sharpen the analysis related to this nature.

## **MATHEMATICS AND SCHOOL MATHEMATICS**

All research in mathematics education deals with issues that have to do with mathematics: “mathematical meaning”, “mathematical activity”, “mathematical outcomes” (of students, teachers, communities, etc). However, the “mathematical” part in these expressions remains rather undefined and one could hardly justify why a meaning, an activity or an outcome can be characterized as ‘mathematical’.

It is generally admitted that school mathematics is different from experts’ mathematics (e.g. Steinbring, 1998), because, in the process of transformation from one type to the other, changes occur both ‘externally’ (from experts’ knowledge to knowledge for teaching) and ‘internally’ (from knowledge for teaching to the taught knowledge). On the other hand, it is also recognized that there are similarities between school mathematics and science mathematics (e.g., concepts, procedures, structure, etc.).

The relation between a ‘teaching object’ and the corresponding ‘mathematical object’ is rather blurred. Firstly, because mathematical objects and approaches had different forms in the history of their development and the correspondence we are looking for is not so obvious. Secondly, as Ernest (2006) reports, “most school mathematics topics are no longer a part of academic mathematics and thus figure in no contemporary academic textbooks” (p. 73).

Whether one agrees or not with the aforementioned comments, it is evident that the study of the knowledge taught in the mathematics classroom requires certain clear criteria for what and if can be considered as ‘mathematics’. As Godino and Batenero (1996) argue, we have to “be based upon an analysis of the nature of mathematics and mathematical concepts... Such epistemological analysis is essential in mathematics education for it would be very difficult to efficiently study the teaching and learning process of undefined and vague objects...” (p. 177).

It is widely accepted today that mathematical meanings or procedures are not something that students have to ‘learn’ and ‘apply’ (e.g., Yackel, 2001, Steinbring, 1998), but something that is constructed, accepted or negotiated in the classroom. Either as a personal or as a social construction, materialized in different contexts and in different ways (e.g., in social interaction), school mathematics knowledge needs an agreement upon whether what is personally or socially constructed is or is not mathematics. Moreover, the study of teaching and learning phenomena in the mathematics classroom and, in particular, the study of children’s activity, under the perspective of developing mathematical meanings, needs agreed detailed criteria with respect to the nature of the actual mathematical knowledge constructed.

## **THE THEORETICAL APPROACHES UTILIZED**

As it was mentioned above, in order to initiate a discussion on the epistemological status of the knowledge emerging interactively in the mathematics classroom, we

resorted to two known research approaches. Our purpose was to exploit the possibilities offered by each of them in trying to identify the particular epistemological features of the subject matter knowledge they claim that is shaped in the classroom, as a consequence of the personal, social and epistemological constraints present. These research approaches are briefly described below.

### **a. Sociomathematical Norms**

The notion of sociomathematical norms was conceived in order to analyze and talk about the mathematical aspects of teachers' and students' activity in the mathematics classroom (Cobb & Yackel, 1996). These norms are collective criteria of values with respect to mathematical activities, which are interactively constituted (Voigt, 1995), not predetermined, but continually regenerated and modified by the interactions taking place between the teacher and the pupils. The sociomathematical norms are established in all types of classrooms and they are context dependent. However, in the relevant literature, the sociomathematical norms were mainly studied in the context of inquiry classrooms, with the focus being on identifying that they are interactively constituted.

The most common sociomathematical norms reported in the literature are especially related to explanations, justifications and solutions. With respect to explanations and justifications, the main sociomathematical norm detected is related to 'what counts as an acceptable mathematical explanation' (Yackel & Cobb, 1996). Specifically, two categories of explanation were identified: explanations as descriptions of actions on experientially real mathematical objects and explanations as objects of reflection. The related sociomathematical norms were respectively: explanations must describe actions on mathematical objects and should not constitute procedural instructions, and explanations should aim at providing an explanation understandable by them (Yackel & Cobb, 1996).

Concerning solutions, the related sociomathematical norms were concerned with 'what is valued mathematically; what a more sophisticated solution is; what is an elegant mathematical solution' (Yackel & Cobb, 1996). Asking for a mathematically different solution (Yackel, Cobb & Wood, 1998) and evaluating the solutions using terms such as "insightful solution, simple solution, discoveries" (Voigt, 1995, p.198), the teacher helps the classroom to elaborate norms about what is mathematically efficient and/or what is mathematically different (Yackel & Cobb, 1996).

The above suggest that sociomathematical norms allow us to study how 'what counts as mathematical' is constructed in the classroom. However, they do not inform us with respect to the nature of what is being accepted as 'mathematical'.

### **b. The epistemological triangle**

Steinbring (2005) focuses on the epistemological status of what is interactively constructed by the students through working on concrete problems, being treated as exemplary embodiments of mathematical structures or relations. He argues that the

identification of this status requires an epistemological analysis of the pupils' statements, which can be achieved through a relational structure called 'the epistemological triangle'.

In particular, he advocates that in the course of classroom interaction, students have to actively construct possible relationships between signs/symbols and reference contexts. This personal construction becomes 'official' in social negotiations with the teacher and the students. Consequently, the analysis of the classroom production of mathematical meaning within an epistemological perspective needs to take into account the interrelation between the following two interdependent dimensions: (a) the construction of meaningful relations for sign systems is regulated by the reference contexts utilized and b) the meaning construction processes are embedded and interfere with the social conditions and conventions functioning in the instruction process (Steinbring, 2005).

The production of mathematical meaning in the interplay between the sign system and the reference context can be described as a process via which possible meanings are transferred from a relatively familiar situation (the reference context) to a still new and unfamiliar sign system. During the developmental process, the roles of the 'reference context' and 'sign system' can be exchanged (Steinbring, 1998). However, the mainly empirical reference contexts for sign systems utilized in the classroom promotes an empirical type of mathematical knowledge, which "accompanied by routinized interactive patterns of communication, such as the funnel pattern, changes meaningful mathematical understanding into conventionalized rules of algorithmic operations" and encourages an interpretation of mathematical symbols "that conflicts with the theoretical epistemology of mathematical knowledge because ... students become accustomed to an artificially concrete understanding of mathematical concepts, and this produces epistemological obstacles to an understanding of the relational character of mathematical knowledge" (Steinbring 1998, p. 523 and 524).

The analysis suggested by the above perspective acknowledges that all mathematical knowledge is context-specific and therefore, the difference between scientific and school mathematics lies in the different types of reference contexts utilized in the course of development.

## **THE STUDY**

In order to study what emerges interactively in the everyday classroom as mathematical knowledge, we adopted Cobb and Yackel's socio-constructivist framework and Steinbring's interactionist perspective and, in particular, the notions of sociomathematical norms and the epistemological triangle respectively. Our intention is to provide a comparative reading of the specific epistemological status of the mathematical knowledge shaped in the context of the interaction taking place in the classroom, through the lenses offered by the two approaches, an issue which barely and rather vaguely crops up in the relevant analyses.

The data utilized in the present work were the lessons provided by two secondary school teachers on the topic of fractional algebraic expressions. These were ‘normal’ teaching sessions of forty five minutes to two different classes of 3<sup>rd</sup> year pupils (14 – 15 years old), which were videotaped and transcribed. Both teachers had a university degree in Mathematics and more than 15 years of teaching experience.

Analyzing the data in the light of the above two perspectives, we followed an interpretive approach. Specifically, we focused on the classroom interaction, trying to identify episodes which highlight sociomathematical norms on the one hand, and processes of constructing referential meaning on the other. We then considered the nature of the knowledge emerging, claimed to be ‘mathematical’, by resorting to the epistemological features of the knowledge shaped.

## DATA ANALYSIS AND DISCUSSION

The analysis that follows concentrates first on the notion of the sociomathematical norms and then on that of the epistemological triangle.

### 1. Sociomathematical norms (SN)

In the characteristic episodes of the two lessons below, we report on instances of explanation, which highlight exemplary types of norms identified through the respective framework.

*EpisodeSNa.* In teacher’s A class, the students are invited to look for the situation when an algebraic fraction is defined (Explanation as procedure).

T(eacher). Consider  $a - \frac{5}{a^2} - 1$ . Tell me what the values of variable  $a$  are, for which this fraction is defined. Vagelis? (she repeats the last phrase). That is, when does it have meaning or what is the value of  $a$ , which we want?

Vagelis. One

T. Well done! How did you find out?

Vagelis.  $1^2$  is 1.

T.  $1^2$  is 1. Well done!  $a \neq 1$ . Is there anyone else who wishes to say something?

Dimitris. And  $a \neq -1$ .

T. Well done, Dimitris! What did you say? That  $a$  should be different from 0. How did you think of it?

Dimitris. Since  $-1$  squared becomes  $+1$ ,  $-1$  times  $-1$  makes 1.

T. Right, this was a little ... We were looking... like fishing in a little muddy sea. Is there a safer way to find the values we don’t want in the denominator?

*EpisodeSNb.* The discussion in teacher’s B below is focused on the values of the variable  $x$ , for which the denominator  $x^2 - 4$  of an algebraic fraction is different from 0 (Explanation as procedure initiated as explanation on object).

Kostis. When we say  $x^2 = 4$ , isn’t it  $x^2 = 2^2$ ? So, since the two squares are equal, isn’t it that their bases are also equal?

T. Well, look. ... Our problem is to solve an equation. And what do we notice in this equation? Both 2 squared and -2 squared give us 4. Thus, the equation is true for both values. That is, we should not lose -2. We always write it this way from here onwards.

*EpisodeSNc.* In a similar episode reported in teacher's B class, the students work on the equation  $x^2 - 4 = 0$ , trying to find the values of  $x$ , for which the fraction with this expression as a denominator is defined (Explanation as action on object).

Argyro. Isn't it that 4 should go to the other side?

T. Correct! Thus, what do we have here?  $x^2 = 4$ . What shall I write now?  $x \dots$

Argyro.  $x$  equals with square root of 4.

T. Fine! Be careful children! This is what we were saying until last year. Because, when we were learning to solve this type of equation,  $x$  was a line segment. We learnt this, if you remember, when talking about Pythagoras' theorem. And there the line segments were always ...

Students. Positive

On the basis of the preceding analysis, it could be argued that the two teachers introduce, through their statements, rather identifiable norms about when explanations in mathematics are acceptable, by continuously providing or requiring explanations which are mainly descriptive in character. Furthermore, they appear to indicate in a fairly clear manner, when and why a mathematical procedure is efficient. Hence, the notion of sociomathematical norms seems to offer an especially useful tool for analyzing classroom interactive patterns related to procedural elements of the mathematical knowledge under construction, showing that there are aspects of these patterns which are specifically connected to mathematics and thus affecting this construction (i.e., it helps us to identify what counts as mathematics in these classes). However, these patterns concern almost exclusively procedural features of this knowledge, paying limited attention to other characteristics, which determine its relation to mathematics. As a consequence, it does not allow us to formulate a coherent view as to the nature of the mathematical knowledge produced in a particular class or across classes.

## 2. Epistemological triangle

Steinbring's approach appears to provide a challenging framework of focusing in a more specific manner on the epistemological status of the knowledge emerging in the context of the mathematics classroom interaction. Especially, reading the data through the lenses provided by Steinbring's concept of the epistemological triangle (ET), we focused on the way in which classroom interactive patterns function with respect to the relation between 'reference context' and 'sign system', which is meditated by the mathematical concept under consideration. The episodes below present aspects of this function.

*Episode ETa.* Teacher B discusses with the class what an algebraic fraction is.

Agy. It is an expression which has a variable as denominator.

T. Very good...Come Alexia, write on the board such a fractional expression (she writes  $\frac{3}{x}$ ). Right! Harry, is  $x$  a variable?

Harry. Yes, it is.

T. Tell me, does it take all values?

Harry. Except 0.

T. Except 0, very nice! Why does not it take the value of 0 Christina?

Christina. Because the denominator becomes equal to 0.

T. I did not understand anything. So? Does it matter?

Christina. We do not want it to be 0.

T. Why don't we want it to be 0?

Christina. There is no fraction with 0 as denominator

T. There is no fraction with 0 as denominator. How did we call this in primary school?

George. Division by 0.

T. Division by 0. I am very proud of you!

It is evident in the above episode that there is a rapid succession of reference contexts: from algebraic numbers to rational numbers and then to an operation, with the sign system remaining the same (algebraic symbols). This is accompanied by an interactive pattern which bears the characteristics of a funnel process, directing the students to a concrete understanding of the mathematical concept concerned.

*Episode ETb.* This is an episode in teacher's A class, following immediately after episode SN1a above.

T. The safe way is to solve this small equation. How do I solve such equations?

Vagelis.  $a^2 - 1 = 0, \Rightarrow a^2 = 1$ .

T. This is one way. Are we sure that we are not going to lose roots here?... And then?

Vagelis.  $a=0$ ?

T. Square root of 1! This way we get the 1. How are we going to get the -1?

Helen. We do not get -1, because minus times minus makes plus...

T. We again fish in muddy waters. Any safer way?

[The teacher finally suggests the expression  $a^2 - 1^2$ ].

T. Does this remind you of something?

Students. Difference of squares ... Let's now factorize it. Olymbia?

Olymbia.  $(a-1)(a+1)$ .

T. And what do I have now? The product of two numbers, of two expressions inside brackets, isn't it? And it is equal to 0. What do we conclude?

Giota. That either one of them or both are equal to 0.

T. Great!

We notice here that as the episode develops, while the sign system remains the same (algebraic), the reference contexts utilized by the teacher change: from deciding, generally, when an algebraic expression is different from 0, to looking for specific numbers that satisfy an equality ( $1^2-1=0$ ,  $(-1)^2-1=0$ ), to solving one particular type of equation, followed by solving equations in general, through factorization. In the teaching course, students appear to follow teacher's intention via routinized interactive patterns, mostly consisted of a succession of short and well focused questions, narrowing down step by step. The result is that they fail to gain a genuine insight into the knowledge targeted, as they either get stuck in one particular reference context (e.g., square difference) or they are unable to successfully relate the elements of the reference context at play each time – mostly identical with the sign system and not very familiar to the majority of the students - with those of the sign system (e.g., many students appear to think arithmetically until the end of the episode).

The analysis of the data through Steinbring's lenses shows that, in an attempt to cover various aspects of the mathematical knowledge under consideration (which implicitly moves from defining an algebraic fraction to solving an equation), both teachers tended to continuously change reference contexts. This variety of reference contexts, not clearly differentiated from sign systems, seems to prevent the construction of a meaningful relationship between the two and allows funnel interactive patterns to distort the mathematical knowledge targeted.

The above suggests that Steinbring's framework allow us to identify epistemological aspects of the mathematical knowledge under negotiation (i.e., whether it is concrete or general in character) and, through the succession of reference contexts and their relation to sign/symbol systems, to attend the route followed by the mathematical content and its management by the teacher and the pupils. This is in contrast with the sociomathematical perspective, which only makes it possible to describe the way in which what counts as mathematical procedure emerges in the classroom, providing limited access to the mathematical knowledge shaped. However, Steinbring's approach does not permit us to characterize epistemologically the knowledge emerging in the classroom in specific. That is, via his lenses, we can decide about the epistemological constraints imposed on this knowledge by the social settings in general, but we are not in the position to specify the actual epistemological features that are constrained or distorted each time (see below).

## **CONCLUDING REMARKS**

The two ways of analyzing the data presented in the previous section concern different aspects of the interaction taking place in the mathematics classroom with respect to the mathematics constructed. The first focuses on the processes adopted in the classroom towards the targeted mathematical knowledge, but implicitly on the

nature of the actual knowledge under construction. Thus, when considering explanations, which particular epistemological features of the interactively constructed mathematical knowledge are we referring to? In episode ETa, for example, the explanation provided by the teacher concerns action on object (rather 'advanced' mathematically), and deals with aspects related to definition, whereas in SNa, the explanation concerns procedures (of fairly low mathematical value), and deals with properties. That is, there seems to be an interplay between the explanations provided and the epistemological status of the knowledge under consideration, which cannot be grasped by an analysis based only on the related sociomathematical norms.

On the other hand, Steinbring's approach, focusing on the interactive construction of mathematical concepts through the interplay between sign/symbol systems and object/reference contexts, does permit evaluation of the epistemological orientation of the knowledge under consideration, but in a rather general manner. In particular, while it makes it possible to decide whether or not this knowledge is relational and context-free in character and also how this has interactively happened, it pays little attention to the specific epistemological status of this knowledge. As a consequence, it does not allow us to locate similarities or differences of the mathematical knowledge constructed in different reference contexts. For example, in episode ETa, the reference contexts utilized changes from algebra to rational numbers and then to an operation. Steinbring's approach enables us to argue that the communicative pattern used worked against highlighting the relational connection of these three contexts. However, it does not allow us to give credit to the fact that, in all three contexts, the knowledge negotiated was of similar nature, i.e., a definition.

It is evident that what happens to the mathematical content and how it happens in the classroom are interrelated. The points raised above suggest that the simultaneous exploitation of the two approaches is especially valid. However, they also indicate that we need to look closer at the particular epistemological features of the mathematical knowledge under construction in the classroom. To this direction, we focused on the management of specific characteristics of mathematics (e.g., definitions, properties, validation procedures, etc) by both teachers and pupils. We believe that, in this way, we are in the position to characterize the differentiating character of the knowledge shaped in the mathematics classroom. For example, utilizing this framework to analyze episodes SNb and ETa, we could claim that the nature of knowledge negotiated changes as the lessons evolve in the former (from properties to 'definition'), but not in the latter episode. Such a perspective makes it easier to be in control of the actual nature of the knowledge under construction in the mathematics classroom, as it allows a well focused study of the management of this knowledge and of its impact on the learning product. Thus, we found that in many mathematics classrooms today there is homogeneity of this knowledge, i.e., relations or properties are not distinguished either from definitions or from processes (Kaldrimidou, et al, 2000).

The didactical phenomena occurring in the mathematics classroom are so complicated with respect to personal, social and epistemological aspects that we need a multiple approach, which will carefully incorporate all the issues raised above, in order to identify the nature of the mathematical knowledge interactively constructed in the classroom context.

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## **SOCIAL INTERACTION IN LEARNING PROCESSES AS SEEN BY THREE THEORETICAL FRAMEWORKS <sup>[1]</sup>**

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*We rely on discussions initiated at CERME4 to compare, contrast and combine in a coherent way different theoretical frameworks currently used in mathematics education, with the eventual aim of networking between theories. Specifically, we chose for this purpose the theory of didactic situations (TDS), the nested epistemic actions (RBC+C) model for abstraction in context, and the theory of interest-dense situations. We focus on how each of these frameworks is taking into account social interactions in learning processes. We identified not only connections and contrasts between the frameworks but also additional insights, which each of these frameworks can provide to each of the others.*

### **INTRODUCTION**

Learning processes are at the centre of interest of mathematics education as a scientific endeavour. They are very complex, taking place in a multi-faceted environment, with many aspects interacting and influencing the process. The different theoretical frameworks used today in the field of mathematics education offer different ways for approaching learning processes and for taking into account environmental conditions and influences on these processes. No single framework is able to provide a full understanding of the complex phenomena at stake, but combining their respective insights in an efficient way is far from trivial. Each theoretical frame obeys its own logic and has its own coherence. It looks at the educational reality through a specific lens, without the ambition of developing a holistic view, a sine qua non condition for efficiency and operability. Trying to combine theoretical perspectives thus presents the researcher with unavoidable problems of coherence and compatibility. It is a crucial question for mathematics education, how to cope with these problems of coherence and compatibility in order for the diversity of existing approaches to support our understanding of teaching and learning processes, and in order for research to give more effective assistance to the teachers who have to handle their complexity.

With the aim of making progress toward answering the question, how to cope with these problems of coherence and compatibility, it is certainly of interest to compare

and contrast different approaches in order to identify possible connections between theories, develop complementary or dialectical theoretical views, investigate when and why theoretical approaches contradict each other and, in the long run, establish a network of theories (Bikner-Ahsbahs and Prediger, 2006).

The idea to compare, contrast and combine different theoretical frameworks was presented (see for example, Artigue, Lenfant and Roditi, 2006b; Kidron, 2006) and discussed in the working group on theoretical perspectives in mathematics education at the 4<sup>th</sup> Congress of the European Society for Research in Mathematics Education in 2005 (Artigue, Bartolini Bussi, Dreyfus, Gray and Prediger, 2006a). The analysis presented in this paper is influenced by the discussion and views expressed in that working group and constitutes a theoretical work of comparison of three theoretical frameworks: the theory of didactic situations (TDS) (Brousseau, 1997), the nested epistemic actions (RBC+C) model for abstraction in context (Hershkowitz, Schwarz and Dreyfus, 2001), and the theory of interest-dense situations (Bikner-Ahsbahs, 2003). More specifically, the aim of this paper is to exhibit, compare and contrast how social interactions, a phenomenon which is more and more considered as an essential dimension of mathematics learning processes, are taken into account by these different theoretical frameworks.

In the next section, the role of social interactions for each framework is briefly presented. The following section is the main section of the paper. In it commonalities and contrasts are noted, and it is analyzed what each framework, may have to offer to the others, with respect to the role of social interaction. Finally, the concluding section presents a wider perspective on the potential benefits and difficulties of networking between theories, and some methodological reflections about the process of networking.

## **SOCIAL INTERACTION IN THREE FRAMEWORKS**

### **Social interaction in the TDS**

In order to understand the way social interactions are dealt with by the Theory of Didactic Situations (TDS; see Warfield, 2006, for an excellent entry level description), it is necessary to return to the origins of this theory and to the essential role that design has played in its development. As recalled by Perrin-Glorian in her analysis of the historical development of the theory (Perrin-Glorian, 1992), TDS's first aim was to lay the theoretical foundations for what Brousseau called at the time (the late sixties) an *experimental epistemology*. This contributes to explain the central role given in this theory to the situation, seen as a system of relationships between students, a teacher and some piece of mathematical knowledge. In essence, the central object of the theory, the situation, incorporates the idea of social interaction. More precisely, from the beginning, the TDS distinguishes between different forms of relationship with mathematical knowledge through the distinction between three

dialectics: the *dialectics of action, formulation and validation* (Brousseau, 1997). To each of these dialectics is associated a particular type of game, and the games associated to the dialectics of formulation and validation cannot be conceived as games played by an individual learner. These are necessarily more collective games involving at least groups of learners, if not the whole class. The notion of situation of communication often associated to the dialectic of formulation, for instance, attests to this characteristic.

In addition, even with respect to the dialectic of action, an analysis of some paradigmatic situations such as the “Race to 20” or the “Enlargement of the puzzle” shows that some organization of social interaction is constitutive of the design. For instance, in the Race to 20, the action phase in the original scenario is based on a succession of plays involving pairs of students. For enlarging the puzzle, students work in groups; first, each student in a group is in charge of the enlargement of a specific piece of the puzzle, and then they have to put together all these pieces to build the enlarged puzzle; usually, they discover that this does not work and, discussing the strategies they have used, they have to understand why. Social interactions thus play an essential role in the didactic functioning of situations, that is to say in making a given piece of mathematical knowledge appear as the means of producing winning strategies through the interactions of the students with a certain *milieu*.

Another point is that, in the TDS, the conceptualization of social interactions is not limited to interactions between students but also includes the teacher. The notion of *didactic contract*, understood as the system of reciprocal expectations (both explicit and implicit) between the teacher and the students as regards mathematical knowledge, but also the notions of *mesogenesis* and *topogenesis* that emerged from the theory of didactic transposition, are the fundamental notions here. As recalled by Sensevy, Schubauer-Leoni, Mercier, Ligozat and Perrot (2005), mesogenesis “describes the process by which the teacher organizes a milieu with which the students are intended to interact in order to learn”, while topogenesis “describes the process of division of the activity between the teacher and the students, according to their potentialities”. The paper just quoted shows how these notions can be used together with those of didactic situation and didactic contract in order to understand teacher-student interactions and the ways these are affected by the mathematical knowledge at stake. Asking two different teachers to carry out Race to 20 lessons but giving them complete freedom in the organization of these lessons, the authors create an intermediate object between a lesson design piloted by the TDS and an ordinary classroom lesson, especially appropriate for such a study.

Social interactions are thus a central focus in the TDS, both interactions between students and student-teacher interactions. In engineering designs built according to the theory, particular attention is paid to the ways the organization of these social interactions can support didactic adaptations through the creation of a milieu

offering rich enough potential for action and retroaction. In ordinary classroom situations, these conditions are hardly fulfilled; adidactic and didactic processes tightly intertwine, and even if the same conceptual tools can be fruitfully used, as attested for instance by the special issue recently published by Educational Studies in Mathematics (Laborde and Perrin-Glorian, 2005) or by our own research work (Artigue et al., 2006b), the analysis becomes much more complex. This has led to specific theoretical developments within the last ten years concerning the notions of didactic contract and milieu (Bloch, 2002; Brousseau, 1997; Margolinas, 2004), or the characterization of practices developed by teachers in order to conciliate ordinary classroom constraints and institutional expectations in terms of mathematical responsibilities to be given to the students, such as the ISD (Interactive Synthesis Discussion) (Hersant and Perrin-Glorian, 2005).

### **RBC: Social interaction as a component of context**

In the nested epistemic actions model of abstraction in context, also called the RBC+C-model (Hershkowitz et al., 2001), contextual aspects are considered to be determining and integral factors of the learning process. Context is regarded in a wide sense, comprising historical, physical and social context. Historical context includes students' prior learning history, physical context includes artefacts such as computers and software, and social context refers to the opportunities, kind and frequency of interaction with peers, teachers and others.

Dreyfus, Hershkowitz, and Schwarz (2001) studied processes of abstraction and social interactions in parallel, and in conjunction. Pairs of students were led to discover a surprising numerical pattern and then asked to justify it. The students were thus collaborating on a task with potential for abstraction; more specifically, the intended constructs were (a) conceiving algebra as a tool for justification and, nested within (a), (b) an algebraic technique.

The researchers independently undertook a cognitive and a social analysis of the interview protocols, with the aim of comparing them. The cognitive analysis used the Recognizing, Building-with and Constructing (RBC) epistemic actions, and allowed to generate diagrams showing episodes of the constructing processes. The social interaction analysis used common categories such as explanation, query, and agreement, as well as diagrammatic reference of each utterance to previous utterances. It allowed generating diagrams showing blocks of interaction. One main result of the research was that the cognitive and social diagrams show essentially the same blocks.

The other main result was the identification of patterns of interaction likely to support abstraction:

- Coherence is a characteristic of interaction that strongly favours abstraction; similarly, lack of coherence inhibits abstraction. Coherence is taken in the sense of

sharing a common motive for an activity; in our case, the motive was to arrive at the (algebraic) justification;

- Symmetric argumentative interactions are likely to lead to construction of knowledge;
- In asymmetric interaction, with one student leading the other, combining guidance with (self)-explanation is particularly fruitful for abstraction.

In more recent work, Hershkowitz, Hadas, Dreyfus, and Schwarz (2006; see also Hershkowitz, Hadas, and Dreyfus, 2006), investigated ways in which the common basis of knowledge of a group of students emerges from the individual students' constructing of knowledge through interaction, and as such enables the group to continue to construct further knowledge. The epistemic actions were observed within a larger continuum of activities to study the Consolidating processes of the abstracted construct, thus expanding the model to RBC+C. Cognitive and interactive processes of constructing knowledge were investigated as a single process. This provided insight and understanding of the ways by which knowledge is abstracted by a group.

Methodologically, the data were considered as "stories" taken from the activities of two groups of three students each, from classrooms in different schools, on problems from an elementary probability unit. These stories use the epistemic actions R, B and C to exemplify flows that describe how shared knowledge was constructed out of the individual knowledge. The study showed that the shared knowledge of the group is characterized by its diversity, each partner expressing her own way of constructing a piece of knowledge. Yet all three group members may benefit from this multifaceted shared knowledge in their common work, when going on to new constructs and/or consolidating constructs in follow-up and assessment activities. As in the earlier study, different patterns of interactive constructing were identified:

- In Story 1, one student acted as the source for the construct and, in a very intensive series of questions and requests for clarification, supported the constructing process of a second student (asymmetric, guidance). In a further interactive phase, both these students supported the third, and thus the three students in the ensemble shared the constructed knowledge.
- In Story 2, the two students co-constructed in interaction and the knowledge was shared by both of them. A third student, objecting to her colleagues' shared construct (argumentative), constructed a unique strategy to solve the same problem.
- In Story 3, the shared knowledge of three other students was constructed in a process of three cycles, from a shared awareness of the need for a construct (coherence), via denial of the correct construct (argumentative), to constructing the shared construct by all three students as an effect of the teacher's demonstration.

