Sierpiński Curve Julia Sets of Rational Maps

by

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Abstract. In this note we prove that the so-called Sierpiński holes in the parameter plane $0 < |\lambda| < \infty$ of the McMullen family $F_\lambda(z) = z^m + \lambda/z^\ell$ ($m \geq 2$ and $\ell \geq 1$ fixed) are simply connected, and determine the total number of these domains.

2000 Mathematics Subject Classification: 37F10, 37F15, 37F45

Key words and phrases: Julia set, quasiconjugation, quasiconformal mapping, Mandelbrot set, McMullen family, Sierpiński hole, Sierpiński curve.

1. The one-parameter family

$$F_\lambda(z) = z^m + \lambda/z^m,\ \lambda \neq 0, \tag{1}$$

($m \geq 2$ fixed), has recently been the object of a series of papers [1, 2, 4, 5, 6, 7, 9, 11, 12, 22]. In [3, 10, 8, 25] certain aspects of the more general family

$$F_\lambda(z) = z^m + \lambda/z^\ell,\ \lambda \neq 0, \tag{2}$$

($m \geq 2$ and $\ell \geq 1$ fixed) are discussed. This family is *quasiconjugate* to the family

$$f_\lambda(z) = \lambda^{m-1} z^m (1 + 1/z)^d,\ d = \ell + m, \tag{3}$$

via

$$f_\lambda(z) = \tilde{f}_\lambda(\lambda z)/\lambda \ \text{ and } \ \tilde{f}_\lambda(z^d) = F_\lambda(z)^d.$$

Besides the critical points $z = \infty$, $z = 0$ (if $\ell > 1$), and $z = -1$ (with critical orbit $-1 \mapsto 0 \mapsto \infty$), f_λ has the *free critical point* $z = \ell/m$ with *free critical value*

$$\xi_\lambda = \frac{d^d}{\ell^\ell m^m} \lambda^{m-1},$$

and as is always the case, the free critical orbit determines the dynamics. For notation and results in complex dynamics the reader is referred to [18] or [24].

2. The rational map f_λ has the super-attracting immediate basin about infinity, denoted $\mathcal{A}_\lambda$. The parameter plane $\mathbb{C}^* = \mathbb{C} - \{0\}$ is divided into sets

$$\mathbf{\Omega}_0 = \{\lambda : \ell/m \in \mathcal{A}_\lambda\},$$

$$\mathbf{\Omega}_N = \{\lambda : f_\lambda^N(\ell/m) \in \mathcal{A}_\lambda\} - \mathbf{\Omega}_{N-1},\ N \in \mathbb{N},$$

and

$$\mathbf{\Omega}_\infty = \mathbb{C} - \bigcup_{N\ge 0} \mathbf{\Omega}_N.$$

Due to the Theorem of Mañé, Sad, and Sullivan [14], the sets $\mathbf{\Omega}_N$ are open, and $\mathbf{\Omega}_\infty$ is compact (even in $\mathbb{C} - \{0\}$). We write also $Q_N(\lambda) = f_\lambda^N(\ell/m)$, hence $Q_0(\lambda) = \ell/m$, $Q_1(\lambda) = \dfrac{d^d}{\ell^\ell m^m}\lambda^{m-1}$, and

$$Q_N(\lambda) = \lambda^{m-1}\frac{(Q_{N-1}(\lambda)+1)^d}{Q_{N-1}(\lambda)^\ell},\ N \ge 2.$$

