

Boutroux's Method vs. Re-scaling

Lower estimates for the orders of growth of the second and fourth Painlevé transcendents

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Abstract. We give a new proof of Shimomura's sharp lower estimates for the orders of growth of the Painlevé transcendents II and IV: $\varrho_{II} \geq 3/2$ and $\varrho_{IV} \geq 2$.

MSC2000: 34M05, 30D35

Keywords: Painlevé differential equations, order of growth, re-scaling, Boutroux's method.

1 Introduction

We are concerned with the transcendental solutions of *Painlevé's second and fourth equation*,

$$w'' = \alpha + zw + 2w^3 \quad (1)$$

and

$$2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2 - \alpha)w^2 + 2\beta, \quad (2)$$

the *second* and *fourth transcendents*. In [Sh1] and [St1] it was shown that any second and fourth Painlevé transcendent w has order of growth $\varrho(w) \leq 3$ and $\varrho(w) \leq 4$, respectively. More precisely, if (p_n) denotes the sequence of non-zero poles of w , it was shown in [St1] that, in the respective cases,

$$\sum_{|p_n| \leq r} |p_n|^{-1} = O(r^2) \quad \text{and} \quad \sum_{|p_n| \leq r} |p_n|^{-2} = O(r^2)$$

hold. In the other direction, Shimomura [Sh3] recently derived the sharp lower estimates $\varrho(w) \geq 3/2$ resp. $\varrho(w) \geq 2$. Equality is attained for particular solutions, called *Airy-* and *Hermite-Weber-*solutions, respectively. For

more details concerning these functions we refer to [GLS]. For $2\alpha \in \mathbb{Z}$ proofs of $\varrho(w) \geq 3/2$ in case (II) can be found in [Sh2] and [St2].

Combining the *re-scaling method* with some modified Shimomura approach [Sh3] we will be able to give a different proof of Shimomura's lower estimates, which may be stated as follows:

Theorem. *Let w be any transcendental solution of any of the differential equations (1) and (2), respectively, with sequence (p_n) of non-zero poles. Then, for some $\kappa = \kappa(w) > 0$ and $r > r_0$*

$$\sum_{|p_n| \leq r} |p_n|^{-3/2} \geq \kappa \log r \quad \& \quad \sum_{|p_n| \leq r} |p_n|^{-1/2} \geq \kappa r$$

and

$$\sum_{|p_n| \leq r} |p_n|^{-2} \geq \kappa \log r \quad \& \quad \sum_{|p_n| \leq r} |p_n|^{-1} \geq \kappa r$$

hold, respectively.

2 Two local methods

We start by describing two methods of investigating the Painlevé transcendents locally, and restrict ourselves to equation (1).

a) Re-scaling. Let w be any transcendental solution of (II) and let (p_n) be any sub-sequence of the sequence of poles of w . Then

$$y_n(\mathfrak{z}) = p_n^{-1/2} w(p_n + p_n^{-1/2} \mathfrak{z})$$

has the series expansion

$$y_n(\mathfrak{z}) = \frac{\epsilon_n}{\mathfrak{z}} - \frac{\epsilon_n}{6} \mathfrak{z} - p_n^{-3/2} \frac{\alpha + \epsilon_n}{4} \mathfrak{z}^2 + h_n p_n^{-2} \mathfrak{z}^3 + \dots, \quad \epsilon_n = \pm 1,$$

about $\mathfrak{z} = 0$ and satisfies

$$y_n''(\mathfrak{z}) = p_n^{-3/2} \alpha + (p_n^{-3/2} \mathfrak{z} + 1) y_n(\mathfrak{z}) + 2y_n^3(\mathfrak{z}),$$

where now $'$ denotes differentiation with respect to \mathfrak{z} . One of the major results of the re-scaling method developed in [St1, St2] was that the sequence $(h_n p_n^{-2})$ has a *uniform bound* only depending on the solution w . Thus choosing a sub-sequence of (p_n) , again denoted (p_n) , such that $\epsilon_n = \epsilon$ is constant and