In a parallel line of research the role of teacher-student interactions in the construction of knowledge is being investigated through the lens of the RBC model (Schwarz, Hershkowitz and Azmon, 2006). In summary, the cognitive development of peers learning together in groups and classrooms is closely linked to the interaction among peers and with the teacher; processes of constructing knowledge and patterns of social interaction strongly influence each other and analyzing them in parallel or as a single process serves to specify, detail and explain processes of knowledge construction.

### **Social interactions in interest-dense situations**

In the project "Interest in mathematics between subject and situation" (Bikner-Ahsbals 2003, 2005) social interactions are not regarded as part of the learning environment but as basis which constitutes learning mathematics itself. In this approach learning is assumed as a social event in which mathematical knowledge is created through social interactions as part of the interaction space and the participants align their behaviour with the behaviour of the other participants. A main assumption is that a thing in the world is closely related to a person's interpretations about this thing. That means: people behave towards a thing according to their meanings about it; meanings are created through interpretations within social interactions with other persons and can be changed during processes of negotiation (Blumer, see Wagner 1999, p. 32). Analyzing scientifically in this sense means reconstructing the social processes by re-interpreting the interpretations according to the research question.

In the project mentioned above, so called interest-dense situations were investigated in class discourses of a sixth grade class during half a year. In a mathematics lesson, interest-dense situations are situations which foster learning mathematics with interest. They consist of an epistemic process, begin with a mathematical problem or question and are closed as far as the mathematical theme is concerned. They can be described by three features: Within an interest-dense situation students get more and more intensively involved in the mathematical activity (*involvement*), they construct farther and further reaching mathematical meanings (*dynamic of the epistemic process*) and the activity leads them to highly regard the mathematics at hand (*mathematical valence*). During the observation of 89 lessons, only 18 lessons contained interest-dense episodes; all of these were far away (in time) from tests.

The aim of the project was to reconstruct the conditions, which foster or hinder the emergence of interest-dense situations. The basic view was provided by the perspective of social interactions; building upon these, a profound analysis from the perspective of the epistemic processes and an analysis from the perspective of constructing mathematical values were carried out. The methodical principle was the comparison of situations which led to an ideal type description of processes of genesis of interest-dense situations. Nine of the interest-dense situations occurred ad-hoc due to a sudden utterance or question of one student. The other nine were socially generated so that processes of genesis could be reconstructed by their analyses.

During interest-dense situations the teacher does not behave according to his own content specific expectations towards the solution of the problem: he does not behave in an *expectation controlled* but in a *situation controlled* way. This means the teacher focuses on the students' utterances, he anticipates mathematical ideas, concepts, rules from the students' viewpoint and the direction, in which the social construction of meanings is about to develop. He supports the students in presenting their own mathematical views and gives assistance in the use of comprehensive words. The teacher will not usually evaluate, he rather poses questions to better understand the students' ideas. The students comment, change, state more precisely etc. Processes of this kind can only be sustained if the students do not orient themselves according to the assumed content specific expectations of the teacher, but rather behave *expectation independent*. In these cases, the social interaction is oriented towards the mathematical content and not towards reproducing the teacher's expectations.

The interaction structure which is shaped this way is very fragile. If suddenly the teacher behaves in an expectation controlled way, a conflict can arise because the students resist the teacher's expectations. In this case, either the interaction process terminates, or the teacher changes his behaviour. The interaction process can go on if the teacher's and students' behaviours are not deeply related to each other. In this case each takes keywords from the other's utterances as starting-points; for instance, the teacher tries to offer help by posing questions although the student does not need any; the student might pretend to accept help by saying "yes" or "alright" but continues along his/her own ideas.

If *expectation independent* student behaviour meets *situation controlled* teacher behaviour, students filter the teacher's utterances in order to find out what the teacher wants them to say, and the teacher takes the students' utterances as an expression of their thinking process. Interactions of this kind look aimless; they do not have a common basis of orientation.

In most of the non interest-dense situations we find a very stable interaction structure in which the teacher arranges his behaviour according to his content specific expectations, gives hints and poses constraining questions (*expectation controlled*) and the students try to use these hints to reproduce what the teacher wants to hear (*expectation dependent*). These interaction processes look like guessing games which do not permit to concentrate on deepening the understanding of the mathematical content. They are easy to manage and this might explain why they occur often and proceed routinely. All the participants know that the problem is solved when the teacher's expectations are reproduced. This could be an explanation of the stability of such interaction structures.

## **MUTUAL BENEFITS AND ADDITIONAL INSIGHTS OFFERED TO EACH OTHER BY THE FRAMEWORKS**

In the last decade, the extension of focus in mathematics education from individual students' mathematical conceptions to social interactions among students and between students and teacher has become a general trend. As set forth in the previous sections, the three frameworks considered in this paper agree on the importance of social interactions for learning processes. Indeed, in the TDS by essence, the central object of the theory, the situation, incorporates the idea of social interactions. In interest-dense situations, social interactions are regarded as basis which constitutes learning mathematics itself. And in RBC processes of constructing knowledge and patterns of social interaction strongly influence each other.

Nevertheless, even if there seems to be an agreement between the theories on the importance of social interactions, there are great differences. For example, in the TDS and interest-dense situations learning situations are central objects while in the RBC approach the focus is on the learner or an interacting group of learners. Moreover, experimental studies carried out in the two first perspectives generally concern classroom situations or at least some kind of institutional design while experiments using RBC consider a greater diversity of learning situations inside or outside the classroom. The RBC approach was used for example as the theoretical perspective in a research study on the learning processes of highly structured, advanced mathematics by a solitary learner (Dreyfus and Kidron, 2006).

Social interactions are also viewed differently by the TDS and interest-dense situations. In interest-dense situations social interactions constitute the epistemic process. Thus, knowing is an outcome of the social processes in which a group of students struggle with a mathematical problem. An interaction structure which is shaped by the teacher and the students supports the emergence of these situations. In the TDS, the conceptualization of social interactions includes interactions between students and also between students and teacher. Social interactions between students are viewed as a contribution to the learning potential of the adidactic milieu. Social interactions between teacher and students are approached through the notions of didactic contract, devolution and institutionalization that structure the links between the adidactic and didactic models of situations. In the TDS, great attention is indeed paid to two crucial roles of the teacher: having the students take the responsibility for the mathematics when solving proposed tasks (devolution process), and conversely, linking what has been achieved by the students in the research phase to the official knowledge aimed at (institutionalization process) (Artigue et al., 2006b).

In a more general way, the different views the three theories have in relation to social interactions force us to reconsider the theories in all their details. The reason for this is that the social interactions, as seen by the different frameworks, intertwine with the other characteristics of the frameworks.

In order to compare, contrast and combine the three theoretical perspectives, it is not sufficient to note commonalities or contrasts. We are interested in examining, what insights each framework can offer to the two others in relation to the way this specific framework views the social interactions. Moreover, the specific aspects of one framework can be viewed in terms of the others and this re-viewing might bring mutual analytic benefits. Investigating these mutual analytic benefits is the core of this section, and this paper.

### **TDS and RBC**

The categories of analysis of the RBC framework are clearly different from those of the TDS. As stressed above the two approaches do not focus on the same objects but on the learner and the situation, respectively. As regards the development of mathematical knowledge, they also use different categories: RBC approaches abstraction through three types of epistemic actions: recognizing, building-with, constructing; TDS distinguishes between three functionalities of mathematical knowledge: for acting, for communicating, for proving, which serve to organize the development of students' conceptualizations through appropriate situations. Thus the a priori analysis of the TDS accords high importance to the mathematical problem at stake, and the nature of the relationships with mathematical knowledge that the students can develop interacting with the milieu and their pairs.

Due to its focus on the learner, it might seem that the epistemic actions in the RBC model are described independently of the characteristics of the contextual components that make them possible. In reality, however, contextual aspects in RBC are determining and integral factors of learning processes. That is why this framework is called abstraction *in context*. Studies within the RBC perspective analyze the influence of patterns of social interactions on the processes of constructing knowledge by the learner. Moreover, on-going research studies within the RBC framework deal with the general question of the influence of contextual arrangements on different patterns of epistemic actions (e.g., Kidron and Dreyfus, 2006). At the same time, this kind of analysis contributes to the development of the analytical nature of the RBC model.

#### *Additional insights offered by TDS to RBC*

The RBC approach, as a research methodology, is used with task sequences that have been designed with well defined conceptual learning objectives in mind. However, it does not proceed from a design phase nor does it impose the kind of a priori analysis that is an essential methodological tool in TDS. The RBC approach could be enriched with the idea of developing a systematic a priori analysis as is the case in the TDS. It would allow the researchers to better take into account from the beginning some of the contextual arrangements and the influence these can have on epistemic processes.

According to Hershkowitz et al. (2001), the genesis of an abstraction passes through three stages: (a) the need for a new structure; (b) the construction of a new abstract

entity and (c) the consolidation of the abstract entity through repeated recognition of the new structure and building-with it in further activities. In order to assure stage (a), we have to develop students' need for a new structure. We may attain this aim by building situations that reflect in depth the mathematical epistemology of the given domain. This kind of epistemological concern is very strong in the TDS, and the notion of fundamental situation has been introduced for taking it in charge at the theoretical level. It could be helpful for RBC.

#### *Additional insights offered by RBC to TDS*

When TDS is used with a design perspective, situations are often modelled in terms of games, and in that case the winning states of these games must be clearly identifiable. It is expected that the students could by themselves identify if they have reached a winning state, in order to favour adidactic adaptations over adaptations piloted by the didactic contract. It is also expected that students, at least in their great majority, be able to reach such a winning state with pair interactions but without substantial help of the teacher. The situation is different in the RBC approach: the accent is not on the design of situations obeying the characteristics of adidactic situations recalled above; the task can be an open exploration task and the "end of the game" might be not very clear. But, as shown by RBC research, even so, it might be a situation offering a rich learning potential, and this vision can be helpful for TDS, especially when TDS is used for analyzing ordinary classrooms situations, which is more and more frequent.

In the RBC approach, the focus is on the learner or the group of learners. The identification of constructs in the RBC perspective enables the researcher to identify details of the constructing process. Even if the intended theoretical element, the "end of the game" has not been reached or has been reached only partially, the evolution of the process of construction and its connections with contextual aspects is important in itself. Such a detailed vision can offer complementary insights to those usually reached with the TDS for identifying the evolution of students' mathematical knowledge in the a posteriori analysis, and for becoming aware of some subtle constructions that could not be anticipated in the analysis a priori.

#### **TDS and interest-dense situations**

Interesting connections between TDS and interest-dense situations are less difficult to identify than between TDS and RBC. In the two approaches, learning situations and classrooms are given a central role. The characteristics of interest-dense situations and adidactic situations seem rather close, and the distinction made between student behaviour according to its dependence or not on teacher's expectation in interest-dense situations can be easily interpreted in terms of didactic contract. Nevertheless the two theoretical frames do not simply overlap. Social interactions are given in interest-dense situations a more fundamental role than in the TDS. As pointed out above, they constitute the epistemic process, which is not the case in the TDS.

The combination of the TDS and the theory of didactic transposition has led to the notion of mesogenesis which “describes the process by which the teacher organizes a milieu with which the students are intended to interact in order to learn”. This notion puts the accent on the dynamic character of the milieu, and the role the teacher plays in piloting this dynamic. Considering this process when analyzing situations could certainly help characterize conditions on situations for making them reasonable candidates for interest-dense situations. Within the framework of interest-dense situations such situations in everyday classrooms are identified and investigated in order to find conditions which hinder or foster their emergence and describe their emergence as ideal types. There is an underlying social contract which seems to allow or forbid the emergence of interest-dense situations.

*Additional insights offered by the theory of interest-dense situations to the TDS*

Regarding the whole process and its outcomes as constituted by social interactions, the theory of interest-dense situations could offer TDS a micro-ethnographic approach, which allows to describe in detail, how the emergence of didactic situations or didactic phases in ordinary situations and its underlying social contracts are hindered or fostered.

*Additional insights offered by the TDS to the theory of interest-dense situations*

Through the notions of didactic contract, didactic situation and fundamental situation, the TDS offers another perspective to reflect on social contracts, on the dynamics of the epistemic process, and on the building of situations reflecting in-depth the mathematical epistemology of a given domain. This last aspect might be very beneficial, especially if there is an intention to extend the project of interest-dense situations from elementary to advanced mathematical thinking.

**RBC and interest-dense situations**

The focus of RBC are the epistemic actions, hence the epistemic process and its outcomes. Social interactions belong to the context. The process is divided into episodes, analysing how these episodes are related to the epistemic actions. As has been pointed out above, analysis shows that social interactions are strongly related to the epistemic process: the epistemic process and the social interactions build the same blocks.

Concerning interest-density, social interactions constitute the epistemic process, thus, knowing is an outcome of the social processes in which a group of students struggles with a mathematical problem: coming to know is part of social interactions in a class discussion. Research about interest-dense situations tries to find patterns which constitute the whole situation. All interest-dense situations seem to be coherent, in the terms of RBC, and thus have a high potential to lead to constructing.

In both frameworks, epistemic actions are used but their genesis processes are different. The two models can be regarded as useful analytical tools for different but related purposes.

Investigating the epistemic processes in more detail might lead to mutual benefits for the two frameworks. For example, the following questions might be of interest: "What are the deeper reasons that the same methodological tools, namely epistemic actions, are useful for both, interest-dense situations and construction of knowledge? Are there (other?) epistemic actions that might be appropriate for investigating both, interest-dense situations and knowledge construction?"

*Additional insights offered by RBC to the theory of interest-dense situations*

RBC deals with contextual influence. The influence of the other components of context in addition to the social interaction component might also be of importance in the framework of interest-dense situations. As part of the context, the nature of the mathematical topics in the given domain could be considered. Taking into account that some constructions are fragile, the issue of consolidation might also be important for research on interest-dense situations. This may help answer the question, under what conditions students are able to use (build-with) the knowledge constructed in interest-dense situations in new situations, which are not necessarily interest-dense.

*Additional insights offered by the theory of interest-dense situations to RBC*

Looking at interest-dense situations as providing motivation for in-depth knowledge construction provides an analytic tool for stage (a), the need for a new structure in the RBC model (here we consider the motivation of the learner and not the issue of design). Since it is based on epistemic actions as well, this analytic tool may be eminently suitable to be combined with the RBC epistemic actions. The perspective of interest-dense situations, its epistemic actions, and its background theory might enrich the analytic nature of the RBC model of abstraction in context including the view of its social constitution.

## **THE PROCESS OF NETWORKING BETWEEN THEORIES: DIFFICULTIES AND METHODOLOGICAL REFLECTIONS**

In the previous section, we demonstrated that the three theoretical frameworks potentially complement and thus enrich each other if links between them can be established. We pointed out potential benefits but we should also point to the problems that will necessarily arise in the process of linking between theories. Therefore, the crucial question is not only *whether* the theories can complement each other but *how* this can be achieved.

### **Difficulties**

Our efforts in answering the question *how* the theories can complement each other force us to make very clear the assumptions underlying each theoretical framework, some of which may be hidden. This is rewarding in itself but let us consider the

difficulties that may arise in the process. Indeed, considering the three frameworks described in this paper, there might be possible contradictions between the underlying assumptions of the theories.

Specifically, we have observed how each theoretical framework has its own way of considering the role of social interactions in the learning process: The social interactions are an important part of the context in RBC; but in relation to interest-dense situations, social interactions are not viewed as a part of the context: they are the basis that constitutes learning mathematics; and in the TDS, social interactions are part of the situation, the system of relationships between teacher, students and mathematics. Given these differences, the question arises *how* it is possible to establish links between the theories without getting embroiled in contradictions between the basic assumptions underlying each theory.

To be more specific about the problems that may arise, let us limit our considerations temporarily to two theories: As a consequence of the above differences, the categories of analysis of the RBC framework are different from those of the TDS. As stressed in the previous sections, the two frameworks use different categories in relation to the analysis of the development of mathematical knowledge. RBC approaches abstraction through three types of epistemic actions: recognizing, building-with, constructing; TDS distinguishes between three functionalities of mathematical knowledge: for acting, for communicating, for proving, which serve to organize the development of students' conceptualizations through appropriate situations. Should we use both categories of analysis? Should we try to find a smallest common denominator between the categories (which might turn out to be empty)?

Similar difficulties arise while using the lenses offered by interest-dense situations and RBC: Although epistemic actions are used by both frameworks, not only are they different actions but they are viewed in different ways. Investigating whether there are (other?) epistemic actions that might be appropriate for considering both, interest-dense situations and knowledge construction is a complex issue.

Having become aware of the substantial difficulties involved in any attempt to connect theories, we raise the question what can (and what cannot) be possible aims of such an effort. Clearly, any attempt at unifying the three theories, or even two of them, into an encompassing theory is doomed to failure before it even starts. Such an attempt would necessarily destroy the basic assumptions of all theories involved, or at least of all but one. What, then, can we aim at? We propose to aim at establishing a network of links between the theories. In networking, we want to retain the specificity of each theoretical framework with its basic assumptions, and at the same time profit from combining the different theoretical lenses. What we aim at is to develop meta-theoretical tools able to support the communication between different theoretical languages, which enable researchers to benefit from their complementarities.

## **Methodological reflections**

During the initial work that lies behind the preparation of this report, the authors and other colleagues (F. Arzarello, M. Bosch, K. Ruthven) were using, comparing, combining, and contrasting theories while attempting to apply them to a common dataset. One might say that we were beginning to "network with theories". Here, the term "networking with theories" is used in a sensitizing way in order to find out how theories can be combined, compared and contrasted. One of our aims is to develop heuristics about how networking with theories takes place and what it could potentially lead to. Through negotiations and methodological and methodical reflections meta-theoretical tools might be developed.

We assumed that researchers networking with each other as theorists produce implicit knowledge about how "networking with theories" could proceed. We further assumed that this implicit knowledge can be uncovered through reflections about the process.

How did we proceed? We chose an aspect of the learning process, which has some relevance in all three theories, namely social interactions; on purpose, we did not specify this aspect very precisely in order to leave it relevant for all three frameworks. We presented different views on this aspect and its roles in the different theories. We compared and contrasted each pair of theories in more detail focusing on benefits, additional insights, and tools which one theory can offer to the other and vice versa.

Our analysis of the complexity of the process of linking between theories led us to the conclusion that the following heuristics might support networking:

- Use a common, but not precisely defined aspect that all the theories share and produce an overview of the theories according to this aspect;
- Find out what ideas each pair of theories share;
- Compare and contrast each pair of theories according to the common aspect; consider the benefit, additional insight, limitations and tools each of the theories can offer for working with the others;
- Connect the results into a set of complementary views taking into account all three theories, and describe how this might be able to assist our understanding of learning processes.

## **Conclusion**

In this paper, starting from the diversity of existing theoretical frameworks in mathematics education, and the impossibility of any one of these to give a full account of the complexity of learning processes in mathematics, we presented the idea of looking for fruitful combinations or networking between theories. For exploring this idea, relying on discussions initiated at CERME4 and continued since that conference, we decided to select theoretical frameworks we were familiar with, and to investigate how these could be compared, contrasted and combined in a coherent way in order to increase our understanding of learning processes in

mathematics. For this purpose, we selected three theoretical frames: the Theory of Didactical Situations, the RBC model of abstraction in context and the approach in terms of interest-dense situations; we discussed in some detail how each of these is taking into account social interactions.

The theoretical frames we have chosen are quite different and thus constitute good examples for illustrating the existing diversity in the field. Two of them are situation centred while the third one is learner centred. One of them began to develop about 30 years ago; it has been used by scores of researchers who have contributed to its development. Understanding the complex object it has become along the years is not easy, and many researchers in mathematics education have only a superficial knowledge of it. The two other frames are more recent constructions, developed and used up to now by rather small communities. They do not have such a large scope, and at least at a first sight it seems easier to become reasonably familiar with their main constructs.

Working collaboratively, we have tried to understand our respective didactical cultures, to identify interesting similarities and complementarities between our perspectives, and boundary objects that could support connections. Even focusing on social interactions, an aspect that plays an important role in all three frames, this was far from being an easy task. It required from each of us a costly effort of decentration. The cost of this effort evidences the strength of the coherences underlying our respective didactical cultures, and the specificities of the educational and research experiences underlying these. Looking back at this emergent work, what seems important is the fact that in spite of the diversity of our experiences and cultures, we share common concerns, and that the theoretical constructs we develop or use are the tools we have for approaching these concerns in an efficient way. Comparing, contrasting, and trying to build connections, we certainly understand better today the functionalities each of us gives to the theoretical constructs (s)he uses, how (s)he uses them and what (s)he is able to produce thanks to them; we also see better the limits of our respective tools and what could be offered by networking them in ways that would not destroy their internal coherence. But what we achieved is just a first step.

In the long run this work will hopefully lead to a clearer meta-theoretical concept, which we might call "networking between theories" and which might enhance the development of the theoretical work in our community regarding the need to grasp the complexity of our research objects better than we are able to do now.

1. Research partially supported by Israel Science Foundation grants number 1166/05 and 1340/05 and by the Müller-Reitz-Foundation.

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# STUDENTS' DIFFICULTIES IN VECTOR SPACES THEORY FROM TWO DIFFERENT THEORETICAL PERSPECTIVES

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*In this paper we present some data drawn from a study aiming at investigating the difficulties which undergraduate and graduate students in Mathematics encounter when solving Vector Space Theory problems. The reported difficulties are analyzed according to two different theoretical frameworks, namely Fischbein's theory of tacit intuitive models and Sfard's theory of process-object duality. Finally, a brief comparison is proposed among the findings of the two analyses.*

## INTRODUCTION

This paper is meant to contribute to the discussion on “Different theoretical perspectives and approaches in research” proposing the analysis of graduate and undergraduate students' difficulties in Vector Space Theory (VST) through the lenses of two different theoretical frameworks: Fischbein's theory of intuitive models (1987, 1989) and the theory of process-object duality (Sfard 1991, Sfard and Linchevski 1994).

The data we are going to present and discuss are drawn from our doctorate research project (Maracci, 2005). Preliminary results from the analysis of students' difficulties in terms of process-object duality have been already discussed in Maracci, 2004; we present here a further development of those results. On the contrary, the analysis of students' difficulties in terms of intuitive models may be considered completely new [1], as well as the comparison between the findings of the two analyses.

## THEORETICAL FRAMEWORKS

We choose to develop our analyses according to Fischbein's and Sfard's theories for different reasons. On the one hand, we share Fischbein's view that there exists an implicit dimension of knowledge, beyond one's consciousness and control, which influences one's thinking processes, and that an explicit analysis of such dimension is both possible and necessary. On the other hand, the teaching of VST in Italian Universities follows an axiomatic approach which strongly stresses the ‘algebraic nature’ of the basic notions of VST itself; consequently the study of students' difficulties in VST may benefit from the adoption of a perspective consistent with Sfard's theory, which revealed efficient to analyze students' difficulties in algebra.

### **The theory of tacit intuitive models**

Starting from the assumption that mathematical concepts and operations are essentially formal and abstract constructs whose meaning and coherence are not directly accessible to the individual and whose can not be ‘spontaneously’ manipulated, Fischbein (1989) states that one needs to produce mental models providing her with a directly accessible, unifying meaning to concepts and symbols.

‘If a notion is not representable intuitively one tends to produce a model which can replace the notion in the reasoning processes. We are referring here especially to substitutes which are able to translate the concept in sensorial behavioural terms. These are *intuitive models*.’ (Fischbein 1987, p.203)

Fischbein’s definition of model is quite general (it also includes mathematical or physical models of concrete phenomena):

‘a system B represents a model of system A if, on the basis of a certain isomorphism, a description or a solution produced in terms of A may be reflected consistently in terms of B and vice versa’ (Fischbein, 1987, p. 121).

A model is neither an isolated rule nor a perception: it is a structural entity, ‘internally consistent’, endowed with its own ‘laws’ and autonomous with respect to the original. In addition *intuitive models* possess characteristics such as concreteness, immediacy, stability and coerciveness which make them efficient substitutes of the formal mathematical notions in the reasoning processes.

Finally, models might be produced either purposely and consciously as a support in problem solving activity, or automatically and beyond the direct conscious control of the individual: in this latter case they are named ‘tacit models’. Tacit models constitute an implicit dimension of the individual’s knowledge of which the individual her/himself is not aware. Such dimension influences all the processes of knowledge constructing and developing: such as the processes of problem solving and discovering. As a consequence, individuals’ systematic errors might be due to the limits of a tacit model.

### **The process-object duality**

As Gray and Tall observe ‘the notion of actions or processes becoming conceived as mental objects has featured continually in the literature’ (Gray and Tall, 1994, p.118). Out of the number of studies which adopt such a perspective we briefly present the framework carried out by Sfard (1991) and Sfard and Linchevski (1994) on which we ground the analysis developed in our study.

Sfard (1991) claims that abstract notions ‘can be conceived in two fundamentally different ways: *structurally* - as objects, and *operationally* – as processes’ (Sfard, 1991, p.1). According to her the fact that the same representation and the same mathematical concept may be conceived both structurally and operationally apparently pervades the whole mathematics:

‘Almost any mathematical activity may be seen as an intricate interplay between the operational and the structural versions of the same mathematical ideas.’ (ibidem, p.28)

The two main features of Sfard’s perspective are that: (i) her distinction refers at once to individual cognitive processes and to the historical formation of mathematical concepts, (ii) ‘the terms operational and structural refer to inseparable, though dramatically different facets of the same thing’ (Sfard 1991, p.9); that is the reason why Sfard speaks of duality rather than dichotomy.

As far as concept formation is concerned, the basic tenet of this theory is that

‘the operational (process-oriented) conception emerges first and that the mathematical objects (structural conceptions) develop afterward through reification of the processes’ (Sfard and Linchevski, 1994, p.191).

The emergence of a structural conception of a mathematical object on the one hand provides information which may be more easily processed and manipulated by the individual than that provided by an operational conception, and on the other hand allows the reorganization of the operational knowledge itself. A certain mathematical notion should be regarded as fully developed only if it can be conceived both operationally and structurally.

Finally, we remark, after Sfard, that the process-object duality reflects partially also in the (external) representations of mathematical concepts: in fact not all the representations seem to be able to evoke with the same degree of immediacy operational or structural conceptions. The attention paid to representations in general (not only to symbols) is one of the reasons why we adopt Sfard’s perspective.

## **OUR RESEARCH PROJECT**

As said in introduction, the data we are going to present and analyze are drawn from our doctorate research project. In this section we briefly present the aims and methodology of that research project.

### **Aims**

Roughly speaking, the general goal of our research project was to identify undergraduate and postgraduate students’ errors and difficulties in solving VST problems. More precisely we focused on basic notions of VST: linear combination, linear dependence/independence, generators and so on. In particular main attention was paid to the notion of linear combination, which, although central in an axiomatic approach (as said, usual in Italian University teaching), appears rarely - if ever - object of specific activities in teaching practice.

### **Methodology**

The study was articulated in two different interlaced phases: (a) the analysis of undergraduate textbooks, and (b) the observation and qualitative analysis of undergraduate and graduate students’ behaviours to solve VST problems. In this contribution we only report on the data collected from this latter phase.

The study involved 15 students in Mathematics: 8 first year undergraduates (FYs), 4 last year undergraduates (LYs) and 3 PhD students. The methodology of investigation was that of the clinical interview (Ginsburg, 1981; Swanson et al., 1981; Cohen & Manion, 1994): each student was presented with two or three different problems to be solved in individual sessions which were audio-taped. No time constraint was imposed over the problem solving sessions, most of which lasted more than one hour.

## The problem

Many students met several difficulties when solving especially one out of the proposed problems. In the present paper we will refer only to this problem:

**Problem.** Let  $V$  be a  $\mathbb{R}$ -linear space and let  $u_1, u_2, u_3, u_4$  and  $u_5$  be 5 linearly independent vectors in  $V$ . Consider the vector  $u = \sqrt{2}u_1 - 1/3u_2 + u_3 + 3u_4 - \pi u_5$ .

- Do there exist two 3-dimensional subspaces of  $V$ ,  $W_1$  and  $W_2$ , such that  $W_1 \cap W_2 = \text{Span}\{u\}$  ?
- Do there exist two 2-dimensional subspaces of  $V$ ,  $U_1$  and  $U_2$ , which do not contain  $u$  and such that  $u$  belongs to  $U_1 + U_2$  ?

The problem could be approached in at least two different ways:

Approach 1. One might try to describe the conditions which the subspaces must fulfil in terms of their possible generators. For instance one could notice that the subspaces  $W_1 = \text{Span}\{u, u_1, u_2\}$  and  $W_2 = \text{Span}\{u, u_3, u_4\}$  verify the conditions posed in the former question, and that the subspaces  $U_1 = \text{Span}\{u_1, u_2\}$  and  $U_2 = \text{Span}\{u_3 + 3u_4, u_5\}$  verify the conditions posed in the latter one; many other pairs of subspaces fulfilling the required conditions can be constructed in analogous ways.

