SUB-NORMAL SOLUTIONS TO PAINLEVÉ’S SECOND DIFFERENTIAL EQUATION

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Abstract
In a recent paper, Aimo Hinkkanen and Ilpo Laine have shown that the transcendental solutions to Painlevé’s second differential equation
\[ w'' = \alpha + zw + 2w^3 \]
have either order of growth \( \varrho = 3 \) or else \( \varrho = \frac{3}{2} \). We complete this result by proving that there exist no sub-normal solutions \( \varrho = \frac{3}{2} \) other than the so-called Airy solutions.

Keywords. Normal families, Nevanlinna theory, Painlevé transcendents, Yosida function, Airy solutions, re-scaling method, Bäcklund transformations

2010 MSC. 30D30, 30D35, 30D45

1. Introduction and Main Results

The solutions to the second Painlevé differential equation
\[ [\Pi_\alpha] \]
are either rational or transcendental meromorphic functions of finite order. More precisely, the so-called second Painlevé transcendents have order of growth \( \varrho \) with \( \frac{3}{2} \leq \varrho \leq 3 \) (see Hinkkanen and Laine [7], Shimomura [10, 11] and the author [13, 14]). In a recent paper, Hinkkanen and Laine [8] proved that the order is either \( \varrho = 3 \) or else \( \varrho = \frac{3}{2} \). This result was commonly expected, but nevertheless marks a great breakthrough.

The aim of this paper is to describe the solutions of order \( \varrho = \frac{3}{2} \), called sub-normal, in more detail, and using this information to prove the main result on non-existence of non-Airy sub-normal solutions.

The description is intimately associated with the properties of the first integral
\[ W = w^4 + zw^2 + 2\alpha w - w'^2, \quad W' = w^2. \]
According to [8], the question whether or not \( w \) has order \( \varrho = \frac{3}{2} \) depends on the cluster set \( \mathcal{C}_\epsilon \) of the function \( W(z)z^{-2} \) as \( z \to \infty \) on \( \mathbb{C} \setminus \mathcal{P}_\epsilon \). Here \( \mathcal{P} \) denotes the set of non-zero poles of \( w \), and \( \mathcal{P}_\epsilon \) denotes the \( \epsilon \)-neighbourhood
\[ \mathcal{P}_\epsilon = \bigcup_{p \in \mathcal{P}} \Delta_\epsilon(p), \quad \Delta_\epsilon(p) = \{ z : |z - p| < \epsilon|p|^{-1/2} \}. \]

The sub-normal solutions are characterised by \( n(r, w) = O(r^{3/2}) \), where \( n(r, w) \) denotes the number of poles on \( |z| < r \), and either (i) \( \mathcal{C}_\epsilon = \{-1/4\} \) or else (ii) \( \mathcal{C}_\epsilon = \{0\} \) for some \( \epsilon > 0 \), and are called of the first and second kind, respectively. Special solutions of the first kind are the so-called Airy solutions, which occur for parameters \( \alpha \in \frac{1}{2} + \mathbb{Z} \) and are obtained by (repeated) application of the so-called Bäcklund transformations to the solutions to the special Riccati equations
\[ w' = \pm(z/2 + w^2). \]
Theorem 1. Equation $[\text{II}_\alpha]$ has no sub-normal solutions other than the Airy solutions, which occur for $\alpha \in \frac{1}{2} + \mathbb{Z}$.

The question whether or not zero may be a deficient value for any Painlevé transcendent is still open. From $\alpha/w = w''/w - z - 2w^2$ and $m(r, w) = O(\log r)$ follows $m(r, 1/w) = O(\log r)$ if $\alpha \neq 0$, hence the value zero is non-deficient. In case $\alpha = 0$ it is well-known and easily proved that $m(r, 1/w) = O(\log r)$ for any transcendental solution (see [3], Thm. 10.3). As a by-product of the Hinkkanen-Laine result and Theorem 1 we obtain

Corollary 1. For every solution to $w'' = zw + 2w^3$ the value zero is non-deficient.

The paper is organised as follows: In section 2 we introduce the re-scaling method developed in [13], which together with the Bäcklund transformations presented in section 6 constitutes the main tool. In sections 4, 5, and 8 the solutions of the first and second kind, respectively, are described in more detail in terms of the distribution of their poles and residues, while sections 7, 9, and 10 are devoted to the proofs of Theorem 1 and Corollary 1.

2. The Re-scaling Method

The re-scaling method was developed in [13] to prove the sharp order estimate $\varrho \leq 5/2$ for the solutions to Painlevé’s first equation $w'' = z + 6w^2$. It also applies to the second and fourth Painlevé equation (see [14]). In the present case, for any fixed solution to equation $[\text{II}_\alpha]$ the family $(w_h)_{|h| \geq 1}$ of re-scaled functions

$$w_h(z) = h^{-1/2}w(h + h^{-1/2}z)$$

is normal in the plane (in the sense of Montel). This will be proved immediately (see Proof of Normality).