$h_n p_n^{-2} \rightarrow h$, we obtain $y_n(\mathfrak{z}) \rightarrow \eta_{rs}(\mathfrak{z})$ (rs stands for *re-scaled*), locally uniformly in \mathbb{C} , where η_{rs} is the unique solution of

$$\eta_{rs}'' = \eta_{rs} + 2\eta_{rs}^3 \text{ with } \eta_{rs}(\mathfrak{z}) = \frac{\epsilon}{\mathfrak{z}} - \frac{\epsilon}{6}\mathfrak{z} + h\mathfrak{z}^3 + \dots \text{ about } \mathfrak{z} = 0.$$

We note that

$$\eta_{rs}'^2 = \frac{7}{36} - 10\epsilon h + \eta_{rs}^2 + \eta_{rs}^4 = c + \eta_{rs}^2 + \eta_{rs}^4,$$

with $|c|$ uniformly bounded, independent of the sequence (p_n) .

Finally, application of Hurwitz' Theorem yields the following

Remark a. *Every pole \mathfrak{z}_0 of η_{rs} is the limit of poles \mathfrak{z}_n of y_n ; thus $p'_n = p_n + p_n^{-1/2}\mathfrak{z}_n$ is a pole of w , and any such sequence p'_n gives rise to a pole $\mathfrak{z}_0 = \lim_{n \rightarrow \infty} (p'_n - p_n)p_n^{1/2}$ of η_{rs} .*

b) Boutroux's Method. Again let w be any transcendental solution of (II). The change of variables

$$\xi = \frac{2}{3}z^{3/2} = \phi(z), \quad \Theta(\xi) = z^{-1/2}w(z),$$

see Boutroux's paper [B], leads to the differential equation (where now $'$ denotes $d/d\xi$)

$$\Theta''(\xi) = 2\Theta^3(\xi) + \Theta(\xi) - \frac{\Theta'(\xi)}{\xi} + \frac{2\alpha}{3\xi} + \frac{\Theta(\xi)}{9\xi^2}.$$

To be more precise, let \mathbb{H} be any half-plane with $0 \in \partial\mathbb{H}$. Then any branch of $\psi(\xi) = (\frac{3}{2}\xi)^{2/3}$ maps \mathbb{H} conformally onto some sector S of angular width $2\pi/3$, and ϕ will denote the inverse map $\psi^{-1} : S \rightarrow \mathbb{H}$.

If $p_n \neq 0$ denotes any pole of w in the sector S , we obtain for $v_n(\mathfrak{z}) = \Theta(\phi(p_n) + \mathfrak{z})$ the differential equation

$$v_n''(\mathfrak{z}) = 2v_n^3(\mathfrak{z}) + v_n(\mathfrak{z}) - \frac{v_n'(\mathfrak{z})}{\phi(p_n) + \mathfrak{z}} + \frac{2\alpha}{3(\phi(p_n) + \mathfrak{z})} + \frac{v_n(\mathfrak{z})}{9(\phi(p_n) + \mathfrak{z})^2}.$$

If we choose $p_n \rightarrow \infty$ (the same sub-sequence as was chosen above) we obtain in the limit the differential equation

$$\mathbf{v}_B'' = 2\mathbf{v}_B^3 + \mathbf{v}_B,$$

where $_B$ stands for *Boutroux*. It is obvious that

$$\Theta(\phi(p_n) + \mathfrak{z}) = (\psi(\phi(p_n) + \mathfrak{z}))^{-1/2} w(\psi(\phi(p_n) + \mathfrak{z})) \sim p_n^{-1/2} w(p_n + p_n^{-1/2} \mathfrak{z})$$

holds as $n \rightarrow \infty$, and hence the functions η_{rs} and \mathfrak{v}_B agree. This phenomenon was already observed in [St1] for Painlevé's first equation. The re-scaling method yields the additional information that Θ and Θ' are *uniformly bounded* outside the union of disks $|\xi - \phi(p_n)| < \delta$ about the poles $\phi(p_n)$ of Θ ; $\delta > 0$ is arbitrary. We thus have

Remark b. *Every pole \mathfrak{z}_0 of η_B is the limit of poles \mathfrak{z}_n of v_n ; thus $p'_n = \psi(\phi(p_n) + \mathfrak{z}_n)$ is a pole of w , and any such sequence p'_n gives rise to a pole $\mathfrak{z}_0 = \lim_{n \rightarrow \infty} (\phi(p'_n) - \phi(p_n))$ of η_B .*

3 Proof of the Theorem

To start with the proof we need the following Lemma, which in similar form also was proved in [Sh2, **Lemma 2.2.**].