To count the so-called Sierpiński holes we need

Lemma 1 *For $\ell \ge 2$, $m \ge 2$, and $N \ge 1$ we have*

$$\deg Q_N = (m-1)d^{N-1},$$

while, for $\ell = 1$ and $m \ge 2$,

$$\deg Q_N = (d-2)[1 + d^2 + \cdots + d^{N-1}] \text{ if } N \text{ is odd},$$

and

$$\deg Q_N = (d-2)[d + d^3 + \cdots + d^{N-1}] \text{ if } N \text{ is even}.$$

Proof. It is obvious that Q_N is a rational function of λ^{m-1}, thus $Q_N(\lambda) = R_N(\lambda^{m-1})$, with $R_1(x) = \frac{d^d}{\ell^\ell m^m}x$,

$$R_N(x) = x\frac{(R_{N-1}(x)+1)^d}{R_{N-1}(x)^\ell},\ N \ge 2,$$

and $\deg Q_N = (m-1)\deg R_N$. For $\ell \ge 2$ and $N \ge 2$, R_N has a pole at the origin and at the point at infinity. Counting the zeros of R_N, hence the zeros of $R_{N-1}+1$, easily shows that $\deg R_N = d \deg R_{N-1}$, thus $\deg R_N = d^{N-1}$.

For $\ell = 1$ the situation is slightly more complicated, since then R_N is regular at $x = 0$, with a simple zero and $R_N'(0) > 0$ if N is odd, and $R_N(0) > 0$ if N is even. Again counting zeros yields

$$\begin{aligned} \deg R_N &= d \deg R_{N-1} && (N \text{ even}) \\ \deg R_N &= 1 + d \deg R_{N-1} && (N \text{ odd}), \end{aligned}$$

and an easy computation gives

$$\begin{array}{rcll} \deg R_{2N+1} & = & 1+d^2+\cdots+d^{2N}, & \text{and} \\ \deg R_{2N+2} & = & d+d^3+\cdots+d^{2N+1} & \text{for } N\geq 1. \quad \blacksquare \end{array}$$

3. Prior results on the McMullen family can be best traced back from either [10] or else [25], see also the list in the first section.

- $\mathbf{\Omega}_0 \cup \{\infty\}$ is a simply connected domain; the Julia set of any corresponding f_λ is totally disconnected (a *Cantor set*), and f_λ acts on $\mathcal{J}_\lambda$ like the *Bernoulli shift* acts on the space of d symbols (see [10]). The sequence of super-attracting immediate basins $\mathfrak{B}_N(\infty)$ of Q_N tends to its *kernel* $\mathbf{\Omega}_0 \cup \{\infty\}$ in the sense of Carathéodory (see [25]).

- $\mathbf{\Omega}_1$ is empty.

- $\mathbf{\Omega}_2$ is empty for $1/\ell + 1/m \geq 1$, while $\mathbf{\Omega}_2 \cup \{0\}$ is a simply connected domain otherwise; in this case the corresponding Julia sets are "Cantor sets of circles", more precisely: for $\lambda \in \mathbf{\Omega}_2$ the Julia set of f_λ consists of uncountably many *quasicircles* about the origin (see [15]). For $\ell = m \geq 3$ the boundary of $\mathbf{\Omega}_2 \cup \{0\}$ is a simple closed curve (see [4]).

- $\mathbf{\Omega}_n$ ($n \geq 3$) is non-empty and consists of finitely many connected components, called *Sierpiński holes*. For $\ell = m \geq 2$ the Sierpiński holes are simply connected, and each contains exactly one zero of the rational map $\lambda \mapsto f_\lambda^{N-1}(1) = f_\lambda^{N-2}(4^m\lambda^{m-1})$ (see [22], where these domains are called *capture zones*, and [6], with slightly different notation; Figure 2 in [22] does not display the correct distribution of the Sierpiński holes or capture zones, denoted $\mathcal{H}_2$ there). The Julia set $\mathcal{J}_\lambda$ is a *Sierpiński curve*, for the definition see section 5.

- $\mathbf{\Omega}_\infty$ contains infinitely many copies $\mathcal{M}_2^{[n]}$ of the Mandelbrot set (see [5, 6], [17] in a general situation, and [25]); here $^{[n]}$ means that the parameters of the main cardioid of $\mathcal{M}_2^{[n]}$ correspond to rational maps with (super-) attracting cycles of length n.