Every limit function

$$w = \lim_{h_n \to \infty} w_{h_n}$$

satisfies

$$w'' = w + 2w^3,$$

hence also

$$w'^2 = w^4 + w^2 + c.$$

Convergence $w_{h_n} \to w$ is uniform on compact sets not containing poles of $w$, and poles of $w$ occur as accumulation points of poles of the functions $w_{h_n}$ only. The constant solutions to (3) are $w = 0$ and $w = \pm \sqrt{-1/2}$, while the non-constant solutions to (3) and (4) are either elliptic or trigonometric functions; the latter only occur in the exceptional cases $c = 1/4$ and $c = 0$:

$$w'^2 = (w^2 + 1/2)^2 : w = \pm \tan(\sqrt{2}/2 + \tau) \sqrt{2} \quad (c = 1/4)$$

$$w'^2 = w^2(w^2 + 1) : w = \pm i/\sin(i\sqrt{2} + \tau) \quad (c = 0).$$

For $c \neq 0, 1/4$ all solutions to equation (4) that occur as limit functions (2) of the re-scaling process are elliptic functions.

Poles. The nature of any solution is determined by the distribution of its poles. The set $\mathcal{P}$ of non-zero poles of some fixed solution of $[\text{II}_\alpha]$ is an infinite set, as follows from $m(r, w) = O(\log r)$; for notation and results of Nevanlinna Theory the reader
is referred to the monographs of Hayman [5] and Nevanlinna [9]. At any pole $p$ the Laurent series developments ($\eta = \text{res } w = \pm 1$)

\[
\begin{align*}
w(z) &= \eta(z - p)^{-1} - \frac{1}{2}\eta p(z - p) - \frac{1}{4}(\alpha + \eta)(z - p)^2 + h(z - p)^3 + \cdots, \\
W(z) &= -(z - p)^{-1} + 10\eta h - \frac{7}{36}p^2 - \frac{1}{4}p(z - p) - \frac{1}{2}(1 + \eta\alpha)(z - p)^2 + \cdots
\end{align*}
\]

hold; the coefficient $h$ remains undetermined. Pre-scribing $\eta \in \{1, -1\}$ and $h \in \mathbb{C}$ at $p$ uniquely determines a solution just like initial values $w_0$ and $w'_0$ at $z_0$ do. The series converge on some disc $\Delta_R(p)$, with $R > 0$ independent of $p$.

**Proof of normality.** Since normality of the family $(w_h)_{|h| \geq 1}$ has never been explicitly stated nor formally proved, we will give a proof here. It is based on the following results taken from [13, 14]. For any transcendental solution to equation \([\Pi_n]\) we have

(i) $w(z) = O(|z|^{1/2})$ and $w'(z) = O(|z|)$ as $z \to \infty$ outside $P$;

(ii) there exists $M > 0$ such that for every pole $p \neq 0$ the coefficients in the Laurent series expansion

\[
w(z) = \eta(z - p)^{-1} + \sum_{k=1}^{\infty} c_k(p)(z - p)^k \quad (\eta = \pm 1)
\]

satisfy $|c_k(p)| \leq M^k|p|^{(k+1)/2}$.

The first results was proved [14], Prop. 3.5, following the pattern used in [13] for Painlevé’s first transcendents, while the second easily follows from the key estimate

\[
|h| = O(|p|^2)
\]

for the free coefficient $h$ (also [14], Prop. 3.5). We also note that the equation

\[
w''_h = h^{-3/2}(\alpha + 3wh) + wh + 2w^3
\]

formally tends to equation (3) as $h \to \infty$. This, of course, does not say anything about convergence of any sequence $(w_{h_n})$ with $h_n \to \infty$, but nevertheless will be one key to the proof.

In the first step we shall prove normality of every sequence $(w_{h_n})$ with $h_n \to \infty$ in a neighbourhood of $z = 0$, and that every limit function solves equation (4) by considering three different cases as follows.

Case a. $|h_n|^{1/2}\text{dist}(h_n, P) \geq \epsilon$ for some $\epsilon > 0$. By property (i) above we may assume that $w_{h_n}(0) = h_n^{-1/2}w(h_n)$ and $w'_n(0) = h_n^{-1}w'(h_n)$ tend to $w_0$ and $w'_0$, respectively. Hence $w_{h_n}$ tends to the solution to equation (3) with initial values $w(0) = w_0$ and $w'(0) = w'_0$ on $|z| < \epsilon$, even with respect to the euclidian metric.

Case b. $(h_n) = (p_n)$ is a sequence of poles. Then

\[
w_{p_n}(\zeta) = \frac{\eta_n}{3} + \sum_{k=1}^{\infty} p_n^{-1/2}c_k(p_n)\zeta^k = \frac{\eta_n}{3} + \eta_3 \zeta \quad (\eta_n = \pm 1)
\]

holds, and $\eta_n$ is bounded by $\frac{M|3|}{1 - M|3|}$ on $|z| < M^{-1}$. We may assume that $\eta_n = \eta$ and $\eta_n \to \eta$, hence $w_{h_n}(\zeta) \to w(\zeta) = \eta_3^{-1} + \eta(3)$, again with respect to the euclidian metric, and again $w$ solves equation (3) on $|z| < M^{-1}$.