Lemma. *Let \mathfrak{L}_c denote the (possibly degenerate) period lattice for the differential equation $\eta'^2 = \eta^4 + \eta^2 + c$, and let Σ be any open sector with vertex at the origin and containing $\{1, i\}$ (or $\{-1, i\}$ or $\{-1, -i\}$ or $\{1, -i\}$). Then given $K > 0$ there exists $R > 0$, such that $\Sigma \cap \{\omega : |\omega| \leq R\} \cap \mathfrak{L}_c \neq \emptyset$ for every c satisfying $|c| \leq K$.*

Remark. For $c \neq 0, 1/4$, every non-constant solution of $\eta'^2 = \eta^4 + \eta^2 + c$ is an elliptic function, closely related to Jacobi's *sinus amplitudinis*. If $\{\omega, \tilde{\omega}\}$ is a suitably chosen basis of the period lattice \mathfrak{L} and if η has a pole at $\mathfrak{z} = 0$, then it has simple poles exactly at $m\omega + (n + \frac{1}{2})\tilde{\omega}$, $m, n \in \mathbb{Z}$, see the famous book [HC, p. 215] by Hurwitz and Courant.

Proof of Lemma. We have to consider separately the points of degeneration, namely $c = 0$, $c = 1/4$ and $c = \infty$. For $c = 0$ and $c = 1/4$ the non-constant solutions η are simply periodic with primitive periods $\omega_0 = \pm\pi/\sqrt{2}$ and $\omega_{1/4} = \pm i\pi$, respectively. Hence, for $\delta > 0$ sufficiently small, we have in the respective cases $|c| < \delta$ and $|c - 1/4| < \delta$ that, by continuity, one of the periods $\pm\omega_c$ belong to $\Sigma \cap \{\omega : |\omega| \leq 4\}$, say.

In case $c \rightarrow \infty$ we set $\mathfrak{u}_a(z) = a\eta(az)$ with $a^4 c = 1$, to obtain $\mathfrak{u}'_a{}^2 = \mathfrak{u}_a^4 + a^2 \mathfrak{u}_a^2 + 1$, and hence, in the limit $c \rightarrow \infty$, the differential equation $\mathfrak{u}'_0{}^2 = \mathfrak{u}_0^4 + 1$.

Thus, for $|c|$ large, the period lattice \mathfrak{L}_c is approximately a square lattice with mesh size $\asymp |c|^{-1/4}$.

In the compact parameter set $\{c : \delta \leq |c| \leq K, |c - 1/4| \geq \delta\}$ each lattice \mathfrak{L}_c has a basis $\{\omega_c, \tilde{\omega}_c\}$ such that $\kappa R \leq |\omega_c| \leq |\tilde{\omega}_c| \leq |\omega_c \pm \tilde{\omega}_c| \leq R$ for some constants $R \geq 4, \kappa > 0$, independent of c . The problem now is equivalent to the following: Let \mathfrak{L} be the lattice spanned by 1 and τ with

$$\operatorname{Im} \tau > 0, \quad -1/2 < \operatorname{Re} \tau \leq 1/2 \text{ and } 1 \leq |\tau| \leq M,$$

$M > 1$ some constant, and let Σ be *any* open sector with vertex at the origin and with angular width $> \pi/2$. Then we have to show that

$$\Sigma \cap \{1, 1 + \tau, \tau, -1 + \tau, -1, -1 - \tau, -\tau, 1 - \tau\} \neq \emptyset.$$