- for $|\lambda| < \lambda^* = \sqrt[m-1]{\dfrac{m^{m-1}\ell^{\ell+1}}{d^d}}$ the boundary of $\mathcal{A}_\lambda$ is a *quasicircle* (see [4] for $\ell = m \geq 2$, and [25] in the general case). Note that $|\xi_\lambda| < \ell/m$ if and only if $|\lambda| < \lambda^*$, and $\xi_{\lambda^*} = \ell/m$, hence f_λ has a finite super-attracting fix-point, if and only if $\lambda = \omega\lambda^*$ and $\omega^{m-1} = 1$.

Remark. In case $\ell = m$ the proofs depend significantly on explicit computations and constructions, which are not possible otherwise. We mention just one point: for $\ell = m$ the rational map f_λ satisfies $f_\lambda(1/z) = f_\lambda(z)$, hence its Julia set is symmetric with respect to the unit circle, and the immediate basin $\mathcal{A}_\lambda$ is either invariant under the inversion $z \mapsto 1/z$ ($\lambda \in \mathbf{\Omega}_0$), or else its pre-image under f_λ is also its pre-image under $z \mapsto 1/z$. This may explain why so many of the papers cited above deal with the symmetric case, although it is likely that the results will remain true in the general case.

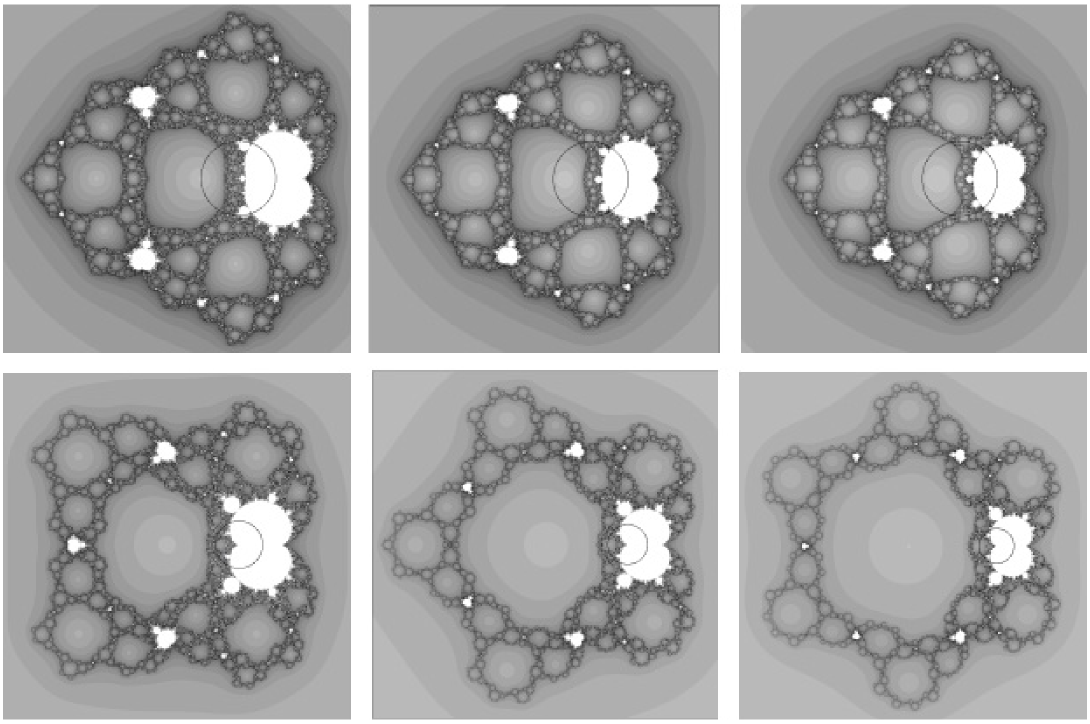


Figure 1 The parameter-c-planes ($c = \lambda^{m-1}$) for $m = 2$, $\ell = 2, 3, 4$, (top, left to right), and $\ell = 1$, $m = 3, 4, 5$, (bottom, left to right), with circles $|c| = (\lambda^*)^{m-1}$.