Case c. $h_n = p_n + o(|p_n|^{-1/2})$ for some sequence of poles $p_n$. Then

\[
w_{h_n}(\zeta) = (1 + o(|p_n|^{-3/2}))w_{p_n}(\zeta + o(|p_n|^{-3/2})),
\]

hence (some sub-sequence of) $w_{h_n}$ tends to some solution to (3), locally uniformly on $|z| < M^{-1}$, this time with respect to the spherical metric.
In the second step we shall prove normality of \((w_{k_n})\) in a neighbourhood of any point \(z_0\). To this end we set \(k_n = h_n + z_0 h_n^{-1/2}\) and obtain
\[
\begin{align*}
w_{k_n}(\zeta) &= (k_n - z_0 h_n^{-1/2}) - 1/2 w(k_n + h_n^{-1/2} (z - z_0)) \\
&= (1 + O(\|k_n\|^{-3/2})) w_{k_n}(z_0 + O(\|k_n\|^{-3/2})),
\end{align*}
\]
locally uniformly with respect to \(z\). Since by the first part the sequence \((v_{k_n})\) defined by \(v_{k_n}(\zeta) = w_{k_n}(\zeta + O(\|k_n\|^{-3/2}))\) is normal on a neighbourhood of \(\zeta = 0\), the sequence \((w_{k_n})\) is normal on a neighbourhood of \(z_0\), and every limit function satisfies equation (3). The case \(h_n \to h_0 \in \mathbb{C}\) is trivial and of no interest.

The cluster set \(\mathcal{C}_\varepsilon\) consists of all limits \(\lim_{h_n \to \varepsilon} h_n^{-2} W(h_n)\) with \(h_n \notin \mathcal{P}_\varepsilon\); it is closed and connected (as always), and also bounded by a constant only depending on \(\varepsilon\), see [14], Prop. 3.5.

**Proposition 1.** The cluster set \(\mathcal{C}_\varepsilon = \mathcal{C}_\varepsilon\) does not depend on \(\varepsilon\). Every limit
\[
\lim_{p_n \to \infty} [10 \eta_n, h_n - \frac{7}{36} p_n^2] p_n^{-2} \quad (\eta_n = \text{res } w),
\]
where \((p_n)\) denotes any appropriate sequence of poles, also belongs to \(\mathcal{C}_\varepsilon\). Conversely, any limit \(\lim_{h_n \to \varepsilon} h_n^{-2} W(h_n)\) with \(0 < a \leq |h_n|\) dist\((h_n, \mathcal{P})\) \(b < \infty\) coincides with some limit (6). If, however, \(|h_n|\) dist\((h_n, \mathcal{P})\) \(\to \infty\), then \(h_n^{-2} W(h_n)\) tends to either 0 or else \(-1/4\).

**Proof.** The assertions are consequences of the following observation. If the limit (2) exists and solves (4), and if \((k_n)\) denotes any sequence such that \(|h_n|\) dist\((h_n, \mathcal{P})\) \(\to \infty\), then there exists \(\epsilon > 0\) such that for any pole \(p\) satisfying \(|p| > \epsilon\) the poles of \(w_{k_n}\) converge to \(w(z_0 + p)\) which solves the same differential equation as does \(w\). In case of \(|h_n|\) dist\((h_n, \mathcal{P})\) \(\to \infty\) the sequence \((w_{k_n})\) tends to \(w = \pm \sqrt{-1/2}\), hence \(h_n^{-2} W(h_n)\) tends to 0 or else \(-1/4\).

We note explicitly \(\lim_{p \to \infty} 180 p^{-2} \text{res } w = \begin{cases} -1 & \text{if } \mathcal{C}_\varepsilon = \{-1/4\} \\ 7/2 & \text{if } \mathcal{C}_\varepsilon = \{0\} \end{cases}\), while the solutions to \(w' = \pm (z/2 + w^2)\) satisfy \(180 p^{-2} \text{res } w = -1\). To describe the global distributions of poles of the second Painlevé transcendents of order \(\varrho = 3/2\) we need the following result on the local distribution of poles; it is based on the distribution of poles of the limit functions \(w = \lim_{p_n \to \infty} w_{p_n}\) \((p_n \in \mathcal{P})\).

**Proposition 2.** Suppose that \(w\) solves \([\Pi_\alpha]\) and has order of growth \(\varrho = 3/2\). Then given \(\epsilon > 0\) and \(\theta > 0\) there exists \(r_0 > 0\) such that for any pole \(p\) satisfying \(|p| > r_0\) the poles of \(w\) on \(\Delta\) \((p)\) \(\{z : |z - p| < R|p|^{-1/2}\}\) may be labelled in such a way that \(p_0 = p\) and
\[
\begin{align*}
&|p_k - (p + k \sqrt{2} \pi p^{-1/2})| < \epsilon |p|^{-1/2} \quad (-k_1 \leq k \leq k_2) \\
&\text{(first kind)} \\
&|p_k - (p + k \pi i p^{-1/2})| < \epsilon |p|^{-1/2} \quad (-k_1 \leq k \leq k_2), \quad \text{(second kind)}
\end{align*}
\]
respectively, hold.

The proof is an immediate implication of the re-scaling method and the known distribution of poles of the solutions (5) to the special re-scaled differential equation.
To determine the asymptotics of the solutions of order \( q = \frac{3}{2} \) more precisely, we shall repeatedly apply the following estimates; the first one is an immediate corollary of the Cauchy integral formula.

**Lemma 1.** Suppose that \( f \) is holomorphic on some sector \( S : a < \arg z < b \) and satisfies \( f(z) = O(|z|^\lambda) \) as \( z \to \infty \) on \( S \). Then \( f^{(k)}(z) = O(|z|^\lambda-k) \) as \( z \to \infty \) holds on every smaller sector \( S(\delta) : a + \delta < \arg z < b - \delta \).