Approach 2. Alternatively one could notice that the definition of  $u$  as linear combination of the vectors  $u_i$  is not necessary to solve the problem (the only relevant information concerns the dimension of  $V$ ), which so can be re-formulated as follows:

*Given a real vector space  $V$ , which dimension is greater than or equal to 5, and a vector  $u$  in  $V$ : do there exist ...*

Such a formulation could lead one to observe for instance that the answer to the second question is positive in  $\mathbb{R}^3$  and as a consequence in any real vector space with dimension greater than or equal to 3.

We sketched these two approaches to give the idea of the wide spectrum of possible solutions of the problem. Of course we did not expect that the students interviewed followed the latter approach – and indeed none did it.

Finally, though the proposed problem could be considered in a sense ‘unconventional’ because more information is provided than needed for the solution, let us remark that such information is not really superfluous: approach 1 shows that all the hypotheses can be effectively used to construct a solution of the problem. It is not necessary at all to neglect any of the hypotheses.

## A BRIEF OVERVIEW OF STUDENTS’ DIFFICULTIES

Above we briefly sketched two different possible approaches to solve the problem, however no matter what solving strategies are chosen, the solution of the proposed problem requires one to cope with many (possibly not stated) ‘sub-problems’, e.g.: to decide whether given vectors are linearly dependent or not, to decide whether a

vector can be written as a linear combination of some other vectors, to construct vector spaces fulfilling specific requirements, to construct/choose a spanning set of a vector space...

The reading of the transcripts of the interviews reveals that many difficulties and errors made by the students may be interpreted in terms of answers – often incorrect – to specific tasks arisen during the problem solving activity. Such tasks are rarely, if ever, made clear during the interviews: on the contrary they are *a posteriori* identified from the analysis of students' solutions [2]. In the next lines these tasks are formulated and for each of them an excerpt is shown from the transcripts of the interviews.

**Task LD:** *Are a linear combination and the vectors defining it Linearly Dependent or independent?*

59 Lau (LY): the point is: are they [ $u_1$  and  $u_2$ ] linearly independent with respect to  $u$ ? no [...]

60 Lau (LY):  $u$  is linear combination of  $u_1, u_2, u_3, u_4, u_5$  thus it is not linearly independent from these other two [ $u_1$  and  $u_2$ ]

**Task LC:** *May a Linear Combination of 5 linearly independent vectors be written as linear combination of 4 linearly independent vectors?*

83 Nic (LY): I think it is not possible... because... because in order to write  $u$  I need 5 linearly independent vectors, in order to write it as [element of the] sum of two 2-dimensional vector spaces I can at most use 4 vectors, because they are linearly independent...

**Task D:** *What Dimension should subspaces containing linear combinations of 5 linearly independent vectors have?*

24 Enr (FY): but  $u$  is constituted by 5 coordinates, and thus by 5 linearly independent vectors; so belonging to the sum [ $U_1+U_2$ ] [...] the dimension [of  $U_1+U_2$ ] has to be at least 5; on the contrary the sum can not have dimension greater than 4

**Task S:** *How can one choose the Spanning set of a vector space containing a given linear combination?*

13 Jas (FY): [...]  $u$  has to belong to their sum [ $U_1+U_2$ ], let's see. I don't think that it is possible, because if I take... let's see [...] in order to get that  $u$  belongs to their sum [ $U_1+U_2$ ] I have to find in this sum at least both  $u_1$  and  $u_2$  and  $u_3$  and  $u_4$  and  $u_5$  [...] but if  $U_1$  has dimension 2 then I get that it does not contain more than 2 linear independent vectors which I can suppose to be  $u_1$  and  $u_2$ ...

Let us underline that 12 (including 2 PhD students) out of 15 students met difficulties when facing (one or more of) these tasks and many of these students did not succeed to solve them.

## ANALYSIS OF STUDENTS' DIFFICULTIES

### Analysis in terms of tacit intuitive models

The observed students' difficulties – together with a tendency to focus on 'special cases' without seemingly grasping their specificities – suggest us the possible influence of tacit models on the solution process. More precisely we formulate the hypothesis that  $n$ -tuples,  $\mathbb{R}^n$ , its 'canonical' basis and its 'coordinate subspaces'[3] may function as tacit intuitive *paradigmatic* models of the concepts of vector, basis, spanning set, vector space and subspace. Fischbein refers to *paradigmatic* model when:

'The meaning subjectively attributed to it [a formally defined concept], its potential associations, implications and various usages are tacitly inspired and manipulated by some particular *exemplar*, accepted as a representative for the whole class.' (1987, p.143)

We now briefly question the internal coherence of the stated hypothesis and later the consistency between this hypothesis and students' behaviours.

The coherency between  $n$ -tuples as paradigm of vectors and  $\mathbb{R}^n$  as paradigm of vector space appears hardly worthy of mention.

As for the notion of basis, one main 'special' feature of  $\mathbb{R}^n$  is the existence of a 'canonical' basis. As a consequence one has privileged (i.e. immediately available) sets of linearly independent vectors and, thus, privileged bases of possible subspaces. In this sense coordinate subspaces are privileged subspaces of  $\mathbb{R}^n$  and consequently may be tacitly assumed as paradigm of the concept of vector subspace.

Moreover, let us consider an  $n$ -dimensional (real) vector space  $V$  and a  $k$ -dimensional subspace  $W$  of its: consistently with our hypothesis,  $V$  and  $W$  might be tacitly thought as  $\mathbb{R}^n$  and  $\mathbb{R}^k$  (being  $W$  a vector space itself). An efficient and meaningful model should allow to express that the latter is a subspace of the former: in fact  $\mathbb{R}^k$  may be 'naturally'[4] represented as a coordinate subspace of  $\mathbb{R}^n$ .

As for linear combinations, a vector  $v$  defined as linear combination of  $n$  linearly independent vectors may be directly represented as an  $n$ -tuple: the  $n$ -tuple of its coordinates with respect to that set of linearly independent vectors. As a consequence it 'naturally' belongs to  $\mathbb{R}^n$ . It can also be thought as an element of  $\mathbb{R}^m$  (with  $m > n$ ) if  $\mathbb{R}^n$  is thought as subspace of  $\mathbb{R}^m$ , or as an element of  $\mathbb{R}^k$  (with  $k < n$ ) if  $n-k$  components are zero: that is if  $v$  is actually linear combination of  $k$  vectors.

As said, the above discussion does not mean to describe the actual mental processes underlying the adoption of such a model, but to show its global internal coherency and to provide elements for investigating the consistency between students' highlighted behaviours and the activation of the supposed model.

In fact, such a model is consistent with the (tacit) re-conceptualization of linear combinations of 5 (respectively 4) linearly independent vectors as quintuplets

(respectively quadruplets) and does not provide any means to transform the former in the latter ones; if this is the case the adoption of such a model may hinder, inhibit the re-conceptualization of the given vector  $u$  as linear combination of 4 linearly independent vectors (*Task LC*). Analogously, if  $u$  is thought as a quintuplet, then it may be tacitly conceived as an element of  $\mathbb{R}^5$  or, maybe, of  $\mathbb{R}^m$  with  $m \geq 5$ : as a consequence the dimension of a vector space containing a linear combination of 5 linearly independent vectors should be greater than or equal to 5 (*Task D*). Finally the hypothesized model may influence student's solutions also limiting the possible choices of the spanning sets of the subspaces to be constructed to subsets of a 'privileged' set of linearly independent vectors, namely  $u_1, u_2, u_3, u_4, u_5$  (*Task S*).

Without entering the details, let us mention that the analysis of textbooks highlights that the vector spaces  $\mathbb{R}^n$  constitute a privileged context where VST notions are introduced or exemplified, and where many problems are posed and solved. One may wonder whether the hypothesized tacit model originates from the teaching practice.

### **Analysis in terms of process-object duality**

Among the different notions of VST, on which our study focus, especially the notion of linear combination can be conceived, consistently with Sfard's theory, both as an object – i.e. the vector resulting from the linear combination – and as a process – the process of 'linearly combining' (through sum and scalar multiplication) those vectors which appear in the linear combination itself. Of course one can expect that this duality reflects also on the conceptualization of those notions which directly involve the notion of linear combination: namely linear independence, spanning set, and basis.

We start our analysis discussing *Task LC*. The students involved in the study refer to the vector  $u$  either by means of signifiers of the natural language – 'linear combination of 5 linearly independent vectors' – or by means of signifiers of the algebraic/formal language – ' $\sqrt{2}u_1 - \frac{1}{3}u_2 + (u_3 + 3u_4) - \pi u_5$ '.

In the former case, the signifier evokes a process ('linear combination of 5 linearly independent vectors') but does not make it explicit. In order to correctly answer the task, one should re-organize this process in the process 'linear combination of 4 linearly independent vectors'. Such re-organization may be hindered because the signifiers do not make the two processes clear. Moreover the term 'linear combination' evokes a class of processes rather than different specific processes of the same class: the two specific processes may remain undistinguishable.

Indeed the only explicit element which distinguishes the two signifiers 'linear combination of 5 linearly independent vectors' and 'linear combination of 4 linearly independent vectors' is the reference to the number of vectors participating in the two linear combinations. According to our hypothesis the number of vectors involved might become the characterizing, distinctive element of the process and of the object as well. Consistently  $u$  might be characterized by the number of vectors which appear

in the linear combination defining  $u$ .

Even students using the algebraic/formal system of representations met difficulties when facing *Task LC*. In this case the Task may be rephrased, for instance, as follows: may the linear combination  $\sqrt{2}u_1 - \frac{1}{3}u_2 + u_3 + 3u_4 - \pi u_5$  be written as  $\alpha_1 v_1 - \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$ ? [5]

The two signifiers still evoke different processes, or better a specific process, the former, which is not directly represented in the class of processes evoked by the latter signifier. The adopted perspective suggests that in order to solve this problem one has (a) to isolate a sub-process from the former process (e.g.  $\sqrt{2}u_1 - \frac{1}{3}u_2$ ), (b) to conceive it as an object and then (c) to take it together with the remaining vectors (e.g.  $u_3$ ,  $u_4$  and  $u_5$ ) as inputs of a new process for constructing  $u$ . The potential effectiveness of the algebraic language is limited on the one hand because  $u$  results from a specific, privileged process (which in fact defines  $u$ ) which should be completely re-organized. On the other hand because such re-organization entails a complex articulation between the procedural and structural conceptions of linear combination.

As mentioned above, the dual nature of the notion of linear combination (process-object) reflects also on the conceptualization of those notions which directly involve the notion of linear combination. The difficulties students met when facing the *Tasks LD*, *D* and *S* may be related to the operational conception of linear combinations.

Conceiving a linear combination as a process to produce a vector may result to be misleading when considering the relation of linear (in)dependence (*Task LD*). In fact it may increase the risk of interpreting the relation of linear (in)dependence among vectors as a relation of ‘functional dependence’ or of common sense dependence. The vector produced by means of the process of linear combination necessarily depends on the vectors defining the linear combination.

As for *Task D*, some students clearly express the belief that the dimension of a vector space containing  $u$  should be greater than or equal to 5, the number of vectors in the linear combination defining  $u$ . Thus the dimension of a vector space containing  $u$  denotes the number of inputs of the construction process of the vector  $u$ . This view is consistent with the discussed ‘impossibility’ of writing  $u$  as linear combination of 4 linearly independent vectors: if it was possible, the dimension of a vector space should not be ‘well-defined’, in a mathematical-like sense. So, though the dimension of a vector space may be inferred by the ‘property’ of a single vector, nevertheless it is still a property of the vector space.

Finally, the spanning set of a vector space is a set of vectors such that through its linear combinations each vector of the space can be expressed. In particular the spanning set of a vector space containing  $u$  should allow to implement the process ‘linear combination’ defining  $u$  and thus should contain the vectors which appear in that linear combination. Possible spanning sets of a vector space containing  $u$  should contain  $u_1$ ,  $u_2$ ,  $u_3$ ,  $u_4$ ,  $u_5$  (*Task S*). Let us remark that, once again, this is consistent with the ‘local’ characterization of the dimension of a vector space as number of

vectors in the linear combination defining  $u$ .

The hypothesis concerning students' difficulties in articulating the operational and the structural conceptions of VST notions, seems consistent with the results of the textbooks analysis. In fact the need of operating such articulations is not made clear in textbooks, on the contrary it may be concealed by the availability of algorithms for solving different problems.

## CONCLUSION

The analyses developed according to the two theoretical frameworks chosen highlight different aspects concerning how basic notions of VST might be conceptualized and how such conceptualization relates to students' difficulties.

The hypotheses formulated might seem hard to compare; indeed a complete comparison of the two developed analyses is out of the possibilities of the present work. Nevertheless interesting elements may be pointed out even from a first comparison.

One main distinguishing aspect concerns the ways in which 'linearity' is taken into account from the cognitive point of view.

In fact, our analysis in terms of process-object duality is centred on and directly concerns the conceptualization of the notion of linear combination. In a sense the idea of 'linearity' as emerged from this analysis refers directly to the possibility and the meaning of performing linear combinations. The global linear structure of vector space remains behind the scene, it is never questioned. In a sense the process-object duality theoretical frame has provided a lens for investigating inside the vector space structure rather than the structure itself.

On the contrary our analysis in terms of tacit model directly deals with the idea of structure: vector spaces and subspaces are taken into account as object of investigation and not merely as 'environments' where linear events occur. Indeed the tacit model suggested by our hypothesis might hide (rather than reveal) the vector structure: in a sense in  $\mathbb{R}^n$  – the supposed paradigm of vector space – relevant linear phenomena might result trivialized because of  $\mathbb{R}^n$  'natural' linear structure. In particular, 'linearity' – meant as the possibility of performing linear combinations – is condensed and hidden if linear combinations are directly perceived as  $n$ -tuples.

Finally we conclude highlighting that, notwithstanding the (even deep) differences of the two analyses, both the stated hypotheses allow to account for a variety of students' difficulties: even more, different students' behaviours facing different tasks are re-organized and framed in 'consistent systems'.

## NOTES

1. Indeed in Maracci 2003, a first analysis of students' intuitive models was developed based on a small subset of data; the analysis drawn in the present work is completely independent from that.
2. Of course, being a posteriori, the identification of such tasks is at some extent artificial; moreover

because they deal with strictly related mathematical contents, in some cases it might be not so clear to what task students' 'answers' may be more appropriately related.

3. Coordinate subspaces of  $\mathbb{R}^n$  are those spaces spanned by a subset of  $\mathbb{R}^n$  canonical basis.
4. In this context we use 'natural' after the mathematical common use of the term. We do not mean to refer to something 'natural' from a cognitive point of view.
5. More precisely: do there exist vectors  $v_1, v_2, v_3, v_4$  and scalar  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  such that... ? Of course the task might be rephrased in many other different ways.

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# USING MIXED-METHODS METHODOLOGY TO INVESTIGATE CYPRIOT PRESERVICE TEACHERS' MATHEMATICS CONTENT KNOWLEDGE

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*This paper focuses on the importance of the analysis of underlying philosophical aspects of the studies and their implications for the choice of data collection methods. Particularly, the aim of this paper is to describe the characteristics and the underlying assumptions of the mixed-methods methodology in the context of my ongoing doctoral study which is centered in understanding the relationship between Cypriot preservice teachers' understanding of mathematics and their teaching practices.*

## **INTRODUCTION**

It is important to take into account different methodologies of educational research in order to provide a context for the methods chosen. Whereas methods are the techniques used for the collection and the analysis of data that will help best to answer research questions, the aim of methodology is first to describe and analyse these methods, throwing light on their advantages and their limitations, and second to help to understand and take a critical view of the research process.

The focus of this paper is on the appropriateness of using-mixed methods methodology for my ongoing doctoral study for which the arguments are based on theoretical definition expressed by Cresswell, Trout and Barbuto (2002). I explain why this approach is better suited to explain teachers' mathematical knowledge for teaching than a study based only on a quantitative or only a qualitative approach.

## **LITERATURE REVIEW**

The object of the study discussed is based on the classic distinction by Shulman (1986) between two aspects of mathematical knowledge, Subject-matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). SMK consists of substantive and syntactic knowledge. Substantive knowledge focuses on the organisation of key facts, theories, and concepts and syntactic knowledge on the processes by which theories and models are generated and established as valid.

Pedagogical content knowledge (PCK) includes the interpretations, examples and applications that teachers use in order to make subject matter comprehensible to students.

From a variety of perspectives, research in the field of preservice teachers' knowledge focuses on their SMK and PCK. Some researchers have investigated preservice teachers' understanding of different topics in mathematics (Ball, 1990) and others have focused on investigating the relationship between SMK and PCK

(Rowland, Huckstep and Thwaites, 2004; Hill, Rowan and Ball, 2005) and have suggested that SMK might affect the process of teaching. These studies have shown that preservice teachers' substantive knowledge of different concepts of mathematics was significantly better than their syntactic knowledge and this was reflected in the ways in which they taught mathematics. Finally, research has suggested that teachers were unsuccessful in promoting mathematical learning outside of the limits of their own understanding and their knowledge was significantly related to student achievement gains. (Hill et al, 2005).

In Cyprus, policy makers' concern about students' achievement gains in mathematics has recently grown and many attempts at improving the instructional practices in public primary schools have been made. The last attempt was to develop new mathematics textbooks and a new curriculum. The attempts to change the curriculum were important but however, it seems that policy makers did not take into consideration that in order to implement the new mathematics curriculum effectively, skilled teachers who understand the subject matter are needed. Indeed, research has suggested that teachers' knowledge of the subject matter is central to their capacity to use instructional material effectively (McNamara, 1991). All, the attempts of improving mathematics teaching in Cyprus have focused on students and the curriculum and none is focused on teachers. Research on teachers' knowledge has been neglected in the Cypriot literature. The few studies in this field (e.g. Philippou and Christou, 1994) focused on investigating aspects of Cypriot preservice teachers' substantive and syntactic knowledge of mathematics and have shown that the participants were poorly prepared to examine different mathematical concepts and procedures conceptually. However, it remains an empirical question what beliefs about mathematics Cypriot preservice teachers hold, and how their SMK is related to their PCK. Moreover, if we want better to understand what goes into teaching mathematics effectively the challenge is to integrate SMK and PCK. Teachers need not only to have comprehensive understanding of mathematics but at the same time they must be in the position to use their understanding to help students learn mathematics. The identification of the relationship between SMK and PCK will help policy makers and university instructors to assess teacher education programmes and improve them where necessary.

The synthesis of the literature reviewed to this point led to the following research questions:

- What aspects of mathematical understanding does the mathematics curriculum in Cyprus expect Cypriot preservice teachers to hold?
- What is the nature of Cypriot preservice primary school teachers' knowledge of mathematics?

Specifically:

Is Cypriot preservice primary school teachers' SMK mostly substantive, or syntactic or both?

What are Cypriot preservice teachers' beliefs about mathematics and its teaching?

-What relationship can be observed between Cypriot preservice primary school teachers' SMK, and their PCK, including particular ways that their teaching of mathematics is informed by their SMK?

Having presented the literature relevant to my study and explaining why my study is worth doing in the light of what has already been done, in the following sections I will analyse the characteristics of the mixed-methods methodology as this is conceptualised in my study, explain why I chose this approach, clarify its theoretical and philosophical assumptions, and finally clearly exhibit how four different methods of data collection will be used to compare, combine, integrate and complement data related to Cypriot preservice teachers' SMK and PCK of mathematics.

## **METHODOLOGY**

### **Conceptualising and justifying mixed-methods methodology.**

An unproductive methodological war posed quantitative against qualitative researchers with the former having a positivistic perspective on educational research and believing that human behaviour is governed by general universal laws on educational research and with the latter having an interpretive perspective in which human behaviour is seen as socially dependent and context related. According to Pring (2000) a 'false dualism' has been created between positivists and interpretivists. The differences between the two paradigms have been exaggerated and opposition between quantitative and qualitative research is mistaken. Differences in epistemological assumptions should not keep an interpretivist from using quantitative methods and a positivist from using qualitative methods.

The qualitative and quantitative debate had a catalytic role in moving beyond the dichotomy of qualitative and quantitative research and in developing a new approach called mixed-methods research. This more productive stance of mixed methods is the one used in the study discussed.

A plethora of definitions of mixed-methods research is provided in the literature and there is some debate amongst researchers as to what would be a precise definition of it (Greene, Caracelli and Graham, 1989; Cresswell et al, 2002). The aim of this paper is not to discuss the various definitions but rather to support the appropriateness of using a mixed-methods methodology based on the theoretical definition of Cresswell et al (2002), according to which,

mixed methods research is research that involves collecting, analysing, and integrating both quantitative and qualitative data in a single study or in multiple studies in a sustained program of inquiry (Cresswell et al, 2002, p.3).

Considering the focus of this paper at this point is important to include some comment on methods used in previous studies in the field of teachers' knowledge. Specifically, through few examples I will support the appropriateness of using mixed-methods methodology to study teachers' knowledge, than a study based on only a

quantitative on only a qualitative approach. Philippou and Christou (1994) used a clearly quantitative approach to investigate Cypriot preservice teachers' conceptual and procedural understanding of fractions. They constructed a test of 30 questions and they used quantitative data analysis methods. For example the multivariate analysis of variance was conducted to identify differences among the subjects regarding their mathematical background at school. A similar approach is the one used in Ball, Hill and Bass (2005) study. In this study the researchers designed a test that includes multiple choice questions that aimed to measure teachers' 'mathematical knowledge for teaching' which refers to what teachers need to know in order to be successful in the classroom.

The quantitative approach used in the studies mentioned above had several advantages. First, in both cases the tests were administered to a large number of participants and the researchers managed to produce a mountain of data in short time and with low cost. This kind of data can be persuasive to policy makers. Second, the researchers managed to get data based on a representative sample of teachers. This, in turn, means that is more likely to generalise statements made on teachers' knowledge of mathematics.

However, the main disadvantage of this approach is that the data that were produced in both studies are likely to lack depth on the topic being investigated. Also, the emphasis on producing data that can be generalised, limits the researchers' ability to check the accuracy of the responses. Here, the use of mixed-methods methodology can fill out what is learned from the quantitative data. For example, in the Philippou and Christou study (1994) the use of interviews could have help to better understand the statistical data. For example if the researchers interviewed a small number of the participants, they would be in the position to explain why there were not significant differences among participants' performance to the test with reference to their mathematical background at school. Furthermore, in the Ball et al (2005) study the researchers could have had a clearer idea of what teachers' need to know in order to be effective in the classroom if they observed mathematical lessons. Even, though their instrument was designed through an extensive development phase and the tasks integrate both aspects of teachers' SMK and PCK, the major advantage of using observations, is that these will provide information about the various things that participants' know and believe about mathematics and how these come together in their teaching in the real context of the mathematics classroom.

In contrast to the quantitative approach used in the studies mentioned above Hutchinson (1997) used a case study approach to explore the relationship between SMK and the acquisition of PCK. For that purpose she used interviews and observations that were analysed qualitatively and supported that in many cases the participant' s limited SMK was evident in her teaching. The main benefit of this approach is that the researcher used a variety of qualitative methods for collecting her data that allowed her to capture the relationship between the participant's knowledge of mathematics and teaching. However, the main point of criticism of this approach is whether its findings can be generalised to other cases. In other words, is it easy to

convince the readers that the case under investigation is it similar, or in contrast with others?

Rowland et al's (2004) mixed-methods approach in investigating the relationship between trainee teachers' SMK and PCK is a good example of how Hutchinson's (1997) study could have been improved. As Hutchinson, Rowland et al used observations for investigating the relationship between teachers' SMK and PCK. The difference was that the selection of the cases that were observed was based on the scores of a 16-item audit that aimed to measure their SMK. The audit was administrated to all the trainees, was marked and used to identify three groups based on the score. Then two cases from each group were observed in order to identify the relationship between participants' SMK and PCK. This, in turn, means that the reader is more likely to be convinced that the trainees observed was more likely to be similar (in respect of SMK with participants classified in the same group), or in contrast with other trainees and thus the results from the cases observed can be in a sense generalised to other cases. Moreover, the use of observations increased the validity of the audit data, as data from them were compared with the data from the audit.

Another example of a mixed-methods study that fits Cresswell's et al conceptualisation is the Teacher Education and Learning to Teach (TELT) study (Kennedy, Ball and McDiarmid, 1993) The TELT study aimed to investigate the relationship between the content and the format of teacher education and what teachers learn about teaching. For the purpose of the TELT study the researchers developed a data collection system that entailed three data collection methods, questionnaires, interviews and observations. The data collected by each method were compared, combined, and integrated and provided a deep understanding of teaching as influenced by teachers' knowledge, skills and beliefs.

Summarising the above the use of a mixed-methods approach in a study on teachers' knowledge can be justified by a number of reasons. First, integrating qualitative and quantitative data can provide strong evidence for conclusions, and provide better inferences on teachers' SMK and PCK (Cresswell, 2003). The results from the qualitative data can help to better understand the statistical findings.

Moreover, triangulating the data from different methods increases the validity of the results and the conclusions. Finally, the strength of one method can be used to compensate the deficits of another method. Further discussion on how the above strengths of using the mixed-methods approach are addressed in my study, will follow in the methods section where I will justify the choice of the data collection methods and the potential strengths of mixing them in a single study.

### **Philosophical and theoretical assumptions that underline my research design.**

The choice of a certain methodology and research design is informed by researchers' theoretical perspective, which is also commonly referred as research 'paradigm'. This includes researchers' world views, values, attitudes and beliefs, which influence all of their decisions in the design of a study. In the literature about mixed-methods research there are different positions on how research paradigms inform the design of

a mixed-methods study. According to one of these, mixed-methods designs should be informed by a single paradigm. For many researchers, pragmatism can be employed as the paradigm for best justifying mixed-methods research. (Tashakkori and Teddlie, 2003). According to this paradigm, researchers can collect both quantitative and qualitative data since they develop a rationale for mixing and they can integrate their data at different stages of the research process. Pragmatists are not committed to any philosophy and they cannot see the importance of discussing assumptions about truth and reality when designing their research. What is important to them is what works in practice. According to Johnson and Onwuegbuzie (2004), pragmatism has been referred to as the anti-philosophy paradigm because it does not clearly raise issues of epistemology and ontology. However, philosophy should matter in mixed-methods inquiry. The aim of my study is to gain knowledge and understanding of teachers' SMK and its relation to their PCK. But what I take knowledge and understanding to be are assumptions that inform my choices in the design of my study. Therefore, the pragmatism paradigm does not underpin my study as it avoids addressing issues of knowledge and reality.

For the purpose of my study I adopted the view according to which multiple paradigms may serve as the foundation for doing mixed-methods research (Cresswell,2003). According to this view, one type of paradigm is best for one type of research, while another paradigm is best when doing another type of research.

Therefore, holding this view the paradigm that underpins my study is interpretive. I believe that human behaviour is the collection of activities performed by human beings and influenced by culture, attitudes, emotions, values, ethics and authority. Therefore, teachers' knowledge can best be understood when observed in the context of their actions. This belief about human behaviour is consistent with the characteristics of the interpretive paradigm of research according to which:

humans actively construct their own meanings of situations; meaning arises out of social situations and it is handled through interpretive processes; behaviour and, thereby, data are socially situated, context-related, context-dependent and context-rich (Cohen, Manion and Morrison et al, 2004, p. 137).

I believe that both SMK and PCK of mathematics represent personal and private knowledge which is influenced by the social and cultural settings in which teachers teach mathematics. Therefore, what I am studying is personal and private knowledge, constructed by teachers, based on their experiences with mathematics. The above characteristics of the interpretive research paradigm are related to the Social Practice Theory (SPT) (Jaworksi,2007) which sees knowledge as being in practice. "Practice" is doing in social context and thus, research should focus on what people do and what is involved in doing it. Cypriot preservice teachers' mathematical knowledge is produced and learned in Cyprus social context. Without taking this context into account it is not possible to understand what preservice teachers know and do.

I expect to find differences in what teachers feel about teaching mathematics, what they know about teaching it, and what they know about it, but at the same time, I

expect to find commonalities, the identification of which would be helpful to future teachers. So, the characteristics of the interpretive research paradigm and the SPT provide the context in which I place my research and my belief that human behaviour must be treated as a collection of activities performed by human beings in a particular social context and influenced by beliefs, values and attitudes. I now turn to how these determine my choice of methods and provide the rationale for why I think that mixing methods will help me to answer my research questions.

