

Comments on “Jordan and Julia”

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As Jochen Becker has pointed out to me, the formulation of Theorem 3 in the paper [1] mentioned in the title was incorrect, since one essential assumption was omitted. The proof, however, was correct, since it used explicitly that assumption. The statement I had in mind is as follows:

Theorem 3 *Let \mathcal{J} be a Jordan arc, and assume that $U = \mathcal{F}$ is a parabolic basin (Leau domain) whose corresponding parabolic fixed point is an endpoint of \mathcal{J} . Then \mathcal{J} is a quasiconformal Jordan arc, i.e., the image of the interval $[0, +\infty]$ under a quasiconformal map ϕ of the sphere. Moreover, the conjugation $f \circ \phi = \phi \circ f_0$ holds on $\mathcal{J}_{f_0} = [0, +\infty]$ and even on some neighbourhood of $(0, +\infty) \setminus f_0^{-1}(\{0\})$, and $d^2 f_0$ is analytically conjugate to T_d :*

$$f_0 = d^{-2} S^{-1} \circ T_d \circ S, \quad S(z) = \frac{1-z}{1+z}. \quad (1)$$

Theorem 3 is actually false without the additional assumption that the parabolic fixed point is an endpoint of \mathcal{J} . It is not hard to determine the quasiconformal representatives in the other case— \mathcal{J} is connected (a Jordan arc), and the Fatou set \mathcal{F} consists of a parabolic basin with fixed point not an endpoint of \mathcal{J} . Then \mathcal{J} is not quasiconformal.

To do this we first look for an odd rational function g of degree $d > 2$ with the following properties:

- (a) $g(0) = 0$ and $g(\infty) = \infty$ or $g(\infty) = 0$.
- (b) g has two critical points $\pm a$ (multiplicity $d - 1$) and critical values $\pm A$.
- (c) g has parabolic fixed points at $z = \pm 1$.

Then $f(z) = (g(\sqrt{z}))^2$ represents a class of quasiconformally conjugated rational functions whose Julia sets are Jordan arcs, and whose Fatou sets are parabolic basins, such that the corresponding parabolic fixed points are not endpoints of \mathcal{J} .¹

¹We note that the condition (a') $g(0) = \infty$ and $g(\infty) = 0$ instead of (a) leads to

We set

$$\frac{A - g(z)}{A + g(z)} = c \left(\frac{a - z}{a + z} \right)^d$$

to fulfill condition (b). Then $g(0) = 0$ if and only if $c = 1$, while $g(\infty) = \infty$ is equivalent with $c = (-1)^{d+1}$ and $g(\infty) = 0$ is equivalent with $c = (-1)^d$, so that (the only possible choice) $c = 1$ leads to $g(\infty) = \infty$ if d is odd and $g(\infty) = 0$ if d is even. In any case g is an odd function.

Condition (c) is equivalent with

$$\frac{A - 1}{A + 1} = \left(\frac{a - 1}{a + 1} \right)^d \quad (2)$$

and

$$\frac{A}{(A + 1)^2} = \frac{da}{(a + 1)^2} \left(\frac{a - 1}{a + 1} \right)^{d-1}. \quad (3)$$

Now (2) can be solved for A ,

$$A = \frac{1 + \left(\frac{a-1}{a+1} \right)^d}{1 - \left(\frac{a-1}{a+1} \right)^d}, \quad (4)$$

and hence (c) is fulfilled for each a satisfying

$$(a + 1)^{2d} - (a - 1)^{2d} = 4da(a^2 - 1)^{d-1}, \quad a \neq 0, \quad (5)$$

provided A is defined by (4). Then

$$f(z) = \left(g(\sqrt{z}) \right)^2$$

is rational of degree d , its Julia set is a Jordan arc with endpoints $z = 0$ and $z = \infty$, and f has a parabolic fixed point at $z = 1$, which is an "inner point" of the Jordan arc \mathcal{J} . We note, however, that this is only possible for $d \geq 4$, since, for $d = 3$, equation (5) has no (non-zero) solution. For $d = 4$, (5) is equivalent with $5a^4 + 2a^2 + 1 = 0$, while for $d = 5$ we obtain $5a^4 + 6a^2 + 5 = 0$.

Reference

1. N. Steinmetz: Jordan and Julia, Math. Annalen **307**, 531-541 (1997)

similar results, but works only in the case d odd. The simplest case is $d = 3$, where we obtain $g(z) = \frac{1}{z} \frac{\bar{a}z^2 - 1}{z^2 - a}$ and $f(z) = \frac{1}{z} \left(\frac{\bar{a}z - 1}{z - a} \right)^2$ with $a = 1 + i\sqrt{8}$.