**Lemma 2.** Let \( (p_k) \) be any complex sequence \( (0 < |p_1| \leq |p_2| \leq \cdots \leq |p_k| \to \infty) \) with counting function \( n(r) = \text{card} \{p_k : |p_k| \leq r\} = O(r^\varrho) \) and \( \varrho \notin \mathbb{N} \). Then \( |z-p_k| \geq \kappa \max\{|z|, |p_k|\} \) for some \( \kappa > 0 \) and every \( k \) implies

\[
\sum_{k=1}^{\infty} \left| \frac{z^h}{(z-p_k)p_k^h} \right| = O(|z|^{\varrho-1}) \quad (z \to \infty, \ h = [\varrho]).
\]

**Proof.** From \( n(r) = O(r^\varrho) \) and \( \left| \frac{z^h}{(z-p_k)p_k^h} \right| \leq \kappa^{-1} r^{\varrho-1} \min \{1, \frac{r}{|p_k|} \} \) on \( |z| = r \) follows

\[
\sum_{k=1}^{\infty} \left| \frac{z^h}{(z-p_k)p_k^h} \right| \leq \kappa^{-1} r^{\varrho-1} \int_{|p_1|}^{r} \frac{dn(t)}{t^h} + \kappa^{-1} r^{\varrho} \int_{r}^{\infty} \frac{dn(t)}{t^{h+1}} = O(r^{\varrho-1}).
\]

3. Solutions in the Yosida Class

If the cluster set \( CL \) contains none of the values \( 0, -1/4 \), then all limit functions (2) are non-constant (actually elliptic functions of Jacobi type), hence \( w \) belongs to the Yosida Class \( \mathcal{Y}_{2,\frac{1}{4}} \), being defined and discussed in [15]. These solutions are traditionally called non-truncated (see Boutroux [1, 2]). Among others it follows that \( T(r, w) \asymp r^3 \) and that the poles are regularly distributed: there exist real numbers \( \epsilon_0 > 0 \) and \( \eta_0 > 0 \), and \( M \in \mathbb{N} \), such that every disc \( \Delta_{\eta_0}(z_0) \) contains at least one pole, and \( \Delta_{\epsilon_0}(z_0) \) contains at most \( M \) poles. This holds in a modified form if the cluster set is restricted to some sector \( S = \{z : \theta_1 \leq \arg z \leq \theta_2\} \): the poles on \( S \) are regularly distributed, and again \( T(r, w) \asymp r^3 \) holds.\(^1\)

4. Solutions of the First Kind

Throughout this section \( w \) will denote any sub-normal transcendent of the first kind.

**Strings of poles of the first kind.** A string in the truncated sector

\[
S_0' : |\arg z| \leq \pi/3, \ \Re z \geq r_0 > 0
\]

is a sequence \((p_k)_{k=0,1,2,...}\) such that (we assume \( \Re p_k^{-1/2} > 0 \))

\[
p_{k+1} = p_k + \sqrt{2}\pi p_k^{-1/2}(1 + o(1)) \quad (k \to \infty);
\]

\( p_0 \) is called the root of the string \((p_k)_{k=0,1,2,...}\).

**Proposition 3.** For \( r_0 > 0 \) sufficiently large, every pole \( p_0 \) in \( S_0' \) is the root of some uniquely determined string of poles \( \sigma = (p_k)_{k=0,1,2,...} \) contained in \( S_0' \). It has the following properties:

a. \( w \) has constant residues on \( \sigma \);

\(^1\) The symbol \( \asymp \) has proved very useful, and will be used extensively throughout this paper: \( \phi(z) \asymp \psi(z) \) on some real or complex region means \( |\phi(z)| = O(|\psi(z)|) \) and \( |\psi(z)| = O(|\phi(z)|) \).
Similarly, we have to show that \( p_{k+1} \in S_0' \). Writing \( p_k = |p_k|e^{i\theta_k} \) it follows that

\[
\text{Re} p_{k+1} > \text{Re} p_k + |p_k|^{-1/2}(\sqrt{2}\pi \cos(\theta_k/2) - \epsilon_1) \\
\geq \text{Re} p_k + (\sqrt{6\pi}/2 - \epsilon_1)|p_k|^{-1/2} > \text{Re} p_k + 3|p_k|^{-1/2}.
\]

Similarly,

\[
|\text{Im} p_{k+1}| \leq |\text{Im} p_k| + |p_k|^{-1/2}(1 - \sqrt{2}\pi|\sin(\theta_k/2)|) < |\text{Im} p_k| + |p_k|^{-1/2}\epsilon_1
\]
holds. With the help of \( \frac{a+b}{c+d} \leq \max \left\{ \frac{a}{c}, \frac{b}{d} \right\} \) for \( a, b, c, d > 0 \) we obtain

\[
(7) \quad |\theta_{k+1}| = \arctan \frac{|\text{Im} p_{k+1}|}{|\text{Re} p_{k+1}|} \leq \arctan \max \left\{ \frac{|\text{Im} p_k|}{|\text{Re} p_k|}, \frac{\epsilon_1}{3} \right\} \leq \max \{|\theta_k|, \epsilon_1\}. 
\]