#### **Four methods of data collection.**

The focus of this section is on how underlying assumptions of the mixed- methods methodology, within an interpretive research paradigm and the characteristics of the SPT, influence the choice of data collection methods. Specifically, in this section I clearly exhibit how four different methods for collecting data will be used to serve the purpose of the mixed-methods methodology to compare, combine, and integrate data related to Cypriot preservice teachers' SMK and PCK of mathematics. Also, for the purpose of this paper my aim is not to detailed describe each method but rather to focus on the advantages and the limitations of each method and discuss how the strengths of one method can be used to compensate for deficits of another method, and how the combination of these four methods will help me to better understand the relationship between preservice teachers' SMK and PCK of mathematics.

An assumption made in the literature review was that teaching of mathematics might be influenced by teachers' knowledge of it, their teaching skills, and their beliefs about mathematics and its teaching, Therefore, in order to better understand the relationship between Cypriot preservice teachers' SMK of mathematics and their PCK, I needed a data collection system that could tap all the above.

The data collection process will start with the use of a questionnaire. According to the interpretive research paradigm teachers' construct their mathematical knowledge based on their beliefs and experiences with mathematics and its teaching. Thus, the aim of the questionnaire is to collect information on participants' experiences with mathematics, their beliefs about mathematics and its teaching, and their substantive and syntactic knowledge of it. The questionnaire will be divided into three sections. The first section aims to elicit information on demographics and on participants' experiences with mathematics. The second section will gather information on participants' beliefs about mathematics and its teaching. This part will be limited to the use of questions asking for degree of agreement (five points Likert scale) on statements related to beliefs about the nature of mathematics and its teaching. Finally, the third part will include ten mathematical tasks that aim to collect data on aspects of participants' substantive and syntactic knowledge of mathematics.

The use of a questionnaire, in the format that is described above, has a number of advantages. First, it presents all respondents with identical items providing a high level of comparability among the respondents. Second, it can be administered to a representative sample and this, in turn, means that statements on teachers' beliefs and SMK of mathematics are more likely to be generalised.

However, the closed format of the questions can determine the depth of the participants' beliefs about mathematics and its teaching only to an extent. The closed-ended questions do not say much of what the respondents mean when agreeing with a statement related to their beliefs about the nature of mathematics. Also, the audit items do not say much about how the participants thought in order to give a correct or an incorrect answer. Here the interview data can fill out what is learned from the questionnaire.

The interview will be divided into two parts. The first part will aim to clarify respondents' answers to the questionnaire items. Particularly, the interviewees will have the opportunity to expand their ideas, explain their views and their way of thinking as these are shown in their responses to the questionnaire items. This discussion will help to better understand, and complement the results from the analysis of the questionnaire. Also, combining the data from the questionnaires and the interviews will make stronger the inferences that were drawn from the statistical analysis of the quantitative data. Finally, an assumption, that is consistent with the interpretive research paradigm, is that participants' responses to the questionnaire items are based on how they interpret the questions. The interview is a way of increasing the validity of the questionnaire data by checking that the participants' interpret in the same way the questionnaire items. Therefore, these strengths of the use of interviews will be used to overcome the weaknesses in the use of a questionnaire.

The aim of the second part of the interview is to gather information on how the interviewees' use various aspects of their SMK in taking decisions about mathematics teaching. I shall address this issue by the use of hypothetical scenarios that represent real classroom situations which every teacher could encounter while teaching.

The interview tasks provide information on both what teachers know and believe about mathematics and on what aspects of their knowledge and skills they draw on in taking teaching decisions. However, while the interview tasks will represent real situations in the mathematics classroom, their context remain hypothetical and cannot provide a sense of what teachers' actually do. What participants say they do cannot be assumed to reflect the truth. According to the SPT (Jaworski, 2007) teachers' teaching practices are context-dependent and context-rich. Therefore, research should study participants' activities and processes in a particular social context. Hence, the only way to understand what preservice teachers do and what is involved in doing it is to observe them in real classroom situations where they interact with their students.

The major advantage of using observations as a method for collecting data is that I will be given the opportunity to look at what is taking place in the classroom rather than having information at second hand of what is happening in the classroom. Also, the use of observations can increase the validity of questionnaires and interviews as data from it can be compared with the data from questionnaires and interviews. The observations will provide information about the various things that participants' know and believe about mathematics and how these come together in their teaching.

Participants' SMK and PCK are influenced by the context in which they teach (mathematics education and curriculum in Cyprus), an assumption related to the interpretive research paradigm and the characteristics of the SPT. Therefore, the data from observations, questionnaires and interviews will be combined and integrated with data from documentary analysis of the mathematics textbooks in Cyprus. An analysis of the content of these books will provide information on what mathematics is taught at primary school level and therefore, what mathematics teachers' need to know in order to use successfully these books.

The combination of the above four methods of data collections will give a better understanding of the relationship between Cypriot preservice primary school teachers' SMK and their PCK. For example, I can learn about participants' teaching skills by observing their teaching, but cannot get much information on what they know and believe. On the other hand, using questionnaires and interviews can provide information on what teachers' know and believe, but cannot know whether these are implemented in practice. Integrating the data from all the methods will provide a clearer idea of how teachers; use their mathematical knowledge and skills while teaching.

## SUMMARY

This paper consists of a case for the importance of analysing philosophical aspects of the studies and its implications for the selection of the data collection methods. Particularly, its focus is on the appropriateness of using the mixed-methods methodology, within an interpretive research paradigm and the characteristics of the SPT, for investigating Cypriot preservice teachers' SMK of mathematics and their PCK.

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## FROM TEACHING PROBLEMS TO RESEARCH PROBLEMS Proposing a Way of Comparing Theoretical Approaches

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*Abstract.* We analyse how researchers with different theoretical backgrounds conceptualise a fuzzy teaching problem and reframe it as a research problem. Does this represent a useful way of comparing different theoretical approaches, and of evaluating their practical significance? This contribution is intended to initiate a discussion of theoretical approaches, first but not only in Working Group 11 of CERME 5.

How should the scientific community in mathematics education deal with what is perceived as diversity of theories within the field? Here we write ‘perceived’, because the community itself may not be well placed to step back and recognise important theoretical commonalities which reflect its shared and taken-for-granted assumptions, being more alert to those features which differentiate theories. Rather than the frequent demand for *unifying* theories, some researchers plead for the primacy of *understanding the differences and commonalities of different theories* (e.g. Dreyfus et al. 2006, Bikner/Prediger 2006). This process of understanding different theoretical approaches has always been an important part of the discourse of CERME-conferences and due to its complexity and due to the richness of different theories, it is far from being finished.

This paper explores one possible way to compare different theoretical approaches and their meaning for research practices and the relation between theory and classroom practice. It follows Charles Sanders Peirce’s pragmatic maxim:

“In order to ascertain the meaning of an intellectual conception one should consider what *practical consequences* might conceivably result by [...it]; the sum of these consequences will constitute the entire meaning of the conception.” (Peirce CP 5.9).

We interpreted Peirce in such a way that we tried to understand different theoretical approaches by considering their *expression in the practice of researchers*. Whereas the practice of research is often discussed only as an issue of research designs and methodologies, we are convinced that research practices are strongly influenced (but of course not completely determined) by the earlier stage of research, namely the way researchers conceptualise their field. That is why we focus our comparison not on different ways of analysing a given piece of data (like Gellert, Halverscheid or Maracci in these proceedings) but on an earlier step in research, conceptualisation of the problem.

Therefore, we asked researchers with different theoretical backgrounds to briefly describe, first how they would conceptualise a given teaching problem, and then how they would design an appropriate research study.

The initial reference point was a teaching problem, which we have often heard expressed along the following lines:

How is it that some students can learn to tackle a particular type of mathematical problem successfully (as shown by their performance in the class), but be unable to do so two weeks or months later?

What strategies can the teacher use to reduce the likelihood of this occurring?

In order to see how different approaches frame this teaching problem as a research problem and devise a research design, we asked the following questions:

- a) How do you –a priori– answer this question and what are your basic assumptions?
- b) How do you transform the raised problem into a research question starting from the question above?
- c) What is your research design?
- d) What type of results would you expect?

The complete responses of eight researchers or research teams can be found in the appendix to this paper.

Some authors informally synthesise different frameworks, although not always explicitly (Ruthven; Bikner-Ahsbabs; and Kaldrimidou, Tzekaki, & Sakonides). Others explicitly adopt some theoretical framework: Jungwirth uses the interactionist perspective, Artigue & Lenfant rely on the Theory of Didactical Situations and the Anthropological Theory of Didactics, Bosch & Gascòn on the Anthropological Theory of Didactics, and Dreyfus & Kidron on the Theory of abstraction in context with the RBC-Model. According to Eisenhart's classification (1991), these frameworks can be classified as theoretical frameworks, whereas Christer Bergsten proposed (in the discussion of the working group) to classify Arzarello & Robutti's use of different aspects of Semiotics, Anthropological Theory of Didactics and the perceptuo-motor approach as a conceptual framework in Eisenhart's (1991) sense.

## CONCEPTUALISATIONS

How do the different authors conceptualise the given teaching problem? Most of the responses accept to a degree the terms in which the problem is posed, representing it as a 'banal phenomenon' (Artigue & Lenfant), a 'natural fact' (Arzarello & Robutti), 'a phenomenon already recognised by everyday commonsense and psychological science' (Ruthven); or couch it in new terms of 'meanwhile the students have worked on other problems [and] have just forgotten how to solve the problem' (Bikner-Ahsbabs); or offer confirming evidence for it, of 'the same students who very successfully factored expressions and solved equations in grade 9, [who] cannot do the same exercises any more a year later' (Dreyfus & Kidron). Nevertheless, all these re-

sponses also suggest that the original terms are inadequate to frame the problem: they identify various issues requiring clarification, or directly elaborate a range of alternative conceptualisations and explanations.

The response from Bosch & Gascón is rather different: it directly proposes a particular reframing, so that the ‘teaching problem’ becomes an ‘institutional problem’:

“We postulate that these facts are different manifestations of a didactic phenomenon that we call ‘*the dis-articulation of the school mathematics (the taught mathematical knowledge)*. ...The kind of mathematical activity the students carry out (for instance, learning to solve a ‘narrowly defined’ type of problems for a short period of time and forgetting it afterwards) is mainly a consequence of the kind of mathematics that exist at school, which are affected by the phenomenon of ‘dis-articulation’.” (Bosch & Gascón)

This conceptualisation, taking an institutional perspective, is far reaching since it implies limitations for improvement strategies at other levels:

“As consequence of our previous postulate, it does not seem that the didactic phenomenon associated with the fact mentioned can be easily modified only by changing teachers’ strategies. The kind of solution we can think of is the implementation of new didactic organisations in a system that has strong traditions and imposes many constraints on the way changes can be carried out ... It is thus necessary to study the mechanism and the scope of the phenomenon.” (Bosch & Gascón)

Bosch & Gascón’s response provides a striking example of how a theoretical framework –in their case, the Anthropological Theory of Didactics– shapes conceptualisation of the given teaching problem and privileges certain types of research question.

Another definite, but distinct position, reframes the ‘teaching problem’ to emphasise that it is also a ‘learning problem’. Following the concerns and perspectives of their RBC model of abstraction in context, Dreyfus & Kidron adopt an individual cognitive perspective with a focus on student learning factors:

“Our research would rather start from the perspective of the student. What we want to know is how things are learned, not only how they are taught. We want to investigate ... what are the learning processes by means of which [...students] arrive at ... connections between knowledge elements [... and] acquire ... (or fail to acquire) explanatory power with respect to a cluster of mathematical concepts or processes.” (Dreyfus & Kidron)

By adding “with respect to a cluster of ...”, they stress the possible domain specificity which is a basic assumption in the theoretical approach of abstraction in contexts used in their research group for analysing processes of knowledge construction. This background guides their formulation of the exact research question in a focused way:

“What are the processes of constructing the knowledge under consideration, and what are students' emerging knowledge constructs? In what are these processes of knowledge construction for a given construct different for the learning processes of students who are successful with this specific construct after a year and those who are not? In what are

these processes of knowledge construction of the same student different for constructs with which the student is successful after a year and those constructs with which she/he is not?”(Dreyfus & Kidron)

In another response, Jungwirth acknowledges that “there are so many explanations”, but “prefer[s] a certain one... due to [her] interactionist stance towards the world”. This distinctive theoretical stance traces the origins of the problem to the way in which “everyday, smooth-running interaction is established by the teacher’s and students’ adjusting to the acting of each other” with the result that “students can successfully participate without an understanding to be located in their ‘heads’; for instance, by answering on questions by short, tentative utterances which seem to indicate understanding so that the teacher completes to the desired answer”. It is typical of an *interactionist perspective* that explanation is sought in terms not of individual cognition but of social constitution of knowledgeableness through classroom interaction.

Although Dreyfus & Kidron, Bosch & Gascón, and Jungwirth all adopt a particular theoretical perspective which ‘privileges’ certain factors as its objects of study, none of the responses denies that other factors may play a part, and that other lines of explanation might be developed; they simply choose not to examine these. By contrast, two other responses identify a much wider collection of potentially important factors.

Bikner-Ahsbas enumerates a wide spectrum of possible causes for the given teaching problem. Whereas the emphasis of most of the responses is on epistemological and cognitive factors, the factors proposed by Bikner-Ahsbahs explicitly include those of student affect and identity which traditional framing of the problem largely ignores. Perhaps, this creates a more holistic model closer to the lived experience of teachers and students, but one less amenable to controlled investigation. In designing a research project, Bikner-Ahsbahs also adopts an *interactionist perspective* on micro-situations in the classroom and poses the following research question: “What kind of conditions in everyday maths classes foster or hinder tackling a similar mathematical problem?” Consistent with her acknowledgement of different lines of explanation, Bikner-Ahsbahs’ framing of a research question is more open and exploratory than that of the similarly interactionist, but more focused and transformative proposal from Jungwirth. Equally, in identifying factors that foster and hinder problem solving, Bikner-Ahsbahs envisages attention to individual students and task characteristics as well as to classroom interaction.

Sakonides, Kaldrimidou & Tzekaki emphasise *a priority of mathematical and epistemological issues* for the conceptualisation of the problem. As long as these issues remain unclear, they cannot develop a concrete research question or a research design. This priority reflects their *epistemological perspective*. They suggest that a constellation of issues must be taken into consideration including the ‘particular type of the mathematical problem’, ‘epistemological features - involving concepts, definitions, properties, procedures, figures, symbols, several modes of representations’, the ‘classroom mathematical culture’ (Sakonides, Kaldrimidou & Tzekaki):

“It will be possible to claim that this particular ‘teaching problem’ is due to difficulties of cognitive nature (stereotypes that persist, or difficulties to treat information of a given way of representing data); of conceptual understanding type; of meta-mathematical nature (it might be that the students thought that something was not very important, so they didn’t learn to ‘tackle the problem’); of didactical nature; of social – cultural nature... [W]e need a hypothesis on why this happens, in order to ... decide whether we need to focus on students (cognition and ways of learning) or on the didactical approach (content knowledge and teaching practices).” (Sakonides, Kaldrimidou & Tzekaki)

In contrast, the response from Ruthven does not accord priority to a specifically mathematical dimension; indeed, this feature distinguishes it from all the other responses. Rather, the main alternative conceptualizations that he proposes frame the problem in generic psychological terms. Most basic of these is that “retention of learned material tends ... to decline over time; in particular, in the absence of further use” (Ruthven); more elaborate, the idea that

“learning is far from complete when students achieve assisted performance in a tightly framed setting; further learning – some of it quite different in character – is required for independent performance in a loosely framed setting.” (Ruthven).

Hence, it makes a difference whether a problem is originally offered in isolation or later in a new situation or combination that demands a related but modified use of learned material. In this case, retention could be improved by giving students experience of solving non-standard problems, tackling mixed revision etc. Other responses acknowledge these same issues, notably those from Dreyfus & Kidron and Artigue & Lenfant, but rather than pursuing these commonalities prefer to adopt a more specifically mathematical focus. Ruthven’s approach to the problem in terms of generic psychological terms which are not specific to mathematics reflects a “*practical theorising approach*” which seeks to find ways of framing the problem which are relatively accessible to practitioners and can be applied “to the design of practical means of addressing it” (Ruthven). Whereas the other responses all emphasise the domain-specificity of their research practices, Ruthven treats the “degree to which generic approaches can be effective, and to which more topic/setting-specific designs are required” as an open question.

Artigue & Lenfant bring out this issue, when they note that their opening suggestions “do[] not have a specific didactic flavour and could lead to look for explanations only at the level of the brain functioning or at the level of personal motivation for studying such or such topic, for learning to solve such or such type of task”, whereas a “didactic approach offers alternative or complementary perspectives, and will not necessarily lead to the same suggestions for improving the situation” (Artigue & Lenfant). Thus they take a quite different path to Ruthven when they suggest that while:

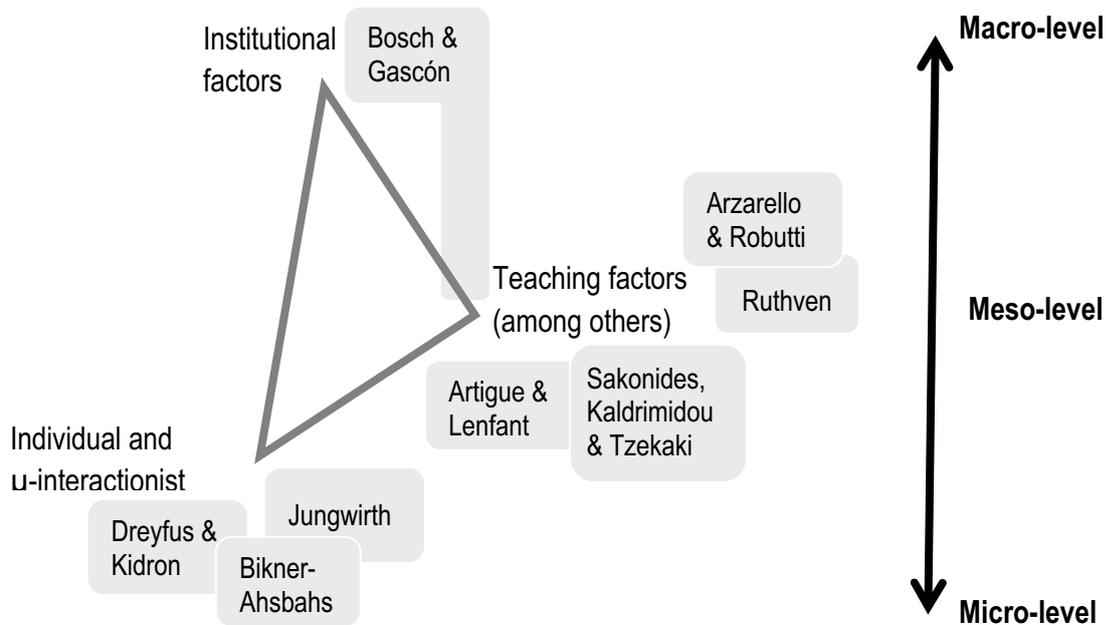
“[t]here is certainly a lot of literature about such issues in cognitive research, [f]rom a didactic perspective, what seems more interesting to us is to transform the raised problem into a research question in such a way that the specificity of mathematics knowledge, of

mathematical and didactical organizations could be taken into account, and that a systemic view could be developed, the ‘forgetting student’ being no longer the exclusive or central object of our attention.” (Artigue & Lenfant)

Due to the situatedness of knowledge and learning, the context of tasks is crucial in their approach. ‘Context’ comprises the situations in which the knowledge was constructed as well as the learning history of the class. They start from the assumption that “the observed phenomenon [of forgetting], if not created, is highly reinforced by didactical choices” (Artigue & Lenfant) concerning the environment of the task. The articulated focus on the mathematical problems and the learning contexts is influenced by the constructivist learning theory –the Theory of Didactical Situations– underlying their framework. Within their conceptualisation in terms of this holistic framework, they suggest that research questions and strategies may vary according to whether the aim is one of improved scientific understanding or of improved teaching practice (see below). They distinguish:

“between research questions orientated towards the understanding of the system functioning and of the influence of its characteristics on the observed phenomenon on the one hand, and research questions associated to the elaboration and evaluation of didactical engineering trying to improve the current situation by playing on one or several levers, on the other hand.” (Artigue & Lenfant)

A distinctive feature of the response from Arzarello & Robutti is the way in which they explicitly put the comparison of teaching strategies –albeit, conceptualised in terms rather different from those current among practitioners– at the centre of their research outline. Adopting a cognitive and semiotic perspective, they conceptualise the problem in terms of three aspects: the level of problems and knowledge (level 1 – knowing, understanding, applying– versus level 2 –analysing, synthesising, evaluating–), the way of thinking (analytical versus spatio-motoric thinking) and the way of teaching. They only focus on two distinct teaching strategies, the traditional one versus the perceptuo-motor approach. The aim of their research outline is to find correlations between the different aspects. One hypothesis is for example that the perceptuo-motor teaching approach produces better long-term effects. For this, teaching experiments and assessments are planned. This response is not only more concretely elaborated than others, but it is also different in the *aim* of giving empirical evidence for the superior long-term-effects of a certain teaching strategy (the perceptuo-motor approach). Long-term knowledge construction is conceptualised to depend on the level of knowledge as well as on the way of teaching and learning. In this respect, the Arzarello & Robutti response aligns with those of Artigue & Lenfant, Dreyfus & Kidron and Ruthven.



One way of thinking about the conceptualisations underlying these responses is in terms of the framing system to which they appeal: a first level system is that of individual learner and associated task environment; a second level system is that of classroom activity and its associated social interaction; at the third level is educational system as a social institution with associated curricular and pedagogical discourses.

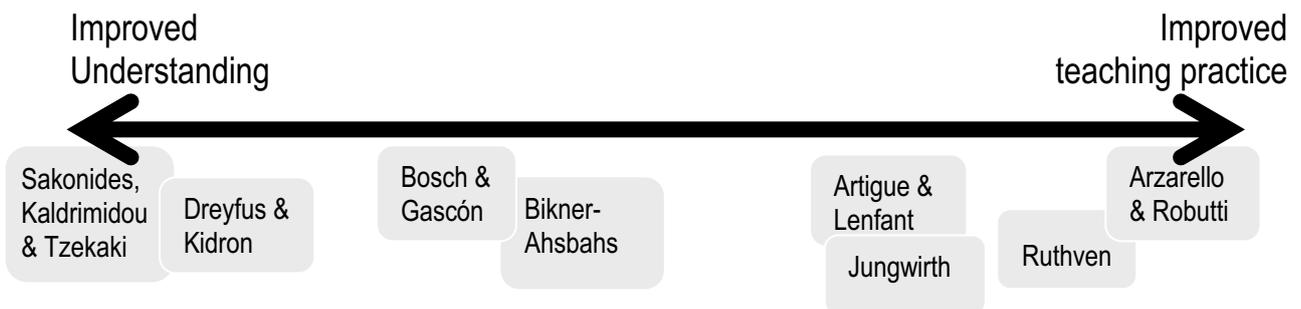
The responses can be located in relation to three idealised poles. One pole, marked by concern with the *micro-level* of individual is most closely represented by Dreyfus & Kidron focusing on domain specific processes of knowledge construction; the micro-interactionist tradition followed by Bikner-Ahsbabs and Jungwirth, which focuses on the fine grain of processes of knowledge construction and communication is probably also most appropriately placed here. Another pole, marked by concern with a *macro-level* of institutional factors is represented by Bosch & Gascón. Finally Artigue & Lenfant, Arzarello & Robutti, Ruthven and Sakonides, Kaldrimidou & Tzekaki define a position which can be thought of as at a *meso-level* in relation to these poles, but also as differing from Bosch & Gascón and Dreyfus & Kidron in a willingness to accept the *particular concern posed with teaching strategies*. In this respect it is notable that such distinctions arise despite the fact that Arzarello & Robutti appeal to the same theoretical perspective as Bosch & Gascón. Equally, Artigue & Lenfant allude to some of the same epistemological and institutional factors as Bosch & Gascón, but tackle questions on the meso-level as well as the macro-level.

## RESEARCH AIMS AND DESIGNS

Understandably, the research outlines could not be very detailed with the given “fuzzy” teaching problem and the restricted space of two-three pages of response. Nevertheless, it is interesting to consider the synopsis (on the next page) of research aims and designs as well as the results expected by the researchers.

Without discussing the expected results of research in detail, we can see differences in the major intention of research, reflecting the dual character of the original request to offer explanation and to advise on teaching strategies. Whereas some responses emphasise the theory-building purpose of mathematics education research, i.e. the increase of understanding for the phenomenon, others stress the theory-applying purpose of developing instructional designs and teaching strategies (see Bergsten 2007 on the double nature of mathematics education research). Depending on the degree to which researchers consider that adequate explanatory frameworks are already available, the balance between seeking improved explanations and converting available explanations into transformative actions can be expected to vary. Ultimately, of course, these two purposes are not opposed but complementary; in particular, research in ‘Pasteur’s Quadrant’ (Stokes 1997) seeks to combine them; indeed all these contributions can be seen as situated there, which is one strong commonality between them.

Without forgetting this complementarity, the responses can, nevertheless, be located at different places on the continuum between pure emphasis on improved scientific understanding or pure emphasis on improved teaching practice in the light of already available explanations.



	Research aims and designs	Explicitly expected results
<b>Artigue &amp; Lenfant</b>	<p>different according to concrete research questions, e.g.</p> <ul style="list-style-type: none"> <li>• “Are different types of tasks equally sensitive to the ‘forgetting phenomenon’?” Explain differences? (by enquiries among teachers and students, questionnaires, analyse syllabus...)</li> <li>• “evaluate the influence on this phenomenon of a specific didactic strategy” (by standard methodology of didactical engineering)</li> </ul>	<p>“A better understanding of the didactic characteristics of this phenomenon, and of the possibility of action.”</p>
<b>Arzarello &amp; Robutti</b>	<p>“RQ1. Does students’ specific knowledge ... change according to the level of the task and how does (can) it change? RQ2: Does the knowledge depend on the way the students learn it and how?”</p> <p>Hypothesis for a concrete design: “correlation between ... ways of learning and ... ways of thinking: ...the perceptuo-motor learning produces long-term effects.”</p> <p>Design: Comparing concurrent teaching styles</p>	<p>“We expect to find some positive correlation between some ... couples of variables:</p> <ul style="list-style-type: none"> <li>• ... Long-time and level 2 knowledge;</li> <li>• Long-time knowledge and B methodology of teaching...</li> <li>• Long-time knowledge and spatio-motoric thinking</li> </ul>
<b>Bikner-Ahsbabs</b>	<p>What kind of conditions in every day maths classes fosters or hinders tackling a similar mathematical problem? qualitative research on classroom videographs (not necessarily in specific learning environments) and interviews afterwards</p>	<p>Improved understanding: Specify “ideal types of situations, task aspects, or personal aspects which disturb or foster solving a problem. ...</p> <p>Improved teaching:</p> <ul style="list-style-type: none"> <li>• enhance diagnostic competences of teachers</li> <li>• give concrete micro-strategies for interaction</li> </ul>
<b>Bosch &amp; Gascón</b>	<p>“(a) Didactic transposition problem: What are the mechanisms of didactic transposition that can explain the phenomenon of disarticulation? ... (b) Ecology of didactic praxeologies: What ... didactic praxeologies can be introduced ... to allow the development of more ‘articulated’ mathematical activities...?”</p> <p><i>Methods:</i> Curriculum analysis and design of epistemological reference model, set up and experimentation of a designed research study course, data analysis</p>	<ol style="list-style-type: none"> <li>1. “<i>Ecology of mathematical praxeologies</i>: new ways of curriculum organisation around powerful generative questions ...</li> <li>2. <i>Ecology of didactic praxeologies</i>: characterisation of possible didactic devices and strategies to manage the different moments and dynamics of the RSC; description of the didactic constraints ...</li> </ol>
<b>Dreyfus &amp; Kidron</b>	<p>Qualitative in-depth-analysis of processes of knowledge construction with respect to the connections to retention.</p>	<ol style="list-style-type: none"> <li>1. characterize differences between the learning processes of students who ‘forget’ and who don’t</li> <li>2. This “might lead to a welcome modification/expansion of the theory”.</li> <li>3. No focus on teaching designs. Hope to derive design principles for constructing and consolidating “in the long run”.</li> </ol>
<b>Jungwirth</b>	<p>I make videos and transcriptions of some lessons of the teacher and analyse them with respect to patterns in the interaction that have been reconstructed by interactionist research before.</p>	<p>I design a teacher education or individual coaching of the teacher ... to make her/him realize the pattern and its routines in order to change her/his part.</p>
<b>Ruthven</b>	<p>“Develop measures for improving retention of material – both in isolation and combination” as a starting point for theorized design of teaching approaches (general approaches and topic/setting specific design) and their analysis (in different research designs: classical experiments, design experiments, action research cycles).</p>	<ol style="list-style-type: none"> <li>1. “Refinement of original theorised measures, and generation of new ones”</li> <li>2. theorised design of teaching approaches</li> </ol>
<b>Sakonides Kaldrimidou, &amp; Tzekaki</b>	<p>Empirical clarification of the extent to which the teaching problem occurs (design depends on concretisation). Afterwards, development of classroom interventions</p>	<p>“provide ... insights into the ... ways in which the classroom learning ... acts, shaping the mathematical knowledge negotiated and ... individual ... learning trajectories.”</p>

## CONCLUSION

Bergsten (2007) raises the question “How does a theoretical basis chosen for a study influence the nature of the purpose, questions, methods, evidence, conclusions, and implications of the study?” In our analysis of eight responses to the starting task, we have initiated some lines of thinking in answer to this question which are worth discussing further. The analysis we have presented is a tentative one, and involving limited testing and iteration. In the working group, these initial thoughts stimulated other members to generate alternative interpretations of the responses, to draw attention to other significant features, and to propose new analyses of the material. We hope that this process continues.