4. Böttcher's function Φ_λ is the solution of *Böttcher's functional equation*

$$\Phi_\lambda \circ f_\lambda(z) = \lambda^{m-1}\Phi_\lambda(z)^m,$$

normalised by $\Phi_\lambda(z) = z + O(1)$ at $z = \infty$. For $\lambda \notin \mathbf{\Omega}_0$, $\mathcal{A}_\lambda$ is simply connected and Φ_λ is given by

$$\Phi_\lambda(z) = \lim_{n\to\infty} \sqrt[m^n]{f_\lambda^n(z)/\lambda^{(m-1)(1+m+\cdots+m^{n-1})}}\,,$$

mapping $\mathcal{A}_\lambda$ conformally onto $|w| > |\lambda|^{-1}$, so that

$$\lambda\Phi_\lambda(z) = \lim_{n\to\infty} \sqrt[m^n]{f_\lambda^n(z)}$$

maps $\mathcal{A}_\lambda$ conformally onto $\Delta = \{w : |w| > 1\}$.

We will come now to the first result of this note.

Theorem 1 *Every connected component of $\mathbf{\Omega}_N$, $N \geq 3$, is simply connected, contains exactly one pole of the rational map Q_N, and*

$$\Xi_N(\lambda) = \lambda\Phi_\lambda(f_\lambda^{N-1}(\xi_\lambda)) = \lambda\Phi_\lambda(f_\lambda^N(\ell/m)) = \lim_{n\to\infty} \sqrt[m^n]{Q_{N+n}(\lambda)}$$

is a proper map of this component Ω onto $\{w : |w| > 1\}$, with a single pole of order $d\ell = \deg \Xi_N$. Every branch of $\Xi_N^{1/d\ell}$ maps Ω conformally onto Δ.

Remark. The pole of Q_N contained in Ω is a zero of Q_{N-1}. Since the zeros of Q_{N-1} in $\mathbb{C}^* = \mathbb{C} - \{0\}$ are d-fold (see the proof of Lemma 2), and since Q_{N-1} has a zero at $\lambda = 0$ if N is even, we obtain on combination with Lemma 1, when applied to Q_{N-1}:

Corollary 1 *For $\ell > 1$ there are exactly $(m-1)d^{N-3}$ Sierpiński holes in the parameter plane, while for $\ell = 1$ this number is $(d-2)[1 + d^2 + \cdots + d^{N-3}]$ if $N \geq 3$ is odd, and $(d-2)[d + d^3 + \cdots + d^{N-3}]$ if $N \geq 4$ is even.*

Remark. In the symmetric case, Theorem 1 and Corollary 1 are due to Roesch [22] ($\ell = m = 2$) and Devaney [6] ($\ell = m \geq 3$), both working with the family (F_λ) and the corresponding Böttcher functions

$$\Psi_\lambda(z) = (\lambda\Phi_\lambda(z^d/\lambda))^{1/d}.$$

The map used in [6] (and called *Roesch-map* there) is a modification of the map used in [22], and is given by

$$\lambda \mapsto \Psi_\lambda(H_\lambda(F_\lambda^{N-1}(v_\lambda))), \tag{4}$$

where $v_\lambda = 2\sqrt{\lambda}$ is one of the two critical values and $H_\lambda(z) = \lambda^{1/m}/z$. Since $F_\lambda \circ H_\lambda = F_\lambda$ only holds for $\ell = m$, this construction makes no sense for $\ell \neq m$. Of course, for $\ell = m$ the maps $\Xi_N^{1/d\ell}$ and (4) agree, since the parameter planes are the same for (F_λ) and (f_λ).