It is obvious that \( \text{Re} p_k \to \infty \) monotonically, and that the sequence \( |\text{Im} p_k| \) decreases as long as \( |\theta_k| \geq \epsilon_1 \). From \( |\theta_k| \leq \frac{|\text{Im} p_k|}{|\text{Re} p_k|} \), however, follows that \( |\theta_{k'}| < \epsilon_1 \)
for some \( k'_1 \), hence \( |\theta_k| < \epsilon_1 \) for \( k \geq k'_1 \) follows from (7). If we denote by \( k_n \) the first index such \( \text{Re} p_k > r_n \), then the above argument shows that there exists some \( k'_n \geq k_n \) such that \( |\theta_k| < \epsilon_n \) holds for \( k > k'_n \). This yields b. To prove c. we consider the conjugate sequence \( \tilde{q}_k = p_k^{-3/2} \). From \( p_{k+1} = p_k + \sqrt{2}\pi p_k^{-1/2} + o(|p_k|^{-1/2}) \) follows \( q_{k+1} = q_k + \frac{3}{2}\sqrt{2}\pi k + o(k) \),

\[
p_k = \left( \frac{3}{2}\sqrt{2}\pi \right)^{2/3} k^{2/3} + o(k^{2/3}) \quad \text{and} \quad n(r) = \frac{\sqrt{2}}{2\pi} r^{3/2} + o(r^{3/2}).
\]

REMARK. For \( p-1 \in S_0' \) the string just constructed may be uniquely extended “to the left” such that \( (p_k)_{k=k_0} \in S_0' \) and \( p_{k_0} \notin S_0' \). Relabelling this string we may assume that \( (p_k)_{k=0,1,2,...} \in S_0' \), but \( p-1 \notin S_0' \). Such a string is called maximal.

There is just one step from local to global distribution of poles.

PROPOSITION 4. Let \( w \) be any second Painlevé transcendent of the first kind. Then up to finitely many the poles of \( w \) form a finite number \( \ell(w) \) of maximal strings with total counting function \( n(r, w) = \ell(w) \frac{\sqrt{2}}{2\pi} r^{3/2} + o(r^{3/2}) \), and such that the following is true:

a. \( w \) has constant residues on each string \( \sigma \);

b. each string \( \sigma \) is asymptotic to one of the rays \( \arg z = 0 \) and \( \arg z = \pm \frac{\pi}{2} \);

c. each string \( \sigma \) is accompanied by a string \( (q_k) \) of zeros

\[
q_k = p_k + \frac{\sqrt{2}}{2}\pi p_k^{-1/2} + o(|p_k|^{-1/2});
\]

d. any two strings \( \sigma, \tilde{\sigma} = (\tilde{p}_k) \) are separated from each other, i.e.,

\[
\lim_{k \to \infty} |p_k|^{1/2} \text{dist}(p_k, \tilde{\sigma}) = \lim_{k \to \infty} |\tilde{p}_k|^{1/2} \text{dist}(\tilde{p}_k, \sigma) = \infty.
\]
Furthermore, \( w \) has Nevanlinna characteristic \( T(r, w) = \ell(w) \frac{\sqrt{2}}{\pi} r^{3/2} + o(r^{3/2}) \), and satisfies \( w(z) \sim \sqrt{-z/2} \) as \( z \to \infty \) on every sector \( S_0''(\delta) \) : \( |\arg z - \pi| < \frac{\pi}{3} - \delta \) and \( S_{\pm 1}(\delta) = e^{\pm 2\pi i/3} S_0''(\delta) \), for some suitably chosen branch of the square-root depending on the sector.\(^2\)

**Proof.** From \( n(r, w) = O(r^{3/2}) \) follows that there exists only a finite number \( \ell(w) \) of strings of poles. Thus \( n(r, w) = \ell(w) \frac{\sqrt{2}}{\pi} r^{3/2} + o(r^{3/2}) \) holds, and \( m(r, w) = O(\log r) \) implies \( T(r, w) = \ell(w) \frac{\sqrt{2}}{\pi} r^{3/2} + o(r^{3/2}) \). The existence and position of the zeros \( \eta_k \) follows from the fact that the limit function \( w = \tan(\sqrt{2}r)/\sqrt{2} \) has zeros and poles at \( z = k\sqrt{2}\pi \) and \( z = (k + \frac{1}{2})\sqrt{2}\pi \), respectively. Also given \( R > 0 \) there exists \( k_0 \in \mathbb{N} \) such that for \( k \geq k_0 \) the disc \( \Delta_R(p_k) \) contains poles of the string \( \sigma \) only, hence \( |p_k|^{1/2} \text{dist}(p_k, \sigma) \geq R \) for any string \( \sigma \neq \sigma \). Finally, the asymptotics for \( w \) follows from the fact that the re-scaling process for any sequence \( (h_n) \) with \( |h_n|^{1/2} \text{dist}(h_n, \mathcal{P}) \to \infty \) leads to the limit functions \( w = \sqrt{-1/2} \).