As an answer to Bergsten’s (2007) final question, “But how does this contribute to compare and integrate the contributions of these studies, and others, to a deepened progression of our didactical knowledge?”, we plead for patience: this exploratory analysis shows that integrating theories is much more than a simple triangulation of research methods. The crucial point is the conceptualisation of the problems in view. Comparing, networking or even integrating theories starts from understanding each other’s problem definitions, something which requires extended communication. At the same time, the complementarity of perspectives gives hints that the process is worth it.

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## APPENDIX

Kenneth Ruthven & Susanne Prediger  
Material for the activity

### COMPARING THEORETICAL APPROACHES WITH RESPECT TO THEIR WAY OF FRAMING TEACHING PROBLEMS AS RESEARCH PROBLEMS LINKED TO RESEARCH DESIGNS

One aim of the working group 11 is to deepen our insights on theories, their underlying assumptions, relationships and differences. For this purpose, the organizing committee agreed to prepare a set of questions based on an exemplary teaching problem.

We invite all participants of working group 11 to submit a report (individually or together with other colleagues) outlining their responses to questions (a) to (d) below. This will provide us with interesting material for discussion during the CERME 5-conference.

The initial reference point is a teaching problem, which we have often heard expressed along the following lines:

How is it that some students can learn to tackle a particular type of mathematical problem successfully (as shown by their performance in the class), but be unable to do so two weeks or months later? What strategies can the teacher use to reduce the likelihood of this occurring?

What we are interested in discussing is your approach to framing this teaching problem as a research problem and devising a research design:

- a) How do you – a priori – answer this question and what are your basic assumptions?
- b) How do you transform the raised problem into a research question starting from the question above?
- c) What is your research design?
- d) What type of results would you expect?

If you decide to participate in undertaking this preparatory task (which is not mandatory), please send us your answers along these questions on max. 2 pages before the 15<sup>th</sup> December. Return them to Susanne Prediger (prediger@math.uni-dortmund.de).

On behalf of the organizing team of Working group 11,

Kenneth Ruthven & Susanne Prediger

**MICHÈLE ARTIGUE & AGNÈS LENFANT:  
THEORY OF DIDACTICAL SITUATIONS**

**a) How do you – a priori – answer this question and what are your basic assumptions?**

This question is a priori a question arising from a rather banal phenomenon: what we learn is most often not definitively learnt, and if we do not use what we have learnt, generally, more or less quickly we forget it. Most of us are certainly no longer able to tackle a lot of mathematical tasks, they were used to tackle years ago, and some of us perhaps share the experience of grasping the content of some math courses they were not especially interested in within a few weeks, getting excellent marks, very little remaining from this learning some months later, and especially not the technical ability quickly developed on some precise tasks. And, we also all know the recurrent complain of teachers saying that during summer holidays or even shorter holidays students forget everything.

What is written just above does not have a specific didactic flavour and could lead to look for explanations only at the level of the brain functioning or at the level of personal motivation for studying such or such topic, for learning to solve such or such type of task.

A didactic approach offers alternative or complementary perspectives, and will not necessarily lead to the same suggestions for improving the situation.

First, it leads us to question the question itself. What is meant by having learnt to tackle a particular type of problem successfully? Up to what point can we say that the task proposed two weeks or months later is the same as the initial task? This cannot be inferred just by looking at the mathematical text of the task, without taking into account the context for this task and the ways the teacher manages it.

Second, a didactic approach leads us to question the didactic strategies used for organizing the students' learning of this particular piece of knowledge, and for organizing its relationships with other related pieces of knowledge, hypothesizing that the observed phenomenon, if not created, is highly reinforced by didactical choices: how this type of task was introduced to the students with what mathematical motivations, how techniques for solving it were developed, how did the respective responsibilities given to the students and the teacher in the solving of this type of task progressively evolved, up to what point some particular techniques were trained and routinized, how the variation around this type of tasks was organized taking into account its didactic variables, up to what point the mathematical knowledge at stake was explicitly pointed out, justified, institutionalized and how the necessary decontextualization of knowledge was worked out, how this type of task was related with other ones in wider mathematical organizations, what opportunities were given to make the students' relationship with this task evolve beyond the necessarily short period of its of-

ficial teaching... All these characteristics of the teaching process can seriously affect the personal relationships the students will develop with this type of task, the resistance to time of their ability of solving it, by using a memorized technique or by reconstructing it. Let us add that from this didactical point of view, a distance of two weeks and several months from initial learning cannot be treated exactly the same; they correspond to different scales in the didactic organization.

Third, if the period of time considered includes some institutional change (change in teacher, change in institution), we can look for other types of answers, relying for instance on the notion of didactical memory, or on the notion of institutional relationship to knowledge, hypothesizing that the introduced change have partially blocked the ordinary functioning of didactical memory, or changed the institutional relationship to this type of task. But we will not develop more this kind of answer as the way the question is phrased does not seem to suppose the possible existence of an institutional change.

**b) How do you transform the raised problem into a research question starting from the question above?**

There is certainly a lot of literature about such issues in cognitive research. From a didactic perspective, what seems more interesting to us is to transform the raised problem into a research question in such a way that the specificity of mathematics knowledge, of mathematical and didactical organizations could be taken into account, and that a systemic view could be developed, the “forgetting student” being no longer the exclusive or central object of our attention.

Several research questions can emerge from the tentative answers proposed above. Moreover it is certainly interesting to distinguish between research questions orientated towards the understanding of the system functioning and of the influence of its characteristics on the observed phenomenon on the one hand, and research questions associated to the elaboration and evaluation of didactical engineering trying to improve the current situation by playing on one or several levers, on the other hand.

We will limit to a few ones.

- Q1: Are different types of mathematical tasks equally sensitive to the “forgetting phenomenon” and what can explain observed differences if any?
- Q2: What are the strategies that mathematics teachers tend to use for limiting or controlling the “forgetting phenomenon”? What is the rationale underlying these and what are their effects?
- Q3: Are there characteristics of the usual mathematical organizations which tend to reinforce the “forgetting phenomenon” and, if so, what are the mechanisms underlying this reinforcement?

Q4: Does an engineering design where specific attention is paid to the balance between the different moments of the study (according to the TAD) and to the completeness of mathematical praxeologies can make a difference?

These remain very general questions that should have to be localized and thus can lead to a great variety of specific research projects.

**c) What is your research design?**

The research design of course depends on the question and on the way this question will be more specifically phrased. For instance, looking for Q1, one could try to create, through enquiries among teachers and students whose extent would be to be defined, a set of potentially contrasted mathematical tasks in that respect, then use another methodology for instance several questionnaires in order to check what tasks are really contrasted, and if so investigate possible explanations for similarities and differences in the nature of the tasks and in their institutional life through the analysis of syllabus and official texts, textbooks, copybooks, teachers' material... Of course, all of this supposes the existence of some regularities... But one could also on the contrary, use the analysis of the characteristics of the institutional life of different types of tasks for conjecturing that they can be more or less affected by this phenomenon and then test these conjectures through adequate questionnaires.

The research design will be different if the question is to evaluate the influence on this phenomenon of a specific didactic strategy, and in this case it could obey for us the standard methodology of didactical engineering.

**d) What type of results would you expect?**

A better understanding of the didactic characteristics of this phenomenon, and of the possibility of action.

**FERDINANDO ARZARELLO & ORNELLA ROBUTTI**

**a) How do you – a priori – answer this question and what are your basic assumptions?**

First a general comment.

In each cognitive performance, particularly in mathematical ones, there are two aspects:

- one is linked to the techniques, which require a continuous training to be performed properly (e.g. how to solve a Riccati differential equation),
- the other is linked to the ideas behind the techniques she is asked to perform, namely the technologies and the theories, in the terminology of the ATD frame (e.g. the basic concepts concerning the differential equations).

Maybe that a person many years after she ended the school remembers something about the theories but has forgotten everything concerning the techniques (it is our case for differential and Riccati equations) and so is not able to solve the problem (if it requires to solve a Riccati equation). Maybe a “feeble” student remembers the technique but not the technology and the theory: so she is not able to solve the problem for different and opposite reasons.

It is a question of level at which the knowledge related to the problem must be known to solve it. It is clear that without a continuous training many abilities linked with techniques and technologies become lower. This may cause lower performances and is a natural fact. Of course this depends on the type of performances asked and on the level of assimilation of the techniques, technologies and theories required by the performance itself.

Hence to tackle the question the teacher must distinguish carefully at which level the performances of a task are situated.

From the point of view of the student, her performances depend on the training she has got in the techniques required to solve the problem and on the level of conceptualisation she got in the ideas and theories related to such techniques. E.g. in acknowledging that a certain technique is suitable to solve that problem, in transferring a technique from a context to another, and so on.

Hence students’ performances depend on the task, on the students and on the didactical story of those students in that classroom.

Moreover students performances are not an abstract concept: the way a teacher (and the student herself) interprets them is intrinsically linked to the methodology of assessment used in the classroom.

There are different levels of performances and implicitly or explicitly a teacher has a taxonomy in her mind. For example, this could be Bloom's taxonomy (or something different). Let us take this for the sake of an example (but what we mean does not depend on which taxonomy we are using, what we are proposing could be based on PISA taxonomies as well).

According to Bloom's taxonomy there are 6 levels of performances. Suppose that we divide the performances in two groups according to their levels:

- (1) knowing, understanding, applying;
- (2) analysing, synthesising, evaluating.

Also this subdivision is arbitrary and is taken here for the sake of simplicity but is meaningful; for example in PISA they use three levels of performances: reproduction, connection, reflection.

A first work should consist in classifying the problems according to the two groups (of course our example is very crude here because we use only two groups), namely according to the level of performances required to solve it. This classification should clarify that teaching and assessing is not only a problem of content but also a problem of levels of performances related to some content.

Hence the teacher could fix her didactical objectives in term of contents to teach and in term of level of performances at which she wants her students perform, for example in problem solving.

A third variable in our discussion could be the methodology of teaching that the teacher is designing for a certain content. For example we could distinguish between:

- A) a traditional approach, based on the sequence: explanation-exercise-repetition-assessment;
- B) a more innovative approach, where the knowledge is constructed by students in suitable learning situations, based on the use of laboratory and ICT.

These two approaches can be analysed according to the different ways of teaching-learning they produce from a cognitive point of view.

For this, two related types of analysis can be developed, based on some recent researches, which point out different modalities of learning and of thinking:

- some researchers distinguish between a perceptuo-motor and a symbolic-reconstructive way (Antinucci, 2001);
- others distinguish between spatio-motoric and analytical thinking (Kita, 2000).

The methodology A is typically based on a symbolic-reconstructive approach, which may produce analytical thinking while the methodology B can be based on a percep-

tuo-motor approach, which may trigger spatio-motoric thinking. For a general discussion on this point, focussed on mathematics learning, see: Nemirovsky et al. (2004) and Arzarello et al. (2005).

Careful observation of teacher's and students' performances can point out the different modalities, according to which teaching and learning happen.

In short, our basic assumptions are based on the analysis of the links among teaching, learning, methodologies and assessment. In fact we have pointed out the following variables:

- different specific mathematical contents  $k_i$  ( $i = 1, 2, \dots$ );
- the level (1 or 2) at which the performances for a specific knowledge  $k_i$  in the task are required;
- the methodology of teaching (A, B) for each  $k_i$ .

**b) How do you transform the raised problem into a research question starting from the question above?**

The research question is:

RQ1. Does students' specific knowledge that we measure as a performance in some task change according to the level of the task and how does (can) it change?

As related questions:

RQ2: Does the knowledge depend on the way the students learn it and how?

RQ3: How can we verify if there is a relationship between the way of learning and the way of thinking?

An hypothesis to validate could be the following:

It exists a correlation between the ways of learning and the ways of thinking, namely:

- i) the perceptuo-motor approach produces more spatio-motoric thinking;
- ii) the perceptuo-motor learning produces long-term effects
- iii) the symbolic-reconstructive one produces short-term effects.

We could investigate the previous question using the data we already have (they concern mainly teaching experiments with methodology B) and designing a teaching experiment as sketched in point c.

**c) What is your research design?**

We prepare two equivalent assessment tests based on structured items, the one to be given immediately after the teaching sessions and the other some months later.

We can suppose to organise the testing and the groups of students to whom the test is given so that we can distinguish among:

- different specific mathematical contents  $k_i$  ( $i = 1, 2, \dots$ );
- the level (1 or 2) at which the performances for a specific knowledge  $k_i$  in the task are required;
- the methodology of teaching (A, B) for each  $k_i$ ;

The data could give us some information on the short- and long-term knowledge and on the ways they correlate with the levels of performances and with the methodology of teaching.

More data should be collected to answer our research hypotheses, namely:

- a) Observation through videos of the two types of teaching in order to point out the perceptuo-motor and the symbolic-reconstructive performances that are required by the students.
- b) Observation of processes in a sample of students while solving problems in order to classify them according to the dichotomy analytical Vs/ spatio-motoric thinking. To get this we should organise some specific problem solving session, where they solve some problem working in group and interacting.

With all these data we could interpret them in order to test our research hypotheses.

**d) What type of results would you expect?**

We expect to find some positive correlation between some of these couples of variables:

- Short-time and level 1 knowledge
- Long-time and level 2 knowledge;
- Short-time knowledge and A methodology of teaching
- Long-time knowledge and B methodology of teaching
- Short-time knowledge and analytical thinking
- Long-time knowledge and spatio-motoric thinking

Of course the research project, as it is stated here is too crude. We should need to elaborate it further before starting the research. In particular we should choose carefully the arguments to teach and to test, in order to avoid the interference of other variables (e.g. epistemological obstacles). An idea could be to start comparing the data got teaching some fresh subject, not usual in the curriculum, e.g. discrete linear dynamic systems, with some standard argument, e.g. second order equations. In any case, the task should graduate carefully the technical abilities, which it requires to be solved.

## References

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**ANGELIKA BIKNER-AHSBAHS**

**a) How do you – a priori – answer this question and what are your basic assumptions?**

A phenomenon like this is often observed by teachers. They think – as a teacher I have thought this as well sometimes – that a topic or an aspect is clear and has been taught so that every student should be able to repeat, know, use, ... it. There are many reasons why this could not be the case:

- (1) Teacher and students have a small basis of understanding: This might depend on the different thinking, working, argumentation styles of the teacher and the student, teacher and students might be of different preference types, might have different cultural backgrounds or languages, different aims, ...
- (2) The students need to know the context the task could be embedded:
  - The student needs to know, why, what for and how the task has to be done.
  - The task is not recognised as known because the story around it is different.
  - The usual tools are not available.
  - The social interaction process appears differently,
- (3) The student needs special cognitive, social, psychic assistance or more challenge:
  - The students might be weak concerning some partial performances like weak figure-ground perception or are dyslexics.
  - They might need an atmosphere without pressure, anxiety, or fear but experienced pressure, anxiety or fear.
  - They might give up their attempts too early because they have low self confidence. That is why they have to be encouraged.
- (4) The problem is posed a little bit differently. The students might need to be supported to become more flexible.
  - The teacher should provide a variety of material which might help or prepare a systematic list of heuristics or specific questions.
  - If the problem is posed more complex then the teacher could reduce complexity. Instead of using variables he/she could use simple numbers.
- (5) Meanwhile the students have worked on other problems. They have just forgotten how to solve the problem.

- The teacher has to prepare assistance to call a proto example to mind. That assumes that the teacher has used paradigmatic examples as prototypes.
  - The teacher could make the necessary knowledge more available by repeating pieces of knowledge before posing the problem.
- (6) The students might be disturbed somehow, are frightened, ill, struggle with personal problems, problems in their family, the social group, etc.
- The teacher could ask, what is wrong or if the student feels well.

I think that there are necessary and unnecessary obstacles of this kind. Unnecessary obstacles of this kind are concerned with the atmosphere in the lesson, and with the memory. They can be changed easily. Obstacles which cannot be avoided depend on the personalities of the teacher and the students, they have to be handled in a suitable way such that the teacher and the students are able to work together. They can depend on the institutional context (for instance just before their holidays), as well. However, there are necessary obstacles which the students have to overcome in order to become more flexible and better problem solvers. Since a teacher works together with different students he/she should arrange the lesson in order to be able to support the students the way they need and provide different similar tasks, provide a list of heuristics, questions, or help according to different categories. Students could be encouraged to choose the kind of help they need. This way they learn how to overcome this kind of necessary obstacles and experience more competence and autonomy.

**b) How do you transform the raised problem into a research question starting from the question above?**

Since this problem is complex, a research study could look at the student and his/her inner world, at the social situations within the lesson and its social interactions and connect both. Since the question posed is oriented towards the development of support I would try to do research in a comparative way. I would try to compare hindering and supporting aspects:

*What kind of conditions in every day maths classes fosters or hinders tackling a similar mathematical problem?*

**c) What is your research design?**

Since the problem is very complex I would prefer carrying out deep analyses with a small group of students in a cyclic way. Every cycle should consist of a data collecting and data analysing step. During the process I would try to narrow the question according for instance to cognitive, mathematical, epistemic, social or other aspects in order to reduce complexity.

### **1. Collecting video data of the lessons**

I would take every day lessons at a starting point because I want to gain knowledge for teachers who have to handle this problem in every day lessons. I would choose a specific topic to teach which is taught according to the curriculum. I would collect video data of all lessons with this topic until the test. I would take the test as another piece of data and find out students who did well and some who did wrong.

### **2. Interviews with students who did surprisingly well or bad and students who were expected to do well or bad, two in each group**

I would ask, what conditions fostered or hindered them to do well solving this problem. The interviews are supposed to be narrative. I would try to build a trusting situation without any harm and inform the students, that I want to find out how students could get suitable help in solving a problem. The analyses of these data would be done in a comparative way and would lead to first hypotheses.

### **3. Analyses of the video data**

With the help of these information, I would analyse the video data, try to prove the hypotheses, and gain new hypotheses.

### **4. Posing a similar problem in the lesson**

According to suitable hypotheses I would develop a similar problem which the teacher poses two weeks later in a lesson. The technical part of the following research design depends on the teacher's kind of teaching (class discourses or group work). I would take video data from the lesson and the students which were interviewed and try to arrange the cameras so that two students of the group could be observed with one camera. This way, I would need at least four cameras which is a lot.

### **5. Stimulated recall.**

I would watch the video together with every student and tell him/her to stop the video when he/she observes something that fostered or hindered him/her solving the problem. During the pauses the interviews would take place. Again, these interviews were video recorded.

### **6. Analyses of the interviews**

Based on the hypotheses these interviews were analysed again.

## 7. Repeating 4, 5, and 6 two and four month later

### d) What type of results would you expect?

Depending on the students I would try to develop ideal types of situations, task aspects, or personal aspects which disturb or foster solving a similar problem. I would take these types as background concepts and reinterpret the stories of the children. The result could be a diagnostic view on every child according to the question what conditions fostered or hindered him/her solving the problem. This could be the basis of investigating the question: How could a teacher be able to handle the problem.

Remarks: I would not build a cyclic design process of finding out the student's problems and implementing and proving a special kind of teacher behaviour because deep analyses takes a long time which practice does not have. Teachers have to react immediately. Therefore we could help the teacher to become more sensitive, offer him/her some suitable possibilities to act in ideal type situations, but whether or not the this kind of action is suitable next time is an open question. The first step is to develop ideal types (see research design above) but one small study is not enough to get an overview. Based on this "theoretical" knowledge we could develop suitable types of teacher behaviour. Again, this work is not done in one study, it is a research programme.

**MARIANNA BOSCH & JOSEP GASCÓN:  
ANTHROPOLOGICAL APPROACH**

**a) How do you – a priori – answer this question and what are your basic assumptions?**

Two questions are raised: (1) “how it is that...” and (2) “what strategies can the teacher use”.

(1) The fact considered here is an aspect of a broader fact that can be described as follows: at school, students are rarely conducted to perform a mathematical activity that goes beyond the resolution of very tightly delimited types of problems, studied in a quite isolated form. They use to work in a narrow ‘mathematical space and time’, where topics come one after the other only weakly connected. Once the study of a topic is finished, all can be forgotten because a completely new activity is starting. We can mention other aspects of the same fact:

- Knowledge built up in the study of previous topics is rarely reinvested in the construction of the new one;
- Students are rarely asked to explore the borders of a type of problems or the limitations of the techniques used to solve them as a way to motivate the passage from one topic to another;
- The identification, description, delimitation, evaluation, connection, etc. of techniques and types of problems is commonly the teacher’s responsibility and rarely “transferred” to the students.
- Problem solving is being assigned by most curriculum reforms as a way to connect different topics and content areas. See for instance the following quotation from the Principles and Standards for School Mathematics of the National Council of Teachers of Mathematics (<http://standards.nctm.org>): “Problems and problem solving play an essential role in students’ learning of mathematical content and in helping students make connections across mathematical content areas”.

We postulate that these facts are different manifestations of a didactic phenomenon that we call “*the dis-articulation of the school mathematics (the taught mathematical knowledge)*”. In other terms, we assume that the kind of mathematical activity the students carry out (for instance, learning to solve a “narrowly defined” type of problems for a short period of time and forgetting it afterwards) is mainly a consequence of the kind of mathematics that exist at school, which are affected by the phenomenon of “dis-articulation”.

(2) As a consequence of our previous postulate, it does not seem that the didactic phenomenon associated with the fact mentioned can be easily modified only by

changing teachers' strategies. The kind of solution we can think of is the implementation of new didactic organisations in a system that has strong traditions and imposes many constraints on the way changes can be carried out – at least if we expect long-term changes, and not only local and temporary modifications. It is thus necessary to study the mechanism and the scope of the phenomenon.

**b) How do you transform the raised problem into a research question starting from the question above?**

According to the phenomenon that we postulate as an “explanation” of the considered fact, we can formulate the two following research problems:

- (a) *Didactic transposition problem*: What are the mechanisms of didactic transposition that can explain the phenomenon of the disarticulation of school mathematics as described above? Why is the current situation as it is? What constraints make things be like this?

We can mention here some generic didactic constraints coming from the necessity for school to show the work done in it. The didactic contract cannot concern the whole mathematical curriculum: the “mathematics to be taught” has to be split up into pieces in order to form a “study programme”.

- (b) *Ecology of didactic praxeologies*: What kind of didactic praxeologies can be introduced at school, and under what conditions, in order to allow the development of more “articulated” mathematical activities, that is, to allow the construction of more “complete” and “connected” mathematical praxeologies?

Some current researches of our team are focusing on these kinds of questions. They are using the notion of “Research and Study Course” (RSC) as a reference didactic praxeology and studying the function of mathematical modelling as a tool to build up more articulated (or connected) mathematical praxeologies [Bosch, García, Gascón, Ruiz Higuera (2006), *Proceedings of PME 30*, Vol. 2, pp. 209-216].

**c) What is your research design?**

We are focusing on problem (b) and on a specific topic or theme.

*Stage 1. Curriculum analysis and design of a “reference epistemological model”*

- Choose a theme or topic in the curriculum; describe the mathematical organisations (MO) that can be put into correspondence with the syllabi instructions and look for “generative questions” that can have some of these mathematical organisations as a possible answer. Describe the way(s) these mathematical organisations can be structured and obtained as the answer of questions that cannot be solved in a previous MO. This leads to the *a priori mathematical design* of a Re-

search and Study Course that may articulate different curricular mathematical organisations, linking them through a dynamic of questions/answers.

- Sometimes this a priori analysis shows that the initial chosen topic was not well delimited or that some curricular constraints were assumed without any questioning. It is thus necessary to come back to the curriculum design and study the transposition phenomena that can explain the particular “map of praxeologies” that is traditionally taught at school.

*Stage 2. Set up and experimentation of the designed “Research and Study Course”*

- Propose a concrete generative question and the necessary didactic resources to make the RSC “viable” at a chosen level and under particular school conditions.
- Experiment the RSC in real classrooms.
- Observe the study process (data collection), with special attention to the way the different *moments of the study process* are managed, the share of responsibilities between teacher and students, etc.

*Stage 3. Analysis of collected data*

- It depends on the kind of data obtained, the initial didactic problem and the available didactic knowledge concerning the problem or the topic considered.

**d) What type of results would you expect?**

- *Ecology of mathematical praxeologies*: new ways of curriculum organisation around powerful generative questions that can give a *raison d’être* to the mathematical praxeologies to be taught.
- *Ecology of didactic praxeologies*: characterisation of possible didactic devices and strategies to manage the different moments and dynamics of the RSC; description of the didactic constraints (coming from different levels of determination) that hinder the experimented study process.

**TOMMY DREYFUS AND IVY KIDRON**

**a) How do you – a priori – answer this question and what are your basic assumptions?**

Ken Ruthven has answered this part so well and comprehensively that we only add a comment:

At least with respect to high school algebra, and in the Israeli curriculum, with which we are familiar, 'consolidation' exercises, 'productive practice', and 'non-standard problems' are rarely used, and 'revision' exercises, though used, appear to have little effect. We surmise that the situation is similar with respect to other content domains, but we specifically relate to algebra because we have research evidence for the fact that the same students who very successfully factored expressions and solved equations in grade 9, cannot do the same exercises any more a year later, even if the first three differences in Ruthven's "Previously" versus "Currently" table are avoided [Hoch and Dreyfus (2006): PME 30, Vol. 3, pp. 305-312].

**b) How do you transform the raised problem into a research question starting from the question above?**

Here as well, Ruthven's remarks are to the point, as far as teaching approaches are concerned: curriculum design, textbooks and teacher action such as coherently organizing the new material, emphasizing key elements, activating students, etc.

Our research would rather start from the perspective of the student. What we want to know is how things are learned, not only how they are taught. What we want to know is whether students' knowledge, their recognition of previously encountered ideas, concepts, processes and strategies, their connections between knowledge elements, explanatory power, and flexibility are excellent, adequate, wrong or lacking. We want to investigate how students reach a state in which, say, their flexibility with respect to a particular cluster of mathematical concepts or processes are excellent or lacking; or what are the learning processes by means of which a student (or a group of students) arrive at excellent (or at only partially correct) connections between knowledge elements; what are the learning processes by means of which a student (or a group of students) acquire (or fail to acquire) explanatory power with respect to a cluster of mathematical concepts or processes.

We included the term 'with respect to a cluster of mathematical concepts or processes' because we surmise that the answers to the above questions are likely to be different for different content domains.

More specifically with respect to the problem raised, we would want to investigate not only the learning processes concerning the relevant cluster of mathematical contents and processes, but also the students abilities to deal with problems requiring this cluster at various points in time after the learning experience.

When investigating such learning processes, we consider the context in which the learning process takes place to be of great importance, and therefore it must be observed and form part of the data. Context is considered in a comprehensive sense, including students' and classes' learning history, social context of learning (classroom, groups, individuals), the physical context of learning (including the availability of manipulatives and/or computer software and the manner in which these are used, etc.

As a side remark, yes, obviously, such a program of research requires instruction, and instruction needs to be designed. However, in the short term, our choice is not to focus on instructional design as a topic to be researched but to use or adapt an existing design, the choice being based on intuition and past experience of team members (the team around Rina Hershkowitz at the Weizmann Institute has well over 30 years of experience). In the long run, we would hope to also derive design principles for constructing and consolidating, derived from experience with RBC analyses.

The above are general aims of our research program; more specific research questions will be formulated in the next part, within the framework of the research design.

### **c) What is your research design?**

#### **Stage 1: Content analysis and instructional decision**

In order to be able to ask questions that make sense, we start from an analysis of the contents (cluster of mathematical concepts and processes) under consideration. We need to analyze the contents in terms of the goals to be achieved: What concepts and strategies do we want the student to have acquired and be able to use, and in which circumstances (contexts)?

As a first step in our research design, we would therefore produce an analysis of the contents to be learned into principles that can form a basis for analysis of the data we will have. In other words, these principles should be operational. Our focus would be on students' constructing, and later building-with (using, or failing to use) these principles in a given context.