Proof of Theorem 1. It is easily shown that Ξ_N is a proper map of Ω onto Δ (just apply the Douady-Hubbard technique, which may be anticipated from [13]). Suppose that Q_N has $s+1$ poles in Ω, with multiplicities $\mu_0, \ldots, \mu_s$. Then $\Xi_N : \Omega \to \Delta$ has $\deg \Xi_N = \sum_{j=0}^{s} \ell\mu_j$, and $\deg \Xi_N - (s+1) + r$ critical points, where r is the number of critical points of Ξ_N which are not poles. Then the Riemann Hurwitz formula yields

$$\#(\Omega) - 2 = (1-2)\deg \Xi_N + \deg \Xi_N - (s+1) + r = r - s - 1,$$

where $\#(\Omega)$ denotes the connectivity number of Ω. In particular, for $r = 0$ follows $\#(\Omega) = 1 - s \leq 1$, hence $s = 0$ and $\#(\Omega) = 1$. To finish the proof we have thus to prove the following:

Lemma 2 *Ξ_N is locally univalent in $\Omega - \{\text{poles of } Q_N\}$, and has poles of order $d\ell$ at poles of Q_N.*

The idea of **proof** is the same as in [22]. Fix $\lambda_* \in \Omega - \{\text{poles of } Q_N\}$, with free critical value $\xi_* = \dfrac{d^d}{\ell^\ell m^m}\lambda_*^{m-1}$ of f_{λ_*}. We assume that the disc $|\xi - \xi_*| < 3\epsilon$ belongs to the Fatou component of f_{λ_*} containing ξ_*, and denote with (η_ξ) any family of diffeomorphisms $\eta_\xi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, which depends analytically on ξ in the disc $|\xi - \xi_*| < \epsilon$, and satisfies

$$\eta_\xi(w) = \begin{cases} w & \text{in } |w - \xi_*| \geq 3\epsilon \\ w - \xi_* + \xi & \text{in } |w - \xi_*| < \epsilon. \end{cases}$$

Then

$$g_\xi = \eta_\xi \circ f_{\lambda_*}$$

is a K-*quasiregular* map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, and analytic outside the closed set $f_{\lambda_*}^{-1}(A)$, with $A = \{w : \epsilon \leq |z - \xi_*| \leq 3\epsilon\}$. Since $f_{\lambda_*} = g_\xi$ outside $f_{\lambda_*}^{-1}(A)$, and since the sets $f_{\lambda_*}^{-n}(A)$ are mutually disjoint, each of the iterate g_ξ^n is also K-quasiregular (g_ξ is also called *uniformly K-quasiregular*), and by Shishikura's *qc-Lemma* [23], g_ξ is quasiconformally conjugate to some rational function

$$R_\xi(z) = h_\xi \circ g_\xi \circ h_\xi^{-1}.$$

If we normalise the quasiconformal map h_ξ to fix the points $-1, 0, \infty$, then h_ξ, as well as R_ξ, depends analytically on ξ. We note that h_ξ is regular in the open set $\mathbb{C} - \overline{\bigcup_{n\in\mathbb{N}} f_{\lambda_*}^{-n}(A)}$, which, in particular, contains $\mathcal{A}_{\lambda_*}$ and the points $-1, 0, \ell/m, \xi_*$. By Lemma 3 below we have

$$R_\xi = f_{\lambda(\xi)}$$

and

$$\lambda(\xi)\Phi_{\lambda(\xi)} = \lambda_*\Phi_{\lambda_*} \circ h_\xi^{-1},$$

where $\xi \mapsto \lambda(\xi)$ is analytic in some disc about ξ_*. Since the free critical point ℓ/m of both f_λ and f_{λ_*} is fixed by h_ξ, it follows that $h_\xi(\xi) = \xi_{\lambda(\xi)}$, hence

$$\Xi_N(\lambda(\xi)) = \lambda_*\Phi_{\lambda_*}(f_{\lambda_*}^{N-1}(\xi)),$$

holds. Thus, if it can be shown that

$$\xi \mapsto \lambda_*\Phi_{\lambda_*}(f_{\lambda_*}^{N-1}(\xi)) \;\; \text{is univalent in a neighbourhood of } \xi_*,$$

and

$$\xi \mapsto \lambda(\xi) \;\; \text{is nonconstant,}$$

one may conclude that λ_* is not a critical point of Ξ_N.