This, however, follows immediately from the fact that the angle between any two consecutive points in the sequence $1, 1 + \tau, \dots, 1 - \tau, 1$ is at most $\pi/2$. \square

To fix ideas we consider (the branches of) $\psi(\xi) = (\frac{3}{2}\xi)^{2/3}$ in the half-plane $\mathbb{H} : -\pi/4 < \arg \xi < 3\pi/4$ with $\psi(\mathbb{H}) = S = \{z : -\pi/6 < \arg z < \pi/2\}$ (the other possibilities being $\psi(\mathbb{H}) = e^{2\pi i/3}S$ and $\psi(\mathbb{H}) = e^{4\pi i/3}S$). We also set, for $z_0 \in \psi(\mathbb{H})$, $D(z_0) = \psi(\phi(z_0) + \mathbb{H})$. Then, if $r > 0$ is sufficiently large, it follows from the Lemma and Remarks a. and b. about the distribution of poles, that to any pole p of w in $D(re^{\pi i/6})$ there exists a pole $\phi(p')$ of Θ in $\phi(p) + \mathbb{H}$ with $|\phi(p') - \phi(p)| \leq 2R$, say. Hence $p' \in D(p) \subset D(re^{\pi i/6})$ is a pole of w satisfying $\frac{3}{2}|p'^{3/2} - p^{3/2}| \geq \frac{1}{2}|p|^{1/2}|p' - p|$, for r and thus $|p|$ sufficiently large, and this gives $|p' - p| \leq 4R|p|^{-1/2}$.

Since $D(p') \subset D(p) \subset D(re^{\pi i/6})$, this process may be repeated to obtain a sequence $\tilde{p}_1 = p, \tilde{p}_2 = \tilde{p}'_1, \tilde{p}_3 = \tilde{p}'_2, \dots$ of *different* poles¹ of w such that $|\tilde{p}_{n+1}| \leq |\tilde{p}_n| + O(|\tilde{p}_n|^{-1/2}) = |\tilde{p}_n|(1 + O(|\tilde{p}_n|^{-3/2}))$ as $n \rightarrow \infty$. This gives $|\tilde{p}_{n+1}| - |\tilde{p}_1| = O(\sum_{\nu=1}^n |\tilde{p}_\nu|^{-1/2})$ and $\log |\tilde{p}_{n+1}| - \log |\tilde{p}_1| = O(\sum_{\nu=1}^n |\tilde{p}_\nu|^{-3/2})$ for every $n \in \mathbb{N}$. The assertion of our theorem in case (II) now follows, since the same method applies to the open half-plane $i\mathbb{H}$ with associated sectors $e^{\pi i/3}S, -S$ and $e^{5\pi i/3}S$; then the domain $\bigcup_{\nu=0}^5 D(re^{(2\nu+1)\pi i/6})$ is some punctured neighbourhood of ∞ .

¹The crucial point was to prove that the construction leads to an *infinite* sequence of poles. It was Shimomura's paper [Sh3] which inspired me to compare Boutroux's method with re-scaling and so to overcome this difficulty.

The proof in case (IV) is almost the same, details will be omitted. We just note that, after some simple calculation, the re-scaling process $y_n(\mathfrak{z}) = p_n^{-1}w(p_n + p_n^{-1}\mathfrak{z})$ leads to the differential equation

$$\eta'_{rs}{}^2 = \eta^4 + 4\eta^3 + 4\eta^2 + c\eta,$$

with $|c|$ uniformly bounded. The degenerate cases correspond to the parameters $c = 0$, $c = 32/27$ and $c = \infty$, which by the substitution $\mathbf{u}(\mathfrak{z}) = a\eta(a\mathfrak{z})$, $a^3c = 1$, again reduces in the limit $c \rightarrow \infty$ to $\mathbf{u}'^2 = \mathbf{u}^4 + 1$. One also has to work with (the branches of) $\psi(\xi) = (2\xi)^{1/2}$ in the half-planes $\mathbb{H} : -\pi/4 < \arg \xi < 3\pi/4$, $i\mathbb{H}$, $-\mathbb{H}$ and $-i\mathbb{H}$. \square

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