We would then choose (and possibly modify) a teaching design that has the potential for constructing these principles. This would be based on long-time experience of designers and teachers but not usually a new design which we want to subject to experimentation (i.e. not a design experiment). In other words, the design is being chosen ad hoc and then possibly refined by successive approximation.

#### **Stage 2:**

Next we identify the elements of the context in which we are interested in observing the learning process, including the students' prior learning experience, the social situation in which we want to make the observations (classroom or laboratory; often we would first collect data in a laboratory situation and then "scale up" to regular

classrooms), the kind of teacher (very experienced or "regular", used to let students work in groups or not, etc.).

Given the above cluster of mathematical concepts and processes, design of instruction, and context, some of our research questions are:

- What are the processes of constructing the knowledge under consideration, and what are students' emerging knowledge constructs? In what are these processes of knowledge construction for a given construct different for the learning processes of students who are successful with this specific construct after a year and those who are not? In what are these processes of knowledge construction of the same student different for constructs with which the student is successful after a year and those constructs with which she/he is not?
- How do contextual factors including prior knowledge and experience, available technology, social interaction, teacher guidance, etc influence the constructing process? In what way, if at all, are these contextual influences different for the learning processes of students who are successful after a year and those who are not.
- Is the knowledge under consideration being consolidated during problem solving and reflecting activities, and possibly during further processes of constructing, and how is it consolidated in cases where the student is / is not successful with the specific construct after a year.
- Are there some constructs which have and others which have not been constructed / consolidated in cases in which a student is not successful after a year?

Some of the hypotheses underlying these research questions are that in cases of lack of success after a year

- constructs may have been constructed but not consolidated,
- constructing may have led to partial knowledge constructs,
- specifically, some "deeper" connecting principles may not have not been constructed or consolidated
- contextual factors such as a student's personal history, peers with whom he or she collaborated, or computerized tools may have had beneficial or detrimental influence on the constructing or consolidating processes.

The kind of data to be collected and the period of data collection have to be determined. In the case at hand, namely students "forgetting" what they have learned in earlier months or even years, this will evidently require long term observation. So far, our experience in the framework of the RBC paradigm is with intensive observation over periods of up to two months. In the present research design, a longer time period is required, and therefore, for practical reasons, we would opt for selective observa-

tion: Intensive observation during the period of initial learning; selected observation during class periods when the planned activities require or give the option to use the relevant concepts and processes.

Additional data would be constituted by all examination questions during the period, in which the relevant contents are similarly required or optional.

Population: If possible, we would choose two populations, one from two or three schools, in which the teachers report that typically the contents under consideration are "forgotten", and a parallel one from a school in which teachers report no such problem of forgetting.

### **Stage 3: Data collection.**

During the learning phase, we would try to have very detailed data. Typically, if we work in a classroom, we might have two video cameras, one focused on a group of students (always the same group) and the other on the teacher (or a student who speaks to the entire class). In addition, a researcher would take classroom notes. Throughout the period following the initial learning phase, and for the time span of interest, possibly about one year, the lessons specified above would be similarly observed. Examinations would be collected.

#### **d) What type of results would you expect?**

We hope to be able to characterize differences between the learning processes of students who are successful after a year and others who are not. For example, we might observe that only some students have reached constructing processes with respect to some of the deeper connecting principles, whereas others have been able to carry out all required task while only building-with the component constructs. And we might see that the same students who have constructed these "deeper principle" are therefore able to make use of opportunities for consolidation which the curriculum offers, whereas other students are not, and that this consolidation leads to the effect that they "remember" a year later.

We may also need to look at re-constructing a principle, i.e. a student going through a second process of constructing the same principle she or he had constructed at an earlier period, possibly a year ago, without having had the opportunity to consolidate this principle, or without having used the opportunities for consolidation that were offered. Thus like most experiments, this one might lead to a welcome modification/expansion of the theory.

## HELGA JUNGWIRTH: AN INTERACTIONIST PERSPECTIVE

### **a) How do you – a priori – answer this question and what are your basic assumptions?**

Why should I have an answer at hand? There are so many explanations, and probably more than one will hold in the respective case. They may focus on students, the teacher, their interaction, on contextual events within the classroom, within the school ...

### **b) How do you transform the raised problem into a research question starting from the question above?**

But I prefer a certain one, anyway. It is due to my interactionist stance towards the world. (Much of my research I have done on this basis). Don't ask me why I favour it. I had an affinity to this stance, from the beginning. It is a viable belief of mine, that is, I have a good rationale for it. In particular, it has proved relevant in initiating steps towards a "better" teaching practice.

My preference in the given case, however, is underpinned by a hint in its description: students did well "in the class"; which I interpret that I cannot assume that they did well in a test, an exam as well. They performed well in the ongoing process. So their performance can be localized there. If it is sensible to understand the process as an interaction being established by the teacher and the students, which is the case presumably not from my point of view only, interactionism will be on the agenda.

### **c) What is your research design?**

My research design is quite simple; my design activities, however, are not confined to research only. I make videos and transcriptions of some lessons of the teacher and analyse them with respect to patterns in the interaction that have been reconstructed by interactionist research before.

### **d) What type of results would you expect?**

In short, the argument is: Interaction –everyday, smooth-running interaction– is established by the teacher's and students' adjusting to the acting of each other. So students can successfully participate without an understanding to be located in their "heads"; for instance, by answering on questions by short, tentative utterances which seem to indicate understanding so that the teacher completes to the desired answer (just "recalling" what the students already "know"). As their competence is a phenomenon of the interaction (as I have called that once), an event existing between people, not in people, it is not surprising that some students cannot repeat neither former solutions nor the solution game later without any break-downs.

I design a teacher education or individual coaching of the teacher where the videos and transcripts are in the centre (this is not hypothetical!) The aim is to make her/him realize the pattern and its routines in order to change her/his part (because if one side does no longer act in the common way the other cannot keep to his; emergence of events in interaction put aside). We develop alternatives for utterances which do not allow students to perform well at the surface. The strategies for the teacher evolve from the concrete, detailed video or transcript reflection together with the teacher.

**KENNETH RUTHVEN: PRACTICAL THEORISING APPROACH****a) How do you – a priori – answer this question and what are your basic assumptions?**

A preliminary caution is that we need to make sure that the phenomenon has been well described. For example, there may have been important differences between the two occasions referred to:

<b>Previously</b>	<b>Currently</b>
The setting is one in which students can take for granted that they are required to solve this particular type of problem.	The setting is one in which students must now identify for themselves that they are required to solve this particular type of problem
The problem type is presented to students in particular forms with which they have become familiar.	The problem type is now presented to students in a variant form with which they are unfamiliar.
Forms of assistance are to hand for students, such as worked examples on the board or in their text or exercise book.	Earlier forms of assistance are no longer to hand for students.
A carefully structured teaching and learning process has led up to students' successful performance.	Students are now expected to achieve successful performance without preliminary 'reteaching' or 'relearning'.

Such differences indicate that learning is far from complete when students achieve assisted performance in a tightly framed setting; further learning –some of it quite different in character– is required for independent performance in a loosely framed setting. Indeed, some design features of existing curriculum materials represent attempts to support such further learning:

- having students tackle 'consolidation' exercises as part of the unit of work on a topic, intended not only to provide practice in solving the problem type in its standard forms, but to give experience of a wide range of variant forms.
- giving students 'productive practice' through the recurrent subsidiary tasks in 'substantial learning environments'.
- having students tackle non-standard problems, which relate in unpredictable ways to more standard problem types, requiring students to learn to match and adapt familiar techniques to unfamiliar situations.

- having students tackle mixed ‘revision’ exercises covering work on a range of topics, providing experience in recognising and solving particular problem types without the various forms of cueing and assistance available during a unit of work itself; and providing opportunities for ‘rehearsal’ and ‘relearning’ of the topic.

However, if no differences of the types noted above are operative, then a simple answer is that this is a phenomenon already recognised by everyday commonsense and psychological science. Both give credence to the idea that retention of learned material tends, other things being equal, to decline over time; in particular, in the absence of further use (or rehearsal, or indeed relearning). At the same time, such sources also suggest that subsequent retention of newly learned material is likely to be more successful if attention is given to:

- avoiding learning being inhibited by students’ lack of fluency with necessary prior material;
- organising new material into what students can appreciate as a coherent and connected system;
- identifying and addressing areas of student uncertainty and confusion about the material;
- engaging students intensively with the material, and in actively thinking about it;
- and most specifically:
  - identifying and emphasising key elements of the material, and giving explicit attention to means of remembering and/or reconstructing them.

Finally, it should be added that the phenomenon under discussion also exercises researchers seeking to evaluate the effects of teaching interventions on student learning. Indeed, in well-designed studies of this type, it is now the norm to administer both immediate and delayed post-tests. These are intended to establish to what degree material is retained beyond the end of the instructional intervention (with some decline normally expected from immediate to delayed performance); however, such measures sometimes provide evidence of further learning (with delayed performance actually superior to immediate performance). The attribution of such improvements to ‘incubation’ seeks to explain them in terms of an extension or stabilisation of cognitive (re)organisation precipitated by the teaching intervention beyond the period of the intervention itself. (It should be noted, however, that the immediate post-test can itself be seen as constituting a further instructional intervention.)

**b) How do you transform the raised problem into a research question starting from the question above?**

A 'practical theorising' approach seeks to link theorisation of the problem to the design of practical means of addressing it. For example, the measures suggested above as means of improving retention of material –both in isolation and combination– provide a starting point for theorised design of teaching approaches and their subsequent analysis and evaluation.

An important issue is the degree to which generic approaches can be effective, and to which more topic/setting-specific designs are required.

**c) What is your research design?**

Analysis and evaluation of theorised designs in action involves forms of experimentation –including classical experiments, design experiments, and action research cycles. The latter two emphasise this as a cyclical process in which a theorised design is repeatedly refined.

**d) What type of results would you expect?**

Refinement of original theorised measures, and generation of new ones.

**HARALAMBOS SAKONIDIS, MARIA KALDRIMIDOU,  
& MARIANNA TZEKAKI**

In discussing and/or answering the four questions (a, b, c, d) in relation to this particular ‘teaching problem’, several issues need to be taken under consideration:

1. Which is the particular type of the mathematical problem? Do we face the same ‘teaching problem’ in all problems of the same type (same content? same strategies? same category? same epistemological features - involving concepts, definitions, properties, procedures, figures, symbols, several modes of representations, several “registers”? same difficulty level -easy, moderate, difficult, complex, usual or unusual?)?
2. Do we have the same teaching problem across the various types of mathematical problems?
3. Who are these “some students”(are they always the same students or do they vary according to the problem at hand)?
4. Which is the ‘usual’ classroom mathematical culture with reference to which we make sense of this particular ‘teaching problem’? (practices of teaching, ways of working with the students, of learning, of argumentation...)
5. Which is the means we exploit in order to assess students’ performance? How these means affect and are affected by our teachers beliefs and conceptions about mathematics, its learning and teaching?

With the above concerns, here is our attempt to express some thoughts along the questions a, b, c, and d.

**a) How do you – a priori – answer this question and what are your basic assumptions?**

It is impossible to answer a priori this question without any reference to the above mentioned issues. Or, to say it better, the a priori answer will vary, depending on the perspective adopted for points 1, 2, 3, 4 and 5.

As for the basic assumptions, they will also vary. For example, it will be possible to claim that this particular ‘teaching problem’ is due to difficulties of cognitive nature (stereotypes that persist, or difficulties to treat information of a given way of representing data); of conceptual understanding type; of meta-mathematical nature (it might be that the students thought that something was not very important, so they didn’t learn to ‘tackle the problem’); of didactical nature; of social – cultural nature...

Depending on the available evidence, we could consider the possibility of dealing with the specific ‘teaching problem’ (that is, to decide whether the teacher would

have to utilize some specific teaching strategies for this particular type of problems or s/he needs a more general re-consideration of his/her teaching practice).

**b) How do you transform the raised problem into a research question starting from the question above?**

The transformation of the raised problem into a research question will also depend on the perspective taken with respect to issues 1, 2, 3, 4 and 5. That is, we need a hypothesis on why this happens, in order to be able to efficiently investigate what we can do and, especially, to decide whether we need to focus on students (cognition and ways of learning) or on the didactical approach (content knowledge and teaching practices).

**c) What is your research design?**

Concerning the research design, we will first have to clarify the extent to which this ‘teaching problem’ occurs. Also, we will need to check the relevance of the assumptions and the hypothesis made, by examining what happens in similar mathematical problems of another part of the curriculum and also in problems with the same mathematical content, but with different linguistic and semantic features. Then, we will be able to work with the teacher of the class to design well thought classroom interventions, which will deal with the features determining the ‘teaching problem’ (or the ones we have identify as such).

**d) What type of results would you expect?**

Based on the above framework, the results should provide some insights into the complicated ways in which the classroom learning - teaching environment acts, shaping the mathematical knowledge negotiated and, thus, individual students’ learning trajectories.

## TOWARDS A CULTURAL THEORY OF LEARNING\*

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*In this paper I sketch a theory of teaching and learning that takes its inspiration from some anthropological and historico-cultural schools of knowledge –the Theory of Knowledge Objectification (TKO). The TKO rests on five main interrelated constructs. The first construct deals with the psychological concept of thinking. Drawing on this concept, the other constructs serve to formulate the problem of learning in a way that does not commit the TKO with rationalist views of cognition and social interaction. The TKO posits the problem of learning as the progressive acquisition of cultural forms of reflection that are objectified as the student engages in joint social activity. Learning, it is argued, arises in the course of sensuous mediated cultural praxes embedded in historically formed epistemes and ontologies.*

### INTRODUCTION: THEORIES OF TEACHING AND LEARNING

Theories of teaching and learning differ from each other mainly in their conceptions about the learner, the content to be learned, and how learning actually occurs. Most contemporary theories adopt the view according to which the student constructs his or her own knowledge. Although in their account of learning these theories do not exclude the role of the social, often they reduce the social to a kind of external environment to which the cognitive activity of the student has to adapt. Much in vein with Piaget's genetic epistemology, these adaptations are seen as universal regulators with no ties with the individual's sociocultural context (see e.g. Piaget & Garcia, 1989, p. 267). In these theories, the idea of the universal mechanisms of knowledge formation –namely the allegedly logical-mathematical structures of thinking– appear as the warrants of the supposedly universal patterns of conceptual development.

However, at the epistemological level, these theories have been criticized, in part for their commitment to a rationalist view of knowing and cognition (Buck-Morss, 1975; Campbell, 2002; Walkerdine, 1988; Wartofsky, 1983). These theories rest indeed on the idea of an intrinsic rational auto-sustained individual maturing as she interprets and refines the feedback that the environment sends to her. As a result, the idea of the learner that these theories convey is the idea of a self-regulated individual acting in a more and more autonomous form, an idea shaped by the Western concept of the scientist.

At the ontological level, other scholars, working within the framework of Realism, find the idea of universal adaptations insufficient to ensure the convergence between

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\* This article is a result of a research program funded by The Social Sciences and Humanities Research Council of Canada / Le Conseil de recherches en sciences humaines du Canada (SSHRC/CRSH).

the individuals' personal conceptual constructions and a reality that precedes all cognitive activity (see e.g. Thom in Piattelli-Palmarini, 1982).

Without denying the existence of a real world, in an interesting move plainly in accordance with Kant's view of human reason, von Glasersfeld (1995) suggested to give up the idea that the individual's conceptual constructions correspond to the objects of the real world. He put forward a much more modest idea of cognition –one in which our ideas are merely *viable constructs*. According with this theory, the so-called *Radical Constructivism* (RC), these subjective constructs are ready to be changed if compelling evidence suggests so. For many, this move is unconvincing. On the one hand, RC cannot avoid the problem of solipsism (Lerman, 1996). On the other hand, to salvage its underlying extremist subjective epistemology, RC gives up ontology and posits the subjective experiential realm as the limits of reason and knowledge.

At the educational level, Radical Constructivism has also been criticized for failing to account for the dissymmetric distribution of knowledge in the classroom. In a recent plenary lecture, Brousseau (2004) argued that “En didactique, le constructivisme radical est une absurdité”. What Brousseau finds absurd in the radical constructivist position is not the claim that legitimate knowledge can only be the result of the individual's own achievement and deeds. On this point, Brousseau, who elaborated his *Theory of Situations* as a response to the general framework of an uncritical learning (learning without meaning), endorses some central tenets of Piaget's constructivism. What he finds erroneous is the idea that the students' constructions necessarily lead to the standard mathematical knowledge (*le savoir savant*). As Brousseau could observe again and again in the classrooms of the Michelet School, the students' subjective conceptual constructs require of an external perspective to, among other things, institutionalize the knowledge arising from classroom mathematical activity. The students cannot be aware of the cultural epistemic status of, say, a method arising as the result of their enquiring activity or, as Brousseau puts the matter, the students may not know that they know. The teacher hence has to highlight those reasonings and methods valued by the mathematicians' community.

These few comments on some current ideas about the learner and how learning occurs provide an idea of some of the theoretical differences in current perspectives in mathematics education. Of course, the differences between theories are subtler as hinted here. My interest is not to delve into these differences. Rather my interest is to recall some of the presuppositions that appear as the focal points from where theoretical differences arise. In the rest of this paper I present some elements of a theory of teaching and learning that takes its inspiration from some anthropological and historico-cultural schools of knowledge. This theory –*The Theory of Knowledge Objectification*– relies on a non-rationalist epistemology and ontology which give rise, on the one hand, to an anthropological conception of thinking, and on the other, to an essentially social conception of learning.

## 1. A NON-MENTALIST CONCEPTION OF THINKING

### 1.1 Thinking as a mediated *praxis cogitans*

Typically, thinking is understood as a kind of interior life, a series of mental processes on ideas carried out by the individual. This conception of thinking, as “mental activity” (de Vega, 1986, p. 439), comes from Saint Augustine’s interpretation of Greek philosophy at the end of the fourth century, an interpretation that brought about, in particular, a transformation in the original meaning of the Greek term *eidōs*. While Homer, among others, used the term *eidōs* in the sense of something external rather than mental—“that which one sees,” for example, the figure, form or appearance—for Saint Augustine, *eidōs* refers to something situated *inside of the individual*. Influenced by this transformation, seventeenth century rationalists such as Descartes and Leibniz believed that mathematics could be practiced even with one’s eyes closed, given that the mind does not need the help of the senses or of experience to reach mathematical truths. As Leibniz put the matter, the principles that we need to understand objects or see their properties, the internal rules of reason, are “interior principles” that is, they are within our interior (Leibniz, 1966, pp. 34-37). Anthropologists such as Geertz have demonstrated the limitations of the conceptualization of ideas as “things in the mind” or of thinking as an exclusively intracerebral process. Geertz (1973, p. 76) claims that “The accepted view that mental functioning is essentially an intracerebral process, which can only be secondarily assisted or amplified by the various artificial devices which that process has enabled man to invent, appears to be quite wrong.” He argues that “the human brain is thoroughly dependent upon cultural resources for its very operation; and those resources are, consequently, not adjuncts to, but constituents of, mental activity. (Geertz, *ibid.*).

The conception of thinking as a kind of interior life has had a great influence in the investigation of cognition in mathematics education. Written questionnaires, interviews, and drawing exercises have often been used to get a glimpse of what is going *in* the head. To avoid the pitfalls of this mentalistic approach, some theories have simply discarded any psychological considerations. They have made “l’économie du sujet.”

The Theory of Knowledge Objectification (TKO) takes off from a non-mentalist position on thinking and intellectual activity. This theory suggests that thinking is a *praxis cogitans*, that is, a social practice (Wartofsky, 1979). To be more precise, thinking is considered to be *a mediated reflection on the world in accordance with the form or mode of the activity of individuals*.

The mediating nature of thinking refers to the role, in the Vygotskian sense, played by artefacts (objects, instruments, sign systems, etc.) in carrying out social practice. Artefacts are not merely aids to thinking (as cognitive psychology would have it) nor simple amplifiers, but rather constitutive and consubstantial parts of thinking. We think with and through cultural artefacts, so that there is an external region which, to

paraphrase Voloshinov (1973), we will call the *zone of the artefact*. It is within this zone that cultural subjectivity and objectivity mutually overlap and where thinking finds its space to act and the mind extends itself beyond the skin (Wertsch, 1991).

The reflexive nature of thinking means that the individual's thinking is neither the simple assimilation of an external reality (as the Empiricists and Behaviorists propose) nor an *ex nihilo* construction (as certain constructivist schools would have it). Thinking is a *re-flection*, that is, a dialectical movement between a historically and culturally constituted reality and an individual who refracts it (as well as modifies it) according to his/her own subjective interpretations, actions and feelings.

One of the roles of culture is to suggest to students ways of perceiving reality and its phenomena, literally, ways of setting one's sights (*manières de viser*), as Merleau-Ponty (1945) would say, or ways of intuiting, as Husserl (1931) might have it. In a more general fashion, the *re-flexivity* of thinking, from the phylogenetic point of view, consists in individuals giving rise to thinking and to the objects that thinking creates. However, at the same time, from the ontogenetic point of view, the individuals' thinking is, from the outset, subsumed by their cultural reality and by the historically formed concepts that they encounter in their environment. This is why, we originate thinking, but at the same time become subsumed by it. (Eagleton, 1997, p. 12)

## 1.2 The anthropological dimension of thinking

In the preceding section, it was said that thinking should be considered as a mediated *re-flection* of the world, in keeping with the form or mode of the activity of individuals. What this means is that the way in which we come to think about and know objects of knowledge is framed by cultural meanings situated beyond the very content of the activities in whose interior the act of thinking itself occurs. These cultural meanings act as mediating links between individual consciousness and objective cultural reality and they make themselves into prerequisites and conditions for individual mental activity (Ilyenkov, 1977, p. 95). These cultural meanings suggest courses of action to our cognitive activity and give it a certain *form*. It is for this reason that thinking is not something that we simply begin to do in a more or less unpredictable way and during which we suddenly come across a good idea. Even though it is true that practical sensual activity, mediated by artefacts, enters into the thinking process, in its very content, the way in which this occurs is subject to the cultural meanings in which the activity is being maintained. Here is an example. The difference between the thinking of a Babylonian scribe and that of a Greek geometer cannot be reduced only to the kinds of problems with which they were respectively occupied, or to the artefacts they used to think mathematically, or the fact that the former was reflecting in a context tied to political and economic administration, whereas the latter was thinking within an aristocratic and philosophical context. The difference between the thinking of the Babylonian mathematician and that of the Greek one has to do with the fact that each one of these forms of thinking is

underpinned by a particular *symbolic superstructure*. This symbolic superstructure, which elsewhere we have called a *Semiotic System of Cultural Signification* (Radford 2003a), includes cultural conceptions surrounding mathematical objects (their nature, their way of existing, their relation to the concrete world, etc.) and social patterns of meaning production. The thinking of the Babylonian scribe is framed by a realist pragmatism where mathematical objects such as “rectangle,” “square,” and so forth – objects which the Greek geometer of Euclid’s time conceptualized in terms of Platonic forms or Aristotelian abstractions – acquire their meaning.

In their interaction with activities (their objects, actions, division of labour, etc.) and with the technology of semiotic mediation (the zone of the artefact), the *Semiotic Systems of Cultural Signification* give rise, on the one hand, to forms or modes of activities, and, on the other hand, to specific modes of knowing or *epistemes* (Foucault, 1966). While the first interaction gives rise to the particular ways in which activities are carried out at a certain historical moment, the second interaction gives rise to specific modes of knowing which allow for the identification of “interesting” situations or problems and which demarcate the methods, reasoning, evidence, etc. that will be considered culturally valid<sup>1</sup>.

From our perspective, cultural diversity in the form of human activity explains the diversity of forms that mathematical activity takes on, something which is demonstrated to us by history. Rather than seeing these historical forms as “primitive” or “imperfect” versions of a kind of thinking that is marching towards a perfected form as represented by current mathematical thought (ethnocentrism), the anthropological dimension of the theory of objectification considers these forms as belonging to human activity and thus resists privileging western rationalism as rationalism *par excellence*.

The manner in which the Babylonian scribe, the Greek geometer and the Renaissance abacist end up thinking about and knowing objects of knowledge, the way in which they approach their problems and consider them to be solved, all are framed by the very mode of the activity and the corresponding cultural episteme (Radford, 1997, 2003a, 2003b).

## **2. THE EPISTEMOLOGICAL AND ONTOLOGICAL BASES OF THE THEORY OF KNOWLEDGE OBJECTIFICATION**

Any didactic theory, at one moment or another (unless it voluntarily wants to confine itself to a kind of naïve position), must clarify its ontological and epistemological position. The *ontological* position consists in specifying the sense in which the theory approaches the question of the nature of conceptual objects (in our case, the nature of mathematical objects, their forms of existence, etc.). The *epistemological* position consists in specifying the way in which, according to the theory, these objects can (or cannot) end up being known.

Contemporary didactic theories that start from an application of mathematics, gradually adopt—even if it is not mentioned explicitly— a realist ontology and approach the epistemological problem in terms of abstractions. Naturally, the situation is not that simple, as Kant himself recognized. As for Realism –which, in an important way, is the Platonist version of the instrumental rationalism (Weber, 1992) which emerged during the Renaissance– the existence of mathematical objects precedes and is independent from the activity of individuals. Like the Platonist, the Realist believes that mathematical objects exist independently of time and culture. The difference is that, while Platonic objects do not mix with the world of mortals, the objects of the Realist govern our world. According to realist ontology, this explains the miracle that is the applicability of mathematics to our phenomenal world (Colyvan, 2001). Naturally, in order to achieve this, Realism makes a leap of faith that consists in believing that the ascent from abstraction to objects is certainly possible. The faith which Plato placed in reasoned social discourse (*logos*) and which Descartes placed in cogitating with oneself are subjected to scientific experimentation by Realism.

The ontological and epistemological position of the theory of objectification moves away from Platonist and realist ontologies and from the Platonists' and Realists' conception of mathematical objects as eternal objects preceding the activity of individuals. By distancing itself from an idealist ontology, the theory also distances itself from the idea that objects are the product of a mind that works folded in onto itself or according to the laws of logic (the Rationalist Ontology). The theory of objectification suggests that mathematical objects are historically generated during the course of the mathematical activity of individuals. More precisely, mathematical objects *are fixed patterns of reflexive activity (in the explicit sense mentioned previously) incrustated in the ever-changing world of social practice mediated by artefacts.*

The conceptual object “circle”, for example, is a fixed pattern of activity whose origins cannot be found in the intellectual contemplation of the round objects which the first individuals would have encountered in their surroundings, but rather must be found in the sensual activity that led said individuals to notice the emergent object:

People could see the sun as round only because they rounded clay with their hands. With their hands they shaped stone, sharpened its borders, gave it facets. (Mikhailov, 1980, p. 199)

This sensual experience of labour has remained fixed in language which encapsulates original meanings, such that

the meaning of the words “border”, “facet”, “line” does not come from abstracting the general external features of things in the process of contemplation (Mikhailov, *ibid.*)

but rather comes from the activity of labour that has been taking place since the origins of humanity. Far from surrendering itself completely to our senses, our

relationship with nature and the world is filtered through conceptual categories and cultural significations which make it so that

man could contemplate nature only through the prism of all the social work-skills that had been accumulated by his predecessors. (Mikhailov, *ibid.*)

### **3. LEARNING AS THE CULTURAL OBJECTIFICATION OF KNOWLEDGE**

In the previous sections we have seen how human activity, from the phylogenetic point of view, can generate conceptual objects, which in turn are transformed as a result of the activities themselves. From the ontogenetic point of view, the central problem is to explain how acquisition of the knowledge deposited in a culture can be achieved: this is a fundamental problem of mathematics education in particular and of learning in general.

As mentioned in the Introduction, classical theories of mathematical education posit the problem in terms of a construction or re-construction of knowledge on the part of the student. The idea of the construction of knowledge originates with the epistemology elaborated by Kant in the eighteenth century. For Kant, the individual is not only an introspective thinker whose mental activity, if it is well carried out, will bring him mathematical truths as upheld by the rationalists (Descartes, Leibniz, etc.); nor is he only a passive individual who receives sensory information in order to formulate ideas, as proposed by the Empiricists (Hume, Locke, etc.). For Kant, the thinker is a being in action: the individual is craftsman of his/her own thinking (Radford, 2006a). Through these ideas Kant expressed, in a coherent and explicit way, the epistemological change that had been gradually taking place since the appearance of manufacturing and the emergence of capitalism in the Renaissance and that Arendt (1958) summarizes in the following way: the modern era is marked by a displacement in the conception of the meaning of knowledge; the central problem of knowledge lies in a movement that goes from ‘the what’ (the object of knowledge) to ‘the how’ (the process), in such a way that, unlike medieval man, modern man can only understand that which he himself has made.