Now the first map is univalent near ξ_* if and only if λ_* is not a pole of Q_N (arising from a zero of Q_{N-1}). If, however, λ_* is a zero of Q_{N-1}, then $f_{\lambda^*}^{N-3}$ is univalent at $z = \xi_*$ (with $f_{\lambda^*}^{N-3}(\xi_*) = -1$), f_{λ^*} has degree d at $z = -1$, and degree ℓ at $z = 0$, so that Q_N has a pole of degree $d\ell$ at ξ_*.

Finally, if the second map $\xi \mapsto \lambda(\xi)$ were constant, then so were $\xi \mapsto h_\xi = h$. This, however, would lead via $f_{\lambda_*} \circ h = f_{\lambda(\xi)} \circ h_\xi = h_\xi \circ g_\xi = h \circ g_\xi$ to the contradiction $g_\xi = h^{-1} \circ f_{\lambda_*} \circ h$. ■

Lemma 3 *$R_\xi = f_\lambda$ for some analytic map $\lambda = \lambda(\xi)$, and $\lambda\Phi_\lambda = \lambda_*\Phi_{\lambda_*} \circ h_\xi^{-1}$ in $\mathcal{A}_\lambda$.*

Proof. By hypothesis, R_ξ has a pole of order ℓ at $z = 0$, a pole of order m at $z = \infty$, and a critical point of order $d - 1$ at $z = -1$. Hence

$$R_\xi(z) = P(z)z^{-\ell},\ P \text{ a polynomial of degree } d \text{ with } P(0) \neq 0,$$

and

$$R'_\xi(z) = z^{-\ell-1}[zP'(z) - \ell P(z)] = mcz^{-\ell-1}(z+1)^{d-1}(z - z_0),$$

with $c \neq 0$ is some constant and $z_0 \neq 0, -1$ is the free critical point. Equating the coefficients of z^ℓ on both sides of

$$zP'(z) - \ell P(z) = mc(z+1)^{d-1}(z - z_0)$$

then yields $\binom{d-1}{\ell-1} - z_0\binom{d-1}{\ell} = 0$, hence $z_0 = \ell/m$, and

$$R_\xi(z) = cz^m(1 + 1/z)^d = f_{\sqrt[m-1]{c}}(z).$$

The last assertion follows from $g_\xi = f_{\lambda_*}$ in $\mathcal{A}_{\lambda_*}$,

$$\begin{aligned}(\lambda_*/\lambda)\Phi_{\lambda_*} \circ h_\xi^{-1} \circ f_\lambda &= (\lambda_*/\lambda)\Phi_{\lambda_*} \circ f_{\lambda_*} \circ h_\xi^{-1} \\ &= (\lambda_*/\lambda)\lambda_*^{m-1}(\Phi_{\lambda_*} \circ h_\xi^{-1})^m \\ &= \lambda^{m-1}((\lambda_*/\lambda)\Phi_{\lambda_*} \circ h_\xi^{-1})^m,\end{aligned}$$

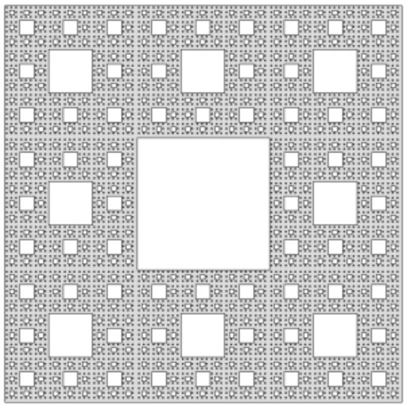
in $\mathcal{A}_\lambda$, with $\lambda = \lambda(\xi)$, and uniqueness of the Böttcher function. ■