**Series expansion.** In [14] it was shown that for every second transcendent

\[
 w(z) = w(0) + \lim_{r \to \infty} \sum_{|p| \leq r} \frac{\eta(p)z}{(z - p)p} \quad (\eta(p) = \text{res} w)
\]

holds, provided \( w(0) \neq \infty \); if \( w \) has a pole at \( z = 0 \), the term \( w(0) \) has to be replaced by \( \eta(0)/z \). In our case the above Mittag-Leffler expansion exists not only as a Cauchy principal value but converges absolutely. Then also

\[
 W(z) = Q(z) - \frac{|\eta(0)|}{z} - \sum_{p \in \mathcal{P}} \frac{z}{(z - p)p}
\]

holds, where \( Q \) is a polynomial of degree at most two (see [14], Thm. 4.3). Lemma 2 applies to \( W - Q \), and from \( |W(z) - Q(z)| = O(|z|^{1/2}) \) as \( z \to \infty \) on each sector \( S_0''(\delta) \) and \( \mathcal{C}L = \{-1/4\} \) then follows \( Q(z) = -\frac{1}{4}z^2 + a_1 z + a_0 \). Also on each sector \( S_0''(\delta) \) we have

\[
 z w^2 = z W' = -\frac{1}{4}z^2 + a_1 z + O(|z|^{1/2})
\]

\[
 w^4 = \frac{1}{4}z^2 - a_1 z + O(|z|^{1/2}),
\]

hence \( z w^2 + w^4 - W = -a_1 z + O(|z|^{1/2}) \) and \( w^2 - 2\alpha w = O(|z|^{1/2}) \) yield \( a_1 = 0 \). We have thus proved

**Proposition 5.** Any first kind transcendent \( w \) satisfies

\[
 w = \sqrt{-z/2} + O(|z|^{-1})
\]

as \( z \to \infty \) in every sector \( S_0''(\delta) \) (for some branch of \( \sqrt{-z/2} \)),

\[
 W = -\frac{1}{4}z^2 + O(|z|^{1/2}) \quad \text{and} \quad Q(z) = -\frac{1}{4}z^2 + a_0.
\]

\(^2\) \( \phi(z) \sim \psi(z) \) as \( z \to \infty \) on some real or complex region means \( \phi(z)/\psi(z) \to 1 \)(\( z \to \infty \)).
5. Solutions of the Second Kind

Now \( w \) will denote any sub-normal solution of the second kind. Again (8) holds, where now \( \deg Q \leq 1 \) follows from \( \mathbf{CL} = \{0\} \) and Lemma 2. Since by [14], Thm. 4.5, the order of \( w \) is \( q \geq 2 \) (hence \( q = 3 \)) if \( \deg Q = 1 \), \( Q \) is a constant.

A string of poles \( (p_k)_{k=0,1,2,...} \) in the sector
\[
S''_0 : |\arg z - \pi| < \pi/3, \quad \Re z < -r_0,
\]
is now characterised by the condition (\( \Im p_k^{-1/2} > 0 \))
\[
p_{k+1} = p_k + i\pi p_k^{-1/2} + o(|p_k|^{-1/2}) \quad (k = 0, 1, 2, \ldots);
\]
it is called maximal in \( S''_0 \) if \( p_{-1} \notin S''_0 \). Similarly we define strings of poles in the sectors \( S''_{\pm 1} = e^{\pm 2\pi i/3} S''_0 \), and obtain the following analog to Proposition 4.

**Proposition 6.** Let \( w \) be any sub-normal solution of the second kind. Then up to finitely many the poles of \( w \) form a finite number \( \ell(w) \) of maximal strings with total counting function \( n(r, w) = \ell(w) \frac{2}{3\pi} r^{3/2} + o(r^{3/2}) \), and such that the following is true:

- the residues on each string alternate, i.e., \( \text{res } w = -\text{res } p_k, p_{k+1} \);
- each string \( \sigma \) is asymptotic to one of the rays \( \arg z = \pi \) and \( \arg z = \pm \frac{1}{3} \pi \);
- any two strings \( (p_k) \) and \( (\tilde{p}_k) \) are separated from each other.

The analog to Proposition 5 for second kind solutions is

**Proposition 7.** Any second kind solution to equation [II\( \alpha \)] satisfies
\[
w(z) = -\alpha/z + O(|z|^{-7/4}) \quad \text{and} \quad W = -\alpha^2/z + O(|z|^{-7/4})
\]
as \( z \to \infty \) on every sector \( S_0(\delta) : |\arg z| < \frac{\pi}{3} - \delta \) and \( S_{\pm 1}(\delta) = e^{\pm 2\pi i/3} S_0(\delta) \).

**Proof.** From \( W = O(|z|^{1/2}) \) follows \( w^2 = W' = O(|z|^{-1/2}) \), \( w' = O(|z|^{-9/4}) \) and \( \alpha + zw = w'' - 2w^3 = O(|z|^{-3/4}) \), hence
\[
\begin{align*}
    w &= -\alpha/z + O(|z|^{-7/4}) \\
    W' &= w^2 = \alpha^2/z^2 + O(|z|^{-11/4}) \\
    W &= b_0 - \alpha^2/z + O(|z|^{-7/4}).
\end{align*}
\]
From \( W = w^4 + zw^2 + 2\alpha w - w^2 = O(|z|^{-1}) \) then follows \( b_0 = 0 \).