According to the theory of objectification, learning does not consist in constructing or reconstructing a piece of knowledge. It is a matter of endowing the conceptual objects that the student finds in his/her culture with meaning. The acquisition of knowledge is a process of active elaboration of meanings. It is what we will later call a process of objectification. For the moment, we need to discuss two important sources for the elaboration of meanings that underlie the acquisition of knowledge.

#### **3.1 The knowledge deposited in artefacts**

One of the sources of the acquisition of knowledge results from our contact with the material world, the world of cultural artefacts which surrounds us (objects, instruments, etc.) and in which is found the historically deposited knowledge from the cognitive activity of passed generations. Although it is true that some animals are

able to use artefacts, nevertheless, for animals, artefacts do not end up acquiring a durable meaning. The wooden stick that a chimpanzee uses in order to reach a piece of fruit loses its meaning after the action has been executed (Köhler, 1951). It is for this reason that animals do not preserve artefacts. Furthermore—and this is a fundamental element of human cognition—unlike animals, the human being is profoundly altered by the artefact: by making contact with it the human being restructures his/her movements (Baudrillard, 1968) and new motor and intellectual skills are formed such as anticipation, memory and perception (Vygotsky and Luria, 1994).

The world of artefacts appears, then, to be an important source for the process of learning, but it is not the only one. Objects cannot make clear the historical intelligence that is imbedded in them. This requires that they be used in activities as well as in contact with other people who know how to “read” this intelligence and help us to acquire it. Symbolic-algebraic language would otherwise be reduced to a group of hieroglyphics. The intelligence that said language carries would not be noticed without the social activity that takes place in the school. It is this social dimension which constitutes, for the theory of objectification, the second essential source for learning.

### **3. 2 Social Interaction**

Even though the importance of the social dimension has been underlined by a great number of recent studies on classroom interaction, there are subtle differences with regards to its cognitive contribution (Yackel and Cobb, 1996; Sierpinska, 1996; Steinbring, Bartolini Bussi and Sierpinska, 1998). Often, interaction is considered as a negotiation of meanings or as a simple environment that offers the stimuli of adaptation that are required for students’ cognitive development. The problem is that the classroom is not a merely material space where the students find an environment to adapt themselves; it is not only a matter of “external” conditions to which the subject must accommodate his/her activity. The crucial point is that the classroom is a symbolic space; it is a space where conceptual objects, activities and the material means that mediate them are endowed with scientific, aesthetic, ethical values, etc. that end up affecting the actions that individuals carry out and the reflections that these actions necessitate. As was mentioned in the first part of this article, the actions that individuals carry out are submerged in cultural modes of activity. It is for this reason that the classroom cannot be viewed as an enclosed space, folded over against itself, where knowledge rules are negotiated. In fact, these rules have a whole cultural history behind them and therefore pre-exist the interaction that takes place in the classroom.

According to the perspective that we are suggesting, interaction plays a different role. Rather than performing a merely adaptive function—a catalyzing or facilitating one—according to the theoretical perspective that we are sketching, interaction is consubstantial to learning.

Therefore, we see that there are elements that play a basic role in the acquisition of knowledge and that these are the material world and the social dimension. The allocation of meaning that rests on these dimensions has a profound psychological importance inasmuch as it is both an awareness of cultural concepts as well as the process of development of the specific capacities of the individual. It is for this reason that, according to our perspective, learning is not merely appropriating something or assimilating something; rather, it is the very process by which our human capacities are formed.

### 3.3 Learning activity

A central element of the concept of activity is its objective (Leont'ev, 1978). Even though the objective may be clear for the teacher, generally speaking, this is not necessarily the case for the students. If the objective were to be clear to them, then there would be nothing left for them to learn. Within the didactic project in the class, the teacher proposes a series of mathematical problems to the students so that a given objective can be achieved. Solving these problems becomes an end that directs the actions of the students. However, from the perspective of the Theory of Knowledge Objectification, doing mathematics cannot be reduced to solving problems. Without devaluing the role of problems in knowledge formation (see, for example, Bachelard, 1986), for us, problem solving is not the end but rather one of the means for achieving the type of *praxis cogitans* or cultural reflection that we call mathematical thinking. So that, behind the objective of the lesson, there lies a greater and more important objective—the generally held objective for the teaching and learning of mathematics—namely, the elaboration on the part of the student of a reflection defined as a common and active relationship with his/her cultural-historical reality.

In other words, learning mathematics is not simply learning *to do* mathematics (problem solving), but rather it is learning *to be* in mathematics. The difference between doing and being is immense and, as we shall see later, it has important consequences not only for the designing of activities but also for the organization of the class itself and the roles that students and teachers play within it.

### 3.4 The objectification of knowledge

The greatest objective of the teaching of mathematics is that the student learn to reflect according to certain historically constituted cultural forms of thinking that distinguish it from other types of reflection (for example, those of a literary or musical kind) inasmuch as in mathematical reflection, the individual's relationship with the world emphasizes ideas regarding form, number, measurement, time, space, etc. It is this emphasis which distinguishes mathematical thinking from other kinds of thinking.

The theory of objectification nevertheless does not see learning as a simple imitation or participation consistent with a pre-established practice, but rather sees it as the fusion between a subjectivity which seeks to perceive the cultural modes of reflecting

and the conceptual objects such a reflection is about. In order to get to know objects and products of cultural development, it is “necessary to carry out a determined activity around them, that is to say, a kind of activity that produces its essential characteristics, embodied, 'accumulated' in said objects.” (Leontiev, 1968, p. 21).

Teaching consists of generating and keeping in movement contextual activities which are situated in space and time and which are heading toward a fixed pattern of reflexive activity incrustated in the culture. This movement, which could be expressed as the movement from process to object (Sfard, 1991; Gray and Tall, 1994) has three essential characteristics. First, the object is not a monolithic or homogenous object. It is an object made up of layers of generality. Second, from the epistemological point of view, these layers will be more or less general depending on the characteristics of the cultural meanings of the fixed pattern of activity in question (for example, the kinaesthetic movement that forms a circle; the symbolic formula that expresses it as a group of points at an equal distance from its centre, etc.). Third, from the cognitive point of view, the layers of generality are noticed in a progressive way by the student. The learning process consists in finding out how to take note of or how to perceive these layers of generality. Just as learning is a re-flection, to learn presupposes a dialectical process between subject and object mediated by culture; a process during which, through his/her actions (sensory or intellectual) the subject takes note of or becomes aware of the object.

Objectification is precisely this social process of progressively becoming aware of the Homeric *eidōs*, that is, of something in front of us—a figure, a form—something whose generality we gradually take note of and at the same time endow with meaning. It is this act of noticing that unveils itself through counting and signalling gestures. It is the noticing of something that reveals itself in the emerging intention projected onto the sign or in the kinaesthetic movement which mediates the artefact in the course of practical sensory activity, something liable to become a reproducible action whose meaning points toward this fixed eidetic pattern of actions incrustated in the culture which is the object itself.

## **4. THE CLASSROOM AS A LEARNING COMMUNITY**

### **4.1 Being-with-others**

The classroom is the social space in which the student elaborates this reflection, defined as a common and active relation with his/her historical-cultural reality. It is here that the encounter between the subject and the object of knowledge occurs. The objectification that allows for this encounter is not an individual process but a social one. The sociability of the process, nevertheless, cannot be understood as a simple business interaction during which each player invests some capital in the hopes of ending up with more of it, or as a kind of game between adversaries (as in the Theory of Situations). Here, sociability means the process of the formation of consciousness

which Leontiev characterized as *co-sapientia*, that is to say, as knowing in common or knowing-with others.

Naturally, these ideas imply a re-conceptualization of the student and his/her role in the act of learning. Insofar as current theories in mathematics education draw on the concept of the individual as formulated by Kant and other Enlightenment philosophers, education justifies itself by guaranteeing the formation of an autonomous subject (understood in the sense of being able to do something for oneself without the help of others). Autonomy is, in effect, a central theme of modern education that has served as a basis for the theorizing of socio-constructivism (see, for example, Yackel and Cobb, 1996) and the Theory of Situations (Brousseau, 1986; Brousseau and Gibel, 2005, p. 22). The rationalism that weighs on this concept of autonomy comes from its alliance with another key Kantian concept: that of liberty. There can be no autonomy without liberty and, for Kant, liberty means the convenient use of Reason according to its own principles so that “it is through reason that we get an insight into principles” (Kant, 1900, p. 34).

Since the Enlightenment did not put forward the possibility of there being a multiplicity of reasons, but rather postulated that western reason was The Reason, community coexistence implies respect for a duty which, in the end, is nothing but a manifestation of that universal reason, whose epitome is mathematics. It was this supposed universality of reason that led Kant to fuse together the ethical, political and epistemological dimension and to affirm that “to do something for the sake of duty means obeying reason.” (Kant, 1900, p. 37).

For the Theory of Knowledge Objectification, classroom functioning and the role of the teacher are not limited to trying to achieve autonomy. It is more important to learn how to live in the community that is a classroom (in its fullest sense), to learn to interact with others, to open oneself up to understanding other voices and other consciousnesses, in brief, *to be-with-others* (Radford, in press).

Just as "the social is irreducible to individuals, however numerous they might be" (Todorov, in Bakhtine, 1984, p. 19), sociability in the classroom means a coming together through links and relations that are prerequisites for that kind of reflection that we mentioned earlier, defined as common and active and which is elaborated by the student along with his/her historical-cultural reality. This sociability not only leaves its mark on the conceptual content being pursued but is furthermore an integral part of it.

The intrinsic social nature of knowledge and mathematical thinking has brought us then to conceiving of the classroom as a learning community whose functioning is oriented toward the objectification of knowledge. Its members work in such a way that: the community allows for the personal achievement of each individual; each member of the community has his/her place; each member is respected; each member respects others and the values of the community; the community is flexible in its ideas and its forms of expression; the community opens up space for subversion in

order to insure: modification, change and its transformation. Being a member of the community is not something that comes as a matter of course. In order to be a community member, students are encouraged to: share in the objectives of the community; involve themselves in the classroom activities; communicate with others. The abovementioned guidelines are not simply codes of conduct. On the contrary, they are indexes of forms of being in mathematics (and, as a consequence, of knowing mathematics) in the strictest sense of the term.

## CONCLUDING REMARKS

Some theories in mathematics education have intentionally excluded the psychological aspects of learning and have occupied themselves with mathematical situations that can favour the emergence of precise mathematical reasoning. Such is the case for the Theory of Situations. On the contrary, other theories have fixated themselves on the mechanisms of the negotiation of meaning in the classroom and the way in which this negotiation explains the construction of representations that the student makes of the world. Such is the case of socio-constructivism. The intellectual debt that the Theory of Knowledge Objectification owes to these two theories is immense and our reference to them should not be seen in a negative light. These theories are sustained by fundamental principles and clear modes of operation that confer upon them an impeccable solidity. Nevertheless, the TKO takes off from other principles. On the one hand, it bases itself on the idea that the psychological dimension of learning has to be an object of study in mathematics education. On the other hand, it suggests that the meanings circulating in the classroom cannot be confined to the interactive dimension that takes place in the class itself; rather, they have to be conceptualized according to the context of the historical-cultural dimension. Therefore, the Theory of Knowledge Objectification proposes a didactic anchored on principles according to which learning is viewed as a social activity (*praxis cogitans*) deeply rooted in a cultural tradition that precedes it. Its fundamental principles are articulated according to five interrelated concepts<sup>2</sup>. The first of these is a concept of a psychological order: the concept of *thinking*, elaborated in non-mentalist terms. The second concept of the theory is of a socio-cultural order. This is the concept of learning. The third concept of the theory is of an epistemological nature and deals with those super-epistemic aspects that frame learning in the form of *semiotic systems of cultural signification* –cultural systems that “naturalize” the ways that one questions and investigates the world. The aforementioned concepts come to be completed by a fourth concept of an ontological nature—that of mathematical objects, which we have defined as *fixed patterns of reflexive activity incrustated in the ever-changing world of social practice mediated by artefacts*. To render the theory operational in its ontogenic aspect, it was necessary to introduce a fifth concept of a semiotic-cognitive nature—that of objectification, or a subjective awareness of the cultural object. In this context, and in light of the previous fundamental concepts, learning is defined as the social process of

*objectification* of those external patterns of action fixed in the culture. Although space constraints did not allow me to illustrate here the students' processes of objectification, these processes have been studied in detail in my classroom research (see e.g. Radford, 2003c, 2006b; Radford et al. 2004, 2006; Sabena et al. 2005).

### Acknowledgements

I wish to thank the two reviewers who commented on a previous version of this paper. I also wish to thank the members of our Working Group at CERME 5, who through their questions and remarks, provided me with insightful feedback.

### End Notes

1. Henceforth, it is not only the action which constitutes the schema of the concept (Piaget)—or its seal or emblem (Kant)—but also the meaning of the action in a precise moment of the socio-cultural activity within which the action occurs (Radford, 2005).

2. I do not have room here to state the way in which these principles frame the fundamental didactic problems of the theory. I can only mention that the problem of learning, as a *practical* problem, is one of the central research problems of the theory (see the references to our classroom-based work). This central problem is considered as deeply rooted in the problem of the student's formation of his or her consciousness—something that happens as the student objectifies the conceptual content that orients the activity and that the theory posits as something happening in the interweaving of the subjective, social and cultural dimensions of knowing and doing.

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## AN ANTHROPOLOGICAL APPROACH TO METACOGNITION: THE “STUDY AND RESEARCH COURSES”

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**Abstract.** *This paper shows how the Anthropological Theory of the Didactic approaches the metacognitive problem and reformulates it in terms of mathematical and didactic praxeologies. We use an empirical study focused on the design and implementation of two “Study and Research Courses” concerning the problem of comparing mobile phones tariffs. The new didactic contract installed during the experience highlights that students can assume responsibilities the current didactic contract assigns exclusively to the teacher (questioning, planning, time and didactic variables management, assessment and institutionalization). These responsibilities can be related to what psychological perspectives usually interpret as the result of the activation of “metacognitive” strategies.*

Research in metacognition is closely related to research in problem solving. Numerous authors emphasize that the deficiencies in metacognitive aspects are the fundamental cause of why students fail when solving problems in general, and more specifically so when it concerns mathematical problems (Schoenfeld 1992). Despite the existence of an agreement on the theoretical concept of metacognition based on Flavell’s researches (1976) specified in “metacognitive knowledge” (the knowledge of cognitive processes) and “regulation” or metacognitive experiences (planning, selecting strategies, monitoring progress, assessing results, revising plans and strategies), many questions remain unanswered, especially the ones focusing on what the term metacognition means in practice. We will propose a new interpretation of both “metacognitive knowledge” and “regulation” using the model of cognition provided by the anthropological approach to the didactic. And we will show its productiveness through the experimentation of a new proposal of instruction.

### PRAXEOLOGIES AND METACOGNITIVE KNOWLEDGE

The Anthropological Theory of the Didactic (ATD) puts forward a model of mathematical activity (especially including school mathematical activity) in terms of praxeologies (Chevallard 1999, Chevallard, Bosch & Gascón 1997). Two aspects may be distinguished as founding elements of the praxeologies: the *praxis* or “know-how”, which includes certain types of tasks as well as the techniques to carry them out; and the *logos* or “knowledge”, which refers to the elements necessary to describe, explain and justify the techniques (that is, the “technology” or discourse – *logos*– about the technique –*techne*– as well as the “theory” or the formal argument that justifies the “technology”). In praxeologies, *praxis* and *logos* are inseparable,

even if we can consider praxeologies with an undeveloped *logos* (we know how to do it but cannot explain it) and theoretical discourses with undeveloped *praxis*. Solving a problem, searching the answer to problematic questions consists in the construction of praxeologies bringing up new ways of doing (new *praxis* or “know-how”) and/or new ways of describing, explaining and justifying them (new *logos* or “knowledge”). Both the process of doing mathematics (solving problems and constructing new knowledge) and the mathematical knowledge produced by this activity are inseparable and equally described in terms of praxeologies. In this context praxeologies are rarely individual: they are shared by groups of human beings organised in institutions. Cognition is thus institutionally conceived.

In order to have more precise tools to analyze institutional didactic processes, Chevallard (1999) classifies mathematical praxeologies as *point*, *local* and *regional* ones: a *point* praxeology is generated by a unique type of problems and is characterized by a unique technique to deal with them; a *local* praxeology is generated by the integration of several point praxeologies within the same technology; a *regional* praxeology is obtained by coordinating, integrating and articulating several local praxeologies in a common mathematical theory. In a simplified way, we can say that what is learnt and taught in an educational institution are point praxeologies more or less articulated into local and regional ones.

As specified in Rodríguez (2005), *metacognitive knowledge* can be explained in this context as those aspects related to the *construction* and the *connection* between praxeologies of *increasing complexity*: location of an isolated question into a *local* mathematical organisation; variations of techniques to study a single type of problems (internal relations of *specific* praxeologies); variations and/or integration of different techniques within a technology (relations between *point* praxeologies within *local* praxeologies), connections between different concepts or other “technological” components (relations between *local* praxeologies) or between different theories (relations between *regional* praxeologies). Metacognitive knowledge can be related to the process of developing a sequence of mathematical organisations of increasing complexity, linking the problem of teaching metacognitive knowledge to the problem of going further in the teaching of a sequence of disconnected mathematical organisations: passing from a point praxeology given by the study of an isolate question to the integration of this question into a type of problems, connecting this type of problems and the corresponding techniques with other problems and techniques within a common theoretical discourse, etc. We can thus talk about *different levels* of metacognitive knowledge and, what is more important, this knowledge can be explicitly described in terms of the concrete mathematical contents involved in a teaching and learning process (Rodríguez 2005).

The use of the *scale of levels of determination* recently introduced in the ATD (Chevallard 2002; see also Bosch & Gascón 2006) can be used to describe a paradox that appears in the introduction of problem solving activities in official curricula. We

have shown in another work (Rodríguez, Bosch & Gascón 2004) that, in most cases, problem solving activities are officially introduced at school at completely different levels of determination, which seems to hinder the connection between different contents or strategies involved in the process of solving a problem. According to textbooks and official syllabi, the study of open questions use to appear either related to a topic or a single issue of a topic – that is, within a *local* or even a *point* mathematical organisation –, or introduced at a very general level, without any apparent connection to a specific content, topic, issue or even discipline. Thus the teacher, and also the students are supposed to attain a very general objective – teaching or learning to solve *any type of problems* – they cannot always concretise in the different levels of curriculum determination but only at the very specific level of the point or local mathematical organisation they are used to work in. We are not developing here the origin of these “gaps” in the levels of determination nor the analysis of the constraints they inflict on the teacher’s and students’ practices. What seems clear is that, paradoxically, this situations leads to treat problem solving, which is proposed as a tool to connect and integrate mathematical contents at school, in a complete isolated way from the rest of mathematical activities (usually structured around “concepts” instead of “types of problems”). As it is happening in many countries, problem solving ends up confined in a single “block” (among “numbers and measure”, “algebra”, “geometry”, etc. appears the new topic “problem solving!”), instead of being the dynamic motor of the whole didactic process.

## **STUDY AND RESEARCH COURSES AS A TEACHING PROPOSAL**

The most recent developments of the ATD (Chevallard 2004 and 2006) put forward the consideration of a new type of didactic mechanism called “Study and Research Courses” (from now on SRC). Changing right round the institutional didactic contract in force, which tends to favour the “study of answers” in detriment of the “study of questions”, the SRC are generated by a question Q with strong generating power, capable of imposing numerous derived questions leading to various bodies of knowledge to teach. Instead of starting from “contents” previously established in the strict framework of a discipline or a body of knowledge, the proposal consists in ideally “covering” school curricula with a set of SRC without a specific connection to the programmed contents. The study of these SRC should cause the encounter with some of these contents (and also many others) but it maintains a high degree of widening comparing to the majority of study processes. We postulate that the intrinsically co-disciplined nature of the SRC should allow attenuating the “thematic confinement” in which teachers and students used to work at school (Chevallard 2001).

### **Choice of the generating question and guidelines for the teacher**

Following Chevallard (2004 and 2006), we will describe the fundamental characteristics of the SRC below, illustrating them with the case we experimented on

two occasions, during the school years 2003-04 and 2004-05 in two secondary schools in Madrid. The SRC arise from the study of problematic questions the solving of which requires the construction of a succession of praxeologies joined together. Using the TSD terminology (Brousseau 1997), we could say that the generating question of a SRC must be able to be formulated initially without resorting to the “knowledge” (or to the praxeologies) one wants to create. In the case of SRC, there is no previously given praxeology towards which the study needs to be directed, that is, which a SRC proposes to build. As said before, the objective of a SRC is to study a problematic question, not to use the question as a means to build a previously determined body of knowledge. The answer to a problematic question will obviously be an answer in the form of a praxeology, but its characteristics, components, “size”, ecology, etc., is to be detailed throughout the study process, without preceding it. Hence we can talk about a bigger opening of the study in contrast to the current ones.

Using the TDS terminology, the initial question needs to be productive enough, which means it needs to provide a great deal of “didactic variables” which contribute to generating the process. In this case, however, unlike what is being propounded in the TDS, the management of these variables has not necessarily been carried out by the teacher, but by the whole class as a “study community”. This will be an important aspect in the modification of the usual didactic contract.

In the case here presented, the chosen generating question – rich, “alive” and relevant for the students – has been specified in “investigate which mobile telephone company and tariff is best for each person”. It is obvious that the answer to this question may have consequences in the phone users’ lives and is therefore not a mere opportunity for certain predetermined mathematical knowledge to appear. Furthermore, given the fact that all the students are real phone-users, they are clearly interested in knowing whether the company tariff they are using is or not the most appropriate and, in case it is not, may change to a better company. The generating power of this question had previously been analysed by the research team during a course of mathematical modelling for first year university students in economics and business administration. It showed in what sense the comparison of more than two different tariffs turned the graphical representation of functions into a relevant tool, being much more powerful than the algebraic work of solving inequations (Rodríguez 2005).

A SRC is essentially determined by the intention to answer the generating question and by the limitations of the people facing its solving and the means used to do so. In this sense, it is not a pre-determined way but rather a plan guided by the context and the need to answer the question. The intermediate questions or stages which will allow obtaining a satisfactory answer do not have to be determined beforehand with a lot of precision. During the study, what will need to be determined is what a satisfactory answer consists in, as this aspect is not determined beforehand either.

In the experimentation we carried out, we showed how a first type of answer can be specified in more or less complex comparisons (between two tariffs, between the tariffs of the same type of each company, among all the tariffs, etc.). Another type of answer may refer to the case of one user in particular. There was even a more general answer in the form of an Excel programme allowing any user to determine which tariff suits him/her best. Included in this last case, different types of answers were considered: a “normal” version, a version for “lazybones” and a version for “extremely lazy people” related to the amount of required details to the exactness of the answer given to the user, that is, the relation between cost and efficiency. Furthermore, the last “version” of the answer was proposed in the form of a web page allowing access to a great amount of users.

For SRC to exist it is necessary that students have enough means to start the study and deal with the initial question. In our case, the students had the necessary mathematical elements, as telephone tariffs are obtained by mathematical models based on straight lines or linear functions defined piece-wise which the students had previously studied. Subsequently, the situation must allow the students to obtain “good means”, that is, elements that allow self-evaluation of the solutions or intermediate answers proposed and the development in the study process. For example, in the experimentation we carried out the students could “simulate” both with pen and paper (with the help of a calculator) and using Excel. Other means, like, for example, their own bills, were also used which gave rise to a statistical study of cases not initially foreseen by the research team.

Finally, a fundamental objective of SRC is to obtain that the students assume the responsibility to answer the question posed, as well as the majority of decisions of the study process. The rest of decisions will have to be agreed upon with them. The researcher assuming the role of teacher (or “director of study”) during the two experiments put as specific objectives to achieve that, in as far as possible, the whole class as a study community would assume the responsibility in the decision-making throughout the entire process. She also made sure the different *study moments* appeared (Chevallard 1999 and 2003), getting the students to spend the appropriate time on them and give them their due attention. This question is closely linked to the general objective of the research, that is, to the aspects of the metacognitive regulation, which will be developed later on.

### **Possibilities and constraints which affect the development of a SRC**

Our objective is to show the application possibilities of this teaching proposal and its efficiency in relation to the incorporation of the different aspects of the metacognitive regulation in the teaching-learning process. We will especially focus on the new distribution of responsibilities defined by the didactic contract (Brousseau 1997). It will be necessary to consider two types of constraints: the ones that make it difficult for the teacher to share responsibilities with the students and the ones that hinder the

students to assume more responsibilities during the study process. It is expected that overcoming some of these constraints, aspects concerning the metacognitive regulation (planning, regulating and evaluating) will emerge.

Another important constraint comes from the fact that, in this teaching proposal, the bodies of knowledge are only objects of study to the extent of answering the question to which an answer needs to be found. It is thus considered that: “[...] knowledge must sacrifice itself, including its possible subsequent uses, from the moment it no longer appears as something that allows answering certain questions, solving certain problems” (Chevallard 2004). A last constraint is the time classes last. In Spanish secondary education they last for about 50-55 minutes. On implanting a SRC in the normal dynamics of a mathematics class, García (2005) found that this aspect represented strong limitations to develop something properly. For this reason, the initially foreseen length of the sessions of our two experimentations was two hours.

### **The distribution of responsibilities: metacognition and didactic contract**

We will study the new distribution of responsibilities which the SRC promotes and what constraints, coming from the usual didactic contract, hinder the assumption of those responsibilities by the different subjects of the institution. We are also interested in observing which decisions need to be made when the study process gets rid of a great deal of constraints, imposed by the school institution in a transparent way for the subjects. We want to analyze to what extent these decisions, despite being “didactic” decisions (in the classical sense, i.e., affecting the running of the teaching-learning process), are an integrating part of the mathematical work. Furthermore, we will see that in many cases they correspond to aspects considered “metacognitive” because they are related to the planning, regulation and evaluation of the learning process. In other words, if “metacognition knowledge” can be related to aspects of the mathematical work which go beyond the limits of the themes studied at school (that is, beyond the level of local praxeologies), the “metacognitive regulation” would correspond to the dimension of the mathematical work which, in the traditional didactic contract, is the sole responsibility of the teacher. It may, therefore, not be considered as a “cognitive” (mathematical) activity of the student, but as a “metacognitive” one. “Metacognitive regulation” can thus be related to the decisions which, in the traditional teaching-learning processes, are usually the exclusive responsibility of the teacher: planning the study, deciding on its chronology and “topology” (distribution of tasks), synthesising and evaluating the results, choosing easier questions or particular cases to start with, formulating new problems, looking for information, analyzing and developing it, etc.

### **ACCOUNT OF THE EXPERIMENTATION**

We carried out two experimental SRC during the second term of 2004 and 2005 with 11<sup>th</sup> grade students (16-17) of two secondary schools in Madrid. In both cases, all

students were invited to participate in what was presented as a “mathematical workshop” organised in after class sessions of 2 hours. A group of 10 to 14 volunteer students participated in each SRC. The total course lasted 18 sessions in both experimentations and the students did a lot of work outside the sessions. One of the researchers was the teacher of both workshops; she took notes of all the sessions, which were also video-recorded. When the students worked in small groups, each group was recorded in audio. In this way, the notes could later on be completed with the observation and transcript of both video and audio.

As for the development of the study of the question, two stages may be considered in the first experience with SRC: one based on the comparisons of fictitious tariffs, guided by the teacher and a second one related to real facts, in which students assumed a bigger responsibility in finding the answer to the questions. An important constraint experienced was what we might call the “temporary economy” of the teacher, who, urged by the need to “advance in the study process”, helped the students along with the answers or proper “hints” instead of waiting and providing the correct means for the students to come up with the answers. This constraint affected the development of the first SRC but was more “controlled” during the second one. There were also inverse situations in which the teacher knew how to put forward “crucial questions” which mean a “turning-point” of what has been covered so far, usually when getting to a deadlocked situation. For instance, when facing the frustrated attempts of the students to find a way to consider the differences in price between tariffs charging per seconds or 30-second lapses, the teacher put forward the possibility of carrying out a study of the proportion of each type of calls statistically considered more likely. In this case, although it could have been left up to the students, the help of the teacher may be considered the most appropriate.

As for the explicit attempt to make the students responsible for a series of dimensions or aspects of the study (planning, regulation and evaluation) which are not normally of their concern in the traditional didactic processes, we only realized later on that this kind of responsibilities were directly assumed by the teacher. In this first SRC the planning aspects which we wanted the students to be co-responsible for were very limited and they only referred to the prevision of usefulness of the techniques or to anticipating results. The students were not made responsible for the temporal and theme-related organization of the course nor for finding the answer to questions such as: where to start, what to deal with first, how much time to spend in each case, etc. This would be one of the main objectives of the second application of SRC.

With regard to the running of the *didactic moments* through which the ATD structures the study process (Chevallard 1999), we will only mention that finding the answer to the generating question made each moment appear in a relatively natural

way.<sup>1</sup> It catches our attention that, on some occasions, it is precisely the intervention of the teacher which limited this process, given her urge to make the study “progress”. This may be clearly observed, for instance, when, as the students were carrying out the comparisons of all the companies’ tariffs to elaborate the final report, once the teacher considered they already knew how to use the comparison technique, she suggested that they could leave it and start another task. The students, surprised, replied: “How can we do the comparison without considering all the cases?!” Obviously carrying out a comparative study means to compare everything with everything. We thus observed how the *moment of the work of the technique* had emerged naturally and was about to be aborted by the teacher! In what concerns the *moment of institutionalisation*, which is currently carried out under the sole responsibility of the teacher, it here took a surprising form when the students proposed to design a website as a way to give a definite answer to the initial question. Determining what materials should be posted on the website and how to present them constituted an important device for the institutionalisation performing and it was carried out in a complete cooperative way between the teacher and the students.

Concerning the second experimentation of SRC during the course 2004/05, we will only mention here that its main objective was to reproduce the experience with a different school and a different group of children, deepening in the modifications of the didactic contract to avoid the constraints borne by the teacher in the first SRC. The students were thus led to assume a bigger amount of responsibility during the whole process of study, especially in the planning of tasks, the organisation of the work in teams and the reformulations of the initial question. The purpose was for the teacher to avoid taking initiatives concerning these aspects of the study process, making them explicit and “negotiating” their organisation with the students.

To evaluate the evolution of the students’ knowledge at the end of the process, the students were asked to answer an individual written test. They had to solve a comparison of *fixed* phone tariffs with some novelties like a “bonus” (pack of calls at a reduced price) and the payment per seconds during the first minute. The students’ performance was good, with an average of 8.5 out of 10 in the first SRC and 8.25 in the second one. To analyse the effect of the SRC on the students’ beliefs and attitudes, the “CAETI- Trait Thinking Questionnaire” (O’Neil & Schacter 1997) was used. In the first SRC we did not find any difference in the students’ results before and after the experience. However, in the second SRC, a student asked if the test dealt with the “mathematics of the current class” or the “mathematics of the SRC”. We then asked the students to answer the test twice: once considering the mathematics done in class and then the mathematics of the SRC. The results showed

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<sup>1</sup> The study process is structured in six different dimensions or “moments”: the first encounter with the problematic question, the exploration of the related type of tasks, the work of developing the technique, the technological-theoretical moment, the evaluation and institutionalisation moments.

that the students' beliefs and attitudes concerning the "SRC mathematics" were significantly different from the "class mathematics" in all items.

#### 4. CONCLUSIONS

Our first conclusion refers to the capacity of the SRC to create connections between different pieces of knowledge, that is, to develop what is traditionally considered as "metacognitive knowledge" or, in terms of the ATD, to increase the studied mathematical organisations beyond the level of the topic, the theme or the domain they belong to. We can say that both SRC have permitted:

- To give functionality to some contents of the block "functions and graphs", such as the construction of the algebraic expression of a function, the use of graphs to solve inequalities, to validate the solutions or to display information.
- To connect different blocks of contents, such as "statistics" and "functions and graphs" to validate the considered functions.
- Even to connect different knowledge areas, such as mathematics and "new technologies" with the use of Excel, the search of information about the tariffs on the web and the design of a web site to display the final results.

In what concerns "metacognitive regulation", our proposal to connect it with the sharing of responsibilities between the teacher and the students during the study process leads to the following conclusions:

- The current didactic contract can explain the students' initial resistance to assume responsibilities concerning the planning, regulation and evaluation of the study process, and how it gradually decreases throughout the process (as soon as a new didactic contract was established)
- What seems more difficult to overcome is the teacher's resistance to share the responsibilities of regulation, assessment and, more than any others, planning. The constraints coming from the current didactic contract were clearly palpable in the cases experimented, the teacher-researcher having strong difficulties not to plan, organise or validate the students' work.

In general, we postulate that the inclusion of metacognitive regulation in school mathematical activity needs a serious transformation of the current didactic contract to overcome the strict separation between what is commonly considered as "the mathematic" and "the didactic". More concretely, it has to allow students to be co-responsible for all aspects of the study process, including those traditionally assigned only to the teacher as, for instance, planning, regulation and evaluation of the study process, or the location of an isolated problematic question in a chosen local mathematical organisation. Obviously this "transfer of responsibilities" cannot take place in a spontaneous and natural way but requires more research and new didactic proposals to achieve that the study of questions becomes the "driving force" of the learning of mathematics. Our present researches on the integration and "viability" of Study and Research Courses as a normalised activity at secondary and university

level show the strong institutional constraints this new didactic contract has to overcome (García, Gascón, Ruiz, Bosch 2006; Barquero, Bosch & Gascón 2007).

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## CONDITIONS AND CONSTRAINTS IN THE TEACHING OF STATISTICS: THE SCALE OF LEVELS OF DETERMINATION

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*This presentation aims to show how the concept of 'scale of levels of didactic determination', developed in the framework of the Anthropological Theory of the Didactic, is able to reveal the space of system of constraints teachers are subject to when defining their educational project. Such a scale acts as a decomposition basis of the teachers' subjections. By way of illustration, we will deal with the teaching of statistics to final year secondary school students in the French educational system (classe de seconde) and we will focus on some generic constraints coming from the way our societies deal with this concrete body of mathematical knowledge.*

### CURRICULUM CHANGES

In September 2000, the training of French citizens to the thinking of variability and to the management of randomness led to introducing a renewed statistics education in the last year of compulsory secondary education, equivalent to grade 10 (15-16 year-old students). Three essential issues form the core of statistics education at this level. The first one is about numerical summaries of quantitative statistical series. The second one deals with the concept of *sampling fluctuation*. This focus on variability, the essence of the science of statistics, breaks an old-fashioned way of teaching statistics, where each statistical series is analyzed separately. The third question is about simulation, carried out using mainly the random generator of pocket calculators. According to the official instructions (Ministère de l'Éducation Nationale, 1999), this issue, linked to randomness in the curriculum, "should not be the theme of a lesson", but should lead to various "statistical studies" whose topics "depend on pupils' interests and topicality", as well as on teachers' tastes. Pupils should write the main elements of such studies down in a "statistical logbook" also recording data processing and simulation experiments, as well as "the reasons leading to carry out simulations or to process data". Ever since this new curriculum was published, mathematics teachers have been in trouble and devoid of statistical knowledge. This is why the French Society of Mathematics Teachers (APMEP) showed such unwillingness to accept this project (APMEP, 1999):

Since future S [scientific section] will not do any more statistics (but will be able to start doing it) and since ES [economics and social sciences section] will do statistics again during the second year, some teachers failed to see the reason to teach statistics because, presently, when statistics is taught, it is done in a slapdash manner at the end of the year. Nevertheless, we are embarrassed to ask to cancel statistics whereas we claim to do citizen mathematics... [our translation]

When the definitive version of the new curriculum was released on August 12<sup>th</sup> 1999, APMEP first requested to postpone the reform, then clamoured for a statistics training course for all mathematics teachers. The reason was that, otherwise, “it would be impossible to teach this new curriculum, the consequence being not to reach the goal of giving to future citizens the tools for understanding the actual world” (Dufossé, 1999) [our translation]. In spite of a petition signed by roughly 10,000 teachers, the new curriculum came into force in September 2000.

## FROM THE TEACHER’S PROBLEM TO THE RESEARCH PROBLEM

The teaching profession, through its association, has expressed its difficulty to elaborate an answer to the problem of building the teaching of statistics at secondary school: *How to develop a teaching process in statistics that is both faithful to the statistical science existing outside the school and relevant to the training of young generations?* The ATD adopts an ecological point of view considering conditions that make the functioning of didactic situations appropriate to the new curriculum possible. Indeed, it is not enough to propose and experiment didactic situations or even to study the economy of such systems by analyzing their functioning. We must understand the conditions of their development, otherwise these didactic situations cannot exist in the “actual classrooms”. Demanding teachers training is not the unique condition of statistics teaching in accordance with the new curriculum. In fact, this need belongs to the wider question of the *conditions* of statistics education. The ATD approach postulates that, when the teacher and the pupils meet around the knowledge to be taught (for instance the statistical knowledge), what can happen is mainly determined by conditions and constraints that cannot be reduced to those immediately identifiable *inside* the classroom: the teacher’s and pupils’ knowledge, the didactic material available, the temporal organisation of teaching, etc. Of course, these “endogenous” conditions and constraints play a great part in determining (or explaining) what happens inside the classroom, but they have to be completed with some “exogenous” conditions and constraints coming from outside the classroom and even from outside the educational system. The teacher’s problem can thus be reworded as the following research question: *how to determine the conditions and constraints under which statistics can be taught in the “classe de Seconde”?*

### Personal and institutional relations

While mathematics, like all human activities, is produced, diffused, employed and taught in social institutions, didactics of mathematics can be considered as the science that studies the conditions and constraints under which mathematical knowledge emerges, diffuses, changes, evolves, etc., within human groups. In line with his previous works on the didactic transposition, Yves Chevallard (1988) has introduced the concept of *institutional relation* to an object of knowledge within an institution, in order to be able to account for the plurality, diversity and complexity of the various forms of “knowledge concerning this object of knowledge” within the various institutions where it is present (see Bosch & Gascón, 2006). The institutional relation

to an object of knowledge refers to the use of this object of knowledge in a given institution, how it operates, how it lives and emerges from the practices where it is involved within this institution. For instance, the institutional relation to the concept of square root within the mathematics classroom is different according to the level of teaching: at the beginning of secondary school (grade 6 in France), it is only defined for positive numbers while in the final year (grade 12), the square root of a negative number is defined as soon as complex numbers are introduced. The institutional relation to an object of knowledge allows clarifying what a subject of an institution must refer to, when one asks the subject to mention this object of knowledge. It is not the relation of a particular individual to an object but rather the relation that subjects of an institution should have with this object of knowledge, according to their position within their institution. Hence, the personal relations of a pupil or his teacher to the concept of square root are different within the same mathematics classroom because they occupy different positions within the institution.

The personal relation to an object of knowledge is defined as the result of the institutional relations that a person has when occupying a given position within various institutions. Yves Chevallard (1988, p. 214) emphasized that these institutional relations “make up the main system of conditions and constraints under which the personal relation of the participants of the institution to the object of knowledge takes shape and evolves” [our translation]. Applied to our problem, this means that the personal relation of French mathematics teachers to statistics knowledge is feed with the institutional relations to objects that he/she successively encountered when being pupils at primary school, then at secondary school, later on at University, and eventually at the teachers training university institute, or like French citizens reading newspapers, etc.

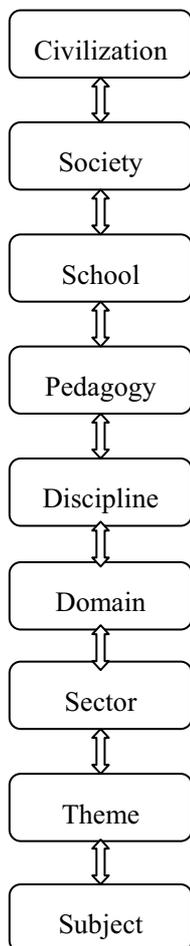


Figure 1

### Scale of levels of didactic determination

The modelling of the teacher as being subject simultaneously or successively to various institutions does allow us to identify the system of constraints that apply to him/her owing to the fact that he/she is subject to these institutions. How to account for it? Yves Chevallard (2002) initiated the concept of scale of levels of didactic determination as an exploratory tool (see figure 1). This scale operates as a filter or a decomposition basis and produces an interpretative framework of the various subjections to institutions. With the prime objective to illustrate this scale, we used this tool to study conditions and constraints acting on mathematics teachers when they want to teach statistics in the “classe de seconde”.

The highest level refers to the concept of civilization that we consider here as a set of conceptual and practical complexes that are common to several societies. These societies are therefore related from the point of view of those complexes. The

civilization is the genus whereas the society that belongs to it is the specific difference: the distinction between society and civilization is thus the same distinction as between “the other one” within “the same”. The pedagogical level refers to the conditions and constraints that affect the teaching and learning of all disciplines at school and is more specific than the school level, which includes all other forms of educational conditions.

We will now see how these “generic” levels of determination (civilization, society, school, pedagogy) can hinder the teaching of a mathematical domain such as statistics. It will illustrate the way the ATD perspective enlarges our “empirical bases” of research, that is, the set of empirical objects we have to look at and not only being students and teachers in a classroom or people doing mathematics outside a classroom. How can we grasp the level of civilization? Where can we find the conditions and constraints our society entails to the teaching of mathematical knowledge? We will present some elements to answer these questions in order to analyse further the problem raised by the French renewed statistics curriculum for the mathematics teacher profession.

### **EMPIRICAL STUDY: “EXOGENOUS” CONDITIONS AND CONSTRAINTS**

The first remarkable fact we can report when observing the practices of teaching statistics in the “classe de seconde” is a sort of *avoiding behaviour*. While the official curriculum suggests roughly 16 hours a week of statistics teaching, a survey of 191 teachers [1] shows that 5% of them have chosen not to teach statistics. Amongst the others, 15% teach less than 7 hours, 50% between 7 and 13 hours and 35% more than 13 hours a week. How can these behaviours be explained? What kinds of constraints, coming from which levels of didactic determination, do allow us to analyse them?

#### **The levels of civilization & society: “The ban of knowledge”**

Alexandre Koyré (1971) emphasized the repugnance of the antique Greek culture towards considering our sublunary world (as the opposite of the celestial world) as quantifiable. According to Koyré, in the terrestrial world accuracy is illusory and approximation is the rule. Ancient metrology was thus little developed. The science of measure, being a condition of possibility for the knowledge of the world, and a set of processes and technical instruments, was slow to evolve. In a more general way, any progress in the measurement of the world encounters three main interdependent obstacles. The first one is a postulate asserting that measuring is impossible, especially when the quantity to measure is considered undefined. The second is a succession of objective difficulties that have existed all through history and are not easily resolved. They are linked to the definition of the quantity to be measured and to the conceptual and instrumental system of measurement. The last obstacle is the traditional interpretation of the measurement project, seen as doing harm to the world because measure would be contemporary with a will to reify the measured.

To explain the historical difficulties encountered by demographic studies to be currently established in our western countries, the “demographer” Jacqueline Hecht (1977, p. 24) observes that “collective memory shall keep the souvenir of malediction tied to census for a long time and the occidental civilization shall finally accept this principle with difficulty. In the Christian Middle Ages, Saint Ambroise and Saint Augustin sentenced the sin of pride made by David.” [our translation]. The fragment cited is the episode where David, encouraged by Yahweh himself during a moment of ire, requests a census against the opinion of Joab, the general in command of his army for whom one cannot count men like herds (2 Samuel, 24). In Israel there were 800,000 warriors, and in Judah there were 500,000 soldiers. David felt guilty after he had numbered the army and said to Yahweh, “I have sinned greatly by doing this! Now, Yahweh, please remove the guilt of your servant, for I have acted very foolishly.” As a punishment, he must choose between three chastisements: seven years of famine, three months of defeat or three days of plague. He opted for three days of plague: seventy thousand people died. This example illustrates the old tension that exists between the established powers that want to know (David personifies them) and the sacred character of “what has to be known” (the human matter), which leads to see the census act like a form of impiety. Hecht (*op. cit.*, p. 70) explains this tension in the following terms:

We have seen that census, at first sight a pure countable and politically and ideologically neutral procedure, appears as an extremely complex and ambiguous operation. Initially endowed with a marked religious character, it has always appeared as a totalitarian and despotic government technique. Men have to be counted to be enslaved, and in spite of the evocative denominations of “fires” and “souls”, they were not counted for anything but animals. [our translation]

We have here a set of obstacles that (the civilization of) Mediterranean societies and others derived from them, have slowly overcome, while at the same time the empire of the measurable enlarged still more.

Within this civilization, a specificity of the French society is its weak sensitivity to statistics. Daniel Schwartz (1994), one of the main actors who introduced and developed medical statistics in France, suggests two main reasons. The first one is intrinsic to mathematics: “French people have a rigorous spirit – they are excellent at pure mathematics – but they are fully imbued with this Cartesian logic which adapts badly to uncertainty” [p. 97, our translation]. The second reason concerns the relations that the French society maintains with the collective and the individual. Indeed, for this author, “the necessity to sharply discern two opposites, the mean and the variance, the collective and the individual”, is the foundation of the statistical thought. Thus, he compares the United States of America, “which are united but remain States”, with France where “the sense of individuality prevails too much over the sense of collectivity” [p. 98, our translation].

In order to objectivize what could be considered as a particular point of view, we studied the diversity of the diffusion of statistical knowledge, towards the society or towards the school, as well as the place allocated to statistics within mathematics when it developed in France. The various analyses made (Wozniak, 2005) converge towards an essential fact: the weak penetration of statistics in the French culture and the non-familiarity with the statistical handling of numeric information must both be considered as fundamental data that strongly constrain statistical education in school training. It is clear, for example, when we compare the small place reserved to graphics dealing with numeric information in French newspapers with the one of an American newspaper as *USA today* which has its own service to produce data. The analysis of the content of this newspaper clearly shows the will to present numerical data as elements of a numerical series within which data is placed.

### **The levels of society and school: the social diffusion of statistical knowledge**

In the French society of the nineteenth century, the organization of the diffusion of knowledge went through constraints imposed by an institutional configuration where three main traditions can be distinguished. The most typical French institution is the organization of civil servants in state professions (engineers, officers, etc.), associated to one or several training schools (schools of engineers, military schools of officers, etc.). For statistics, this led, in 1946, to the creation of the profession of the civil administrators of the *Institut National de la Statistique et des Études Économiques* (INSEE) and its associated school, the *École Nationale de la Statistique et de l'Administration Économique*, (ENSAE) in 1960. A second tradition is the one at universities. Statistics as a knowledge to be taught was recognized lately with the creation, in 1922, of the *Institut de Statistique de l'Université de Paris* (ISUP), officially under the scientific leadership of the four Parisian colleges (law, science, medicine, literature). This institute offered high-level courses in “statistical method and its applications” but the audience was scarce: during the period 1925-1939, only 46 people obtained their diploma and 68 % of which were foreigners. It was not until 1951 that statistics was taught in upper secondary school within the context of the ‘techniques of economy’. The third tradition mentioned refers to other ways of diffusing knowledge to a broad audience used by some societies, generally through the bourgeoisie even if other classes are sometimes touched. For lack of space, we will not describe this third tradition here (see Bédarida, 1977).

Since statistics is not something that everyone knows, must know or even can know in France, it acquires the status of ‘special knowledge’. This status can be useful and is sometimes indispensable in some human groups organized around a given activity. It is not knowledge for everybody but only for some. According to a classical opposition, we make a distinction between the general diffusion of statistical knowledge and special diffusions that are targeted diffusions, officially motivated by the needs of some activities, in particular in the framework of professional training. Special diffusions of statistics are numerous, various and occur at very different

levels of education or training. A curriculum study of these trainings (Wozniak 2005) establishes that statistics has a social and cultural status equivalent to knowledge that seems to be only disseminated to identify professional groups with supposed needs in the field but of weak cultural influence. The diffusion through vocational trainings of a high level was nonetheless a fact in some sectors: through the teaching of methodology, statistics got into the literary academic world in the 1970s in France, more particularly in psychology, sociology and linguistics. It is noticeable that, as we go along the scholar and social scale of diploma, the “concrete” references, imprint of a specific professional universe, tend to disappear.

### **The levels of discipline & domain: the status of statistics within mathematics**

The account for the study of the social diffusion of statistics knowledge leads to the question of the place and status of statistics within mathematics. How to tackle this issue? What kind of material can we use to carry out this study? We will now briefly quote the tracks we have explored (Wozniak, 2005). Works of historians of mathematics, showing how statistics has been constituted as a field of mathematics, are obviously valuable help. As emphasized by Grattan-Guinness (1997, p. 738), Kolmogorov’s axiomatization has allowed probability and statistics in its wake, to get a position within mathematics:

In 1933, two years after Gödel’s theorem appeared, the Soviet mathematician Andrei Kolmogorov (1903-1987) furthered the cause of axiomatization by publishing a system for probability theory. This landmark achievement at last placed the subject within the sphere of “orthodox” mathematics, for he drew upon set theory – another late arrival, but by then impeccably placed in the rainbow.

However, the place of statistics remains singular, as emphasized by the same author when mentioning the classification of mathematics according to the Mathematical Reviews (Ibid., p. 721):

[...] this taxonomy is somewhat perfunctory on probability and statistics, which are however covered in detail in Statistical Theory and Method Abstracts; and mathematical education, omitted almost entirely, is handled by the Zentralblatt für Didaktik der Mathematik.

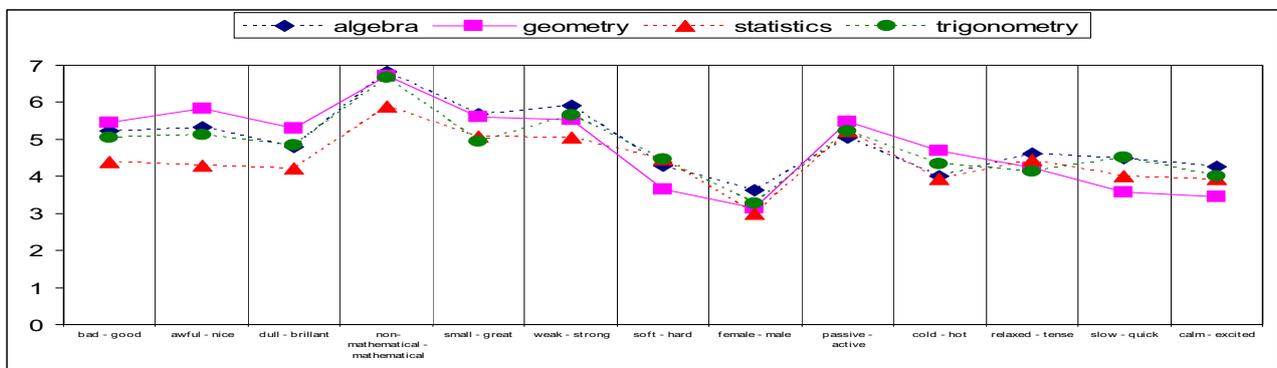
The analysis of some debates within the French mathematicians community about the legitimacy to integrate statistics into mathematics, allows us to account for the small place reserved to statistics within mathematics. We illustrate this by quoting André Weil (1906-1998) [2]:

... although we know that statistical mathematics have been of considerable importance to science (and in particular to biological sciences), it needs to be mentioned that statistical books amount in fact to a collection of recipes and precepts we would like to believe to be well chosen. Being written in a highly algebraic form, sometimes using logarithms, exponentials and integrals, they all have the prestige of mathematical exactitude to the untrained eye while the so-called demonstrations statistics are wrapped in, even highly

sophisticated, most often make no sense for the mathematician and are simply made of more or less convincing heuristic considerations. [our translation]

The recent institutional recognition of this field of mathematics in France explains why the creation of the first *Diplôme d'Étude Approfondie* [3] in statistics does not appear until 1970. Other empirical material corroborate this affirmation: an analysis of the prefaces of some books about statistics for a large public shows for instance the systematic minimization of the necessary mathematical knowledge for understanding statistical tools; a study of the topics tackled during the annual congress of mathematics teachers shows they do not give great importance to this teaching.

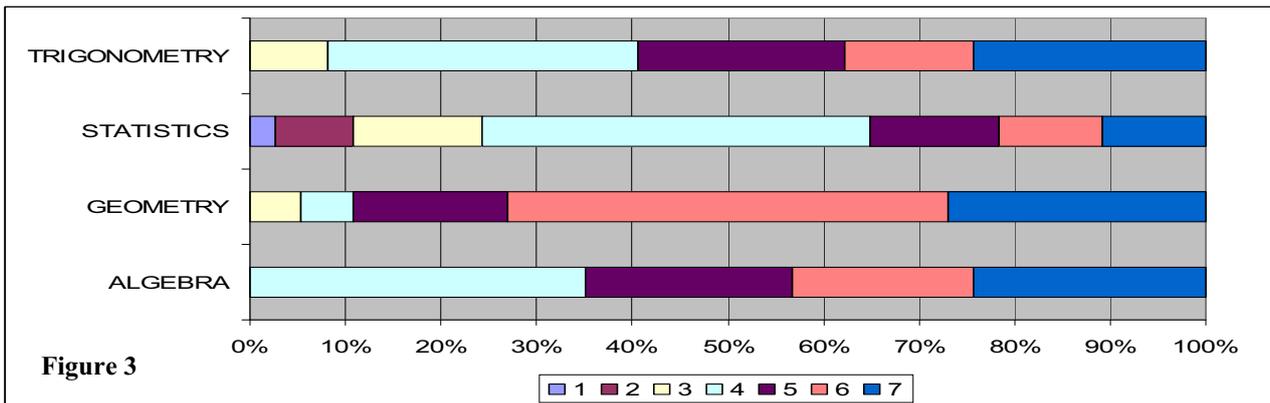
At this point, the question is to know if all these general conditions depicted will ever reach school and affect the institutional relation of mathematics teachers to statistics. We decided to study how a sample of mathematics secondary school teachers place statistics within the field of mathematics. A questionnaire based on Osgood's semantic differential was handed out to 41 future teachers, all in the same final year of training, at IUFM of Aix-Marseille [4]. In four domains of mathematics (algebra, geometry, statistics and trigonometry), 13 items were measured along a scale of seven levels. The items take three dimensions into account: the assessment (bad – good; awful – nice; dull – brilliant; non-mathematical – mathematical; small – great), the power (weak – strong; soft – hard; female – male) and the activity (passive – active; cold – hot; relaxed – tense; slow – quick; calm – excited). Figure 2 displays the main measurements obtained for each item in the four domains.



**Figure 2**

At first glance, we can see the lines moving in quite a parallel way, and statistics being regarded as “mathematics” and “strong”. In any case, statistics is almost always located at the lowest level (except in 4 out of 13 items), and, in 5 items, it is far less than algebra, geometry or trigonometry. It is weakly brilliant and hardly ever nice.

Figure 3 details the percentage of answers obtained in the item “awful - nice”. Statistics is the only one that has values 1 and 2 corresponding to “awful”, just over 20% of values 6 and 7 (“really nice”), while geometry has more than 70% and algebra and trigonometry around 40%.



## CONCLUSION

This last brief empirical study with the teachers reveals that school is permeable to the negative consideration of statistics in past times of our civilization, the weak social diffusion of statistical knowledge in France – set apart as a specialised knowledge in opposition to “knowledge for everybody”– and the pejorative and lowered status inside the mathematical world. We postulate that these “exogenous” constraints coming from outside the classroom are fundamental data to explain some avoiding behaviour of French mathematical teachers and silently hinder their practices and the type of activities that can be done at school.

The few examples of constraints of the highest levels of didactic determination presented above are obviously not sufficient, let alone to explain the teachers’ practices. In previous works (Wozniak 2005, 2006), we carried out a more exhaustive study, exploring the whole scale of didactic determination and including the study of the didactic transposition phenomenon. Here, we wanted to show the diversity and complexity of the space of conditions and constraints within which the teacher is embedded when he/she elaborates his/her educational project. To grasp didactic phenomena with a wider perspective than the one traditionally used, such a complexity needs a plurality of empirical analyses and the implementation of a variety of methodological tools. However, the extent and the ambition of our study tell us to remain modest. We thus conclude quoting the French sociologist Camille Tarot (1999) [our translation]:

The global social fact is surely not the exhibited whole of a Society, because who would be able to manage such a totality, even in the best monography? It is this unique property of facts that is indeed the subject of human science, to be significant, that is to be partial, contingent, arbitrary but connected, linked, always dependent on something which is inside them, both revealed and hidden by them.

## NOTES

1. Survey requested by the *Société Française de Statistique* (SFdS), carried out by six students of the DESS *Statistique et Informatique Socio-Économiques* at the University of Lyon 2, and granted by SFdS, IREM of University of Rennes, France.

2. Remarks reported by Meusnier (2004) and originally published in his article “Calcul des probabilités, Méthode axiomatique, Intégration” of 1940 in the *Revue Rose*.
3. It is the pre-doctoral level, equivalent to a master level allowing students to start a PhD thesis.
4. We did not want to ask future mathematics teachers directly, explicitly and openly what they think about statistics as does the Survey of Attitudes Toward Statistics (SATS) introduced by Candace Schau (see Estrada, Batanero, Fortuny and Diaz, 2005).

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