6. Bäcklund Transformations

The so-called *Airy solutions* are obtained from the solutions to any of the special Riccati equations (1) by successive application of so-called Bäcklund transformations. Generally speaking, a *Bäcklund transformation* is a change of variables that transforms equation [II\( \alpha \)] into itself or into some equation [II\( \alpha_1 \)] with different parameter. Simple examples are \( w_1(z) = -w(z) (\alpha_1 = -\alpha) \) and \( w_1(z) = \mu w(\mu z) (\mu^3 = 1, \alpha_1 = \alpha) \). More sophisticated Bäcklund transformations are
\[
(10) \quad w_1 = -w - \frac{\alpha + 1/2}{w' + w^2 + z/2} \quad \text{and} \quad w_{-1} = -w + \frac{\alpha - 1/2}{w' - w^2 - z/2},
\]
which change \( \alpha \) to \( \alpha_1 = \alpha + 1 \) and \( \alpha_1 = \alpha - 1 \), respectively. It is obvious that the Bäcklund transformations (10) preserve the order \( q \), and by Propositions 4 and 6 even preserve the first and second kind solutions.
A special Bäcklund transformation. In [3] the authors describe the connexion between equations \([\Pi_0]\) and \([\Pi_1]\). If \(y\) is a non-trivial solution to \([\Pi_0]\), then

\[
w(z) = -\frac{d}{dz} \log y(-2^{-1/3}z)
\]

solves \([\Pi_1]\) and is not an Airy solution \((w' \neq z/2 + w^2)\); conversely, if \(w\) solves \([\Pi_1]\) and is not an Airy solution, then the function \(y\), being defined locally by

\[
-2^{1/3}y^2(-2^{-1/3}z) = w'(z) - z/2 - w^2(z)
\]

is a non-trivial solution to \([\Pi_0]\). The poles and zeros of \(y\) correspond to poles of \(w\) with residues 1 and \(-1\), respectively. The Airy solutions to \([\Pi_1]\) correspond to the trivial solution \(y = 0\). It is obvious that the order of growth is preserved. Moreover follows from (12) \(2^{2/3}Y(-2^{-1/3}z) = w(z) - z^2/4 - W(z) + \text{const} (Y' = y^2)\), hence

\[
\mathcal{CL}(y) = -\mathcal{CL}(w) - \frac{1}{4}.
\]

Thus if \(w\) is of the first and second kind, then \(y\) is of the second and first kind, respectively, and vice versa.

7. Proof of Theorem 1: Second Kind Solutions

From the Mittag-Leffler expansion (8) for \(W\) \((Q\) a constant) follows \(W = -h'/h\), with \(h\) an entire functions of order \(g = 3/2\). Proposition 7 thus gives \(h'/h = \alpha^2/z + O(|z|^{-7/4}) (z \to \infty)\) on every sector \(S_k^\prime(\delta) : |\arg z - 2k\pi/3 < \pi/3 - \delta (-1 \leq k \leq 1)\), hence

\[
h(z) = O(|z^{\alpha^2}|).
\]

Since \(h\) has finite order, this also holds true on the remaining sectors \(\Sigma_k(\delta) : |\arg z - \pi - 2k\pi/3 < \delta (-1 \leq k \leq 1)\) by the Phragmén-Lindelöf principle. Then however, \(h\) has to be a polynomial, this contradiction showing that second kind solutions do not exist.

8. The Distribution of Residues

Let \(w\) be any first kind transcendental. We shall denote by \(n_{\oplus}(r)\) and \(n_{\ominus}(r)\) the counting function of poles with residues 1 and \(-1\), respectively. If the circle \(|z| = r\) contains no poles, then

\[
\frac{1}{2\pi i} \int_{|z|=r} w(z) \, dz = n_{\oplus}(r) - n_{\ominus}(r) = (\ell_{\oplus}(w) - \ell_{\ominus}(w)) \frac{\sqrt{2}}{3\pi} r^{3/2} + o(r^{3/2})
\]

holds, where \(\ell_{\oplus}(w)\) and \(\ell_{\ominus}(w)\) count the number of maximal strings with residues +1 and \(-1\), respectively. We choose \(\delta > 0\) sufficiently small and replace any arc of \(|z| = r\) that intersects some disc \(\Delta_\delta(p)\) by a suitable sub-arc of \(\partial \Delta_\delta(p)\) to obtain a simple closed curve \(\Gamma_r\). Then also

\[
\frac{1}{2\pi i} \int_{\Gamma_r} w(z) \, dz = n_{\oplus}(r) - n_{\ominus}(r)
\]

holds. If \(\gamma_r\) and \(\gamma_r'\) denote the part of \(\Gamma_r\) in \(|\arg z| < \delta\) and \(\delta \leq \arg z \leq \frac{2}{3}\pi - \delta\), respectively, then from \(|w(z)| = O(|z|^{1/2})\) on \(\gamma_r\) follows

\[
\left| \frac{1}{2\pi i} \int_{\gamma_r} w(z) \, dz \right| < K\delta r^{3/2},
\]

where \(K\) is a constant depending on \(\delta\).
Let already known, we obtain non-existence of first kind non-Airy solutions from (13):

\[ \frac{1}{2\pi i} \int_{\gamma_L} w(z) \, dz = \frac{1}{2\pi i} \int_{\gamma_L} \sqrt{-z/2} \, dz + o(r^{3/2}) = \mu \frac{\sqrt{2}}{3\pi} r^{3/2} + O(\delta r^{3/2}), \]

holds with \( \mu = \pm 1 \) depending on the branch of \( \sqrt{-z/2} \). This yields

**Proposition 8.** Any sub-normal solution of the first kind either satisfies

\[ |\ell_\oplus(w) - \ell_\ominus(w)| = 1 \quad \text{or else} \quad |\ell_\ominus(w) - \ell_\ominus(w)| = 3. \]

**Remark.** The following results deduced from Proposition 8 for the solutions to the Riccati equation \( w' = z/2 + w^2 \) are well known, see, e.g. [4]. Since all residues equal \(-1\) we have \( \ell_\ominus(w) = 0 \), hence either \( \ell_\ominus(w) = 1 \) or else \( \ell_\ominus(w) = 3 \). There exist three distinguished solutions \( w_1, w_2(z) = e^{2\pi i/3}w_1(z e^{2\pi i/3}), \) and \( w_3(z) = e^{-2\pi i/3}w_1(z e^{-2\pi i/3}) \) with \( \ell_\ominus(w_p) = 1 \). The labelling is chosen in such a way that \( w_1(z) \sim \psi(z) = \sqrt{-z/2} \) with \( \text{Im} \, \psi(z) > 0 \) holds on \( 0 < \arg z < 2\pi \). By symmetry and uniqueness, the poles of \( w_1 \) are real and positive. For any solution \( w_0 \neq w_k \) we have

\[ w_0(z) \sim \begin{cases} 
\psi(z) & (0 < \arg z < \frac{\pi}{2}) \\
-\psi(z) & (\frac{\pi}{2} < \arg z < \frac{3\pi}{2}) \\
\psi(z) & (\frac{3\pi}{2} < \arg z < 2\pi).
\end{cases} \]

9. Proof of Theorem 1: First Kind Solutions

Let \( w \) be any sub-normal solution of the first kind to \([\Pi_a] \). We first assume \( 2\alpha \in \mathbb{Z} \), and may restrict ourselves to the cases \( \alpha = 0 \) and \( \alpha = 1/2 \). Since \( \mathcal{CL}(w_a) \neq \{0\} \) is already known, we obtain non-existence of first kind non-Airy solutions from (13): \( \mathcal{CL}(w_0) = -\mathcal{CL}(w_{1/2}) - \frac{1}{4} \), hence \( \mathcal{CL}(w_{1/2} + \alpha) = \mathcal{CL}(w_{1/2}) \neq \{ -\frac{1}{4} \} \).

For the rest of the proof we thus shall assume \( 2\alpha \notin \mathbb{Z} \) and set

\[ V = w' + w^2 + z/2, \quad w_1 = -w - \frac{\alpha + 1/2}{V}, \quad \text{and} \quad \Delta(w) = \ell_\oplus(w) - \ell_\ominus(w). \]

Then \( w_1 \) solves \([\Pi_{a+1}] \), and the poles \( \oplus \) and \( \ominus \) of \( w \) and \( w_1 \), and the zeros \( \ominus \) of \( V \) that are not poles of \( w \) are related as follows:

\[ \begin{array}{ll}
\oplus & \text{res } w = -1 \Rightarrow V(p) = \infty \text{ (doubly) and res } w_1 = 1 \\
\ominus & \text{res } w = 1 \Rightarrow V(p) = 0 \text{ and res } w_1 = 0 \\
\ominus & \text{res } w = 0 \text{ and } V(p) = 0 \Rightarrow \text{res } w_1 = -1,
\end{array} \]

this leading to the following picture (\( w \) to the left and \( w_1 \) to the right):

\[ \begin{array}{cccccccc}
\oplus & \ominus & \oplus & \ominus & \ominus & \oplus & \oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus
\end{array} \]

(15)

First of all we obtain \( \ell_\ominus(w_1) = \ell_\ominus(w) \) from (15), while

\[ m(r, 1/V) \leq m(r, w_1) + m(r, w) + O(1) = O(\log r), \]
hence
\[ N(r, 1/V) = N(r, V) + O(\log r) = 2N_{\ominus}(r, w) + O(\log r) \]
\[ = N_{\ominus}(r, w) - [N_{\ominus}(r, w) - N_{\ominus}(w, r)] + N_{\ominus}(r, w) + O(\log r) \]
\[ = N_{\ominus}(r, w_1) + N_{\ominus}(r, w) + O(\log r) \]
implies
\[ \ell_{\ominus}(w_1) = \ell_{\ominus}(w) - \Delta(w) \text{ and } \Delta(w_1) = \Delta(w). \]
Repeated application (see [14], Thm. 6.2), and Nevanlinna’s First Main Theorem we obtain (again with
\[ m(r, 1/w) = N(r, w) - N(r, 1/w) + O(\log r) \]
\[ = N_{\ominus}(R, w_1) - N_{\ominus}(R, w_1) + O(\log r) = O(r^{3/2}), \]
and this requires $\Delta(w) \leq 0$ for any $2\alpha \notin \mathbb{Z}$ and any solution of the first kind, hence
\[ \Delta(w) \leq -1 \text{ by Proposition 8}. \]
Replacing $w$ by $-w$ and $\alpha$ by $-\alpha$, however, gives $\Delta(-w) = -\Delta(w) \geq 1$; this contradiction proves Theorem 1 completely. \[ \square \]

10. PROOF OF COROLLARY 1

We have just to consider solutions to equation [IIb] of order $\varrho = 3$. From the special
Bäcklund transformation $w_1(z) = -\frac{d}{dz} \log w(-2^{-1/3}z)$, the estimate
\[ N_{\ominus}(r, w_1) - N_{\ominus}(r, w) = O(r^{3/2}) \]
(see [14], Thm. 6.2), and Nevanlinna’s First Main Theorem we obtain (again with
\[ R = 2^{-1/3}r \]
\[ m(r, 1/w) = O(\sqrt{T(r_k, w)}) \text{ holds on some sequence } r_k \to \infty. \]
Thus zero is non-deficient for every non-non-trivial solution to $w'' = zw + 2w^3$. \[ \square \]

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