5. A Sierpiński curve is a non-empty, compact, connected, locally connected, and nowhere dense subset of the complex plane, with complementary domains bounded by simple closed and mutually disjoint curves. By a theorem of Whyburn [26] any two Sierpiński curves are homeomorphic. The classical construction may be described as follows: Let $\mathcal{H}$ be the space of non-empty compact subsets of the plane, endowed with the *Hausdorff metric,* and define a map $\Phi : \mathcal{H} \to \mathcal{H}$ by the following procedure. Set $\phi_{\mu\nu} = \frac{1}{3}(z + \mu + i\nu)$, $\mu, \nu \in \{0, 1, 2\}$, and

$$\Phi(C) = \bigcup_{(\mu,\nu)\neq(1,1)} \phi_{\mu\nu}(C),\ C \in \mathcal{H}.$$

Then Φ is a contraction on the complete metric space $\mathcal{H}$, and its unique fix-point is the Sierpiński curve. We note that the usual construction of the *Sierpiński triangle* does not lead to a Sierpiński curve, since different complementary domains may and will have common boundary points.

Figure 2 An Approximation to the classical Sierpiński curve.



6. Sierpiński curve Julia sets were first discovered by Milnor and Tan Lei [19], and, almost at the same time, also by Pilgrim [21]. Only in 2003 it was observed by Devaney et al [1, 2, 3, 7, 9] that Sierpiński curve Julia sets occur for rational maps F_λ with parameter $\lambda \in \bigcup_{N\geq 3} \mathbf{\Omega}_N$, but not limited to that case. Morosawa [20] considered the map

$$R_{Morosawa}(z) = \frac{27z^2(z-1)}{(3z-2)^2(3z+1)} \tag{5}$$

with critical orbit $2/3 \Rightarrow \infty \Rrightarrow 1 \to 0 \Leftrightarrow 0$; the dynamical plane is displayed in Figure 3 below. Using the following result combined with an argument, which is generalised in Lemma 5, he was able to show that the Julia set is a Sierpiński curve.

Lemma 4 (**[20]**, Lemma 6) *Let R be a sub-hyperbolic rational function with attracting immediate basin A. If there exists a domain E, complementary to $\overline{A}$, which contains some Fatou component D and also $R^{-1}(D)$, then the boundary of R is a simple closed curve.*

Remark. In Morosawa's example A is the immediate basin about 0, D and $R^{-1}(D)$ are the Fatou components containing 1 and ∞, respectively, and E is the component of $\widehat{\mathbb{C}} - A$ containing 1. For $\lambda \in \mathbf{\Omega}_N$, $N \geq 2$, Lemma 4 applies to the map $R = f_\lambda$ (as well as F_λ) with $A = \mathcal{A}_\lambda$, D and $R^{-1}(D)$ being the Fatou components containing 0 and -1, respectively, and E being the component of $\mathbb{C} - \overline{\mathcal{A}_\lambda}$ containing 0. A combination of Lemma 4 and Lemma 5 below then yields immediately:

Theorem [2, 3, 10] *For $\lambda \in \mathbf{\Omega}_N$, $N \geq 2$, the boundary of $\mathcal{A}_\lambda$ (for f_λ and F_λ) is a simple closed curve, and for $\lambda \in \mathbf{\Omega}_N$, $N \geq 3$, the Julia set of f_λ (and F_λ) is a Sierpiński curve.*

A couple of years ago I considered (unpublished and for other reasons) the rational map

$$R_{\rm mod}(z) = \frac{4}{27}\frac{(1-z+z^2)^3}{z^2(1-z)^2} \tag{6}$$

(its dynamical plane is displayed in Figure 3), which is well-known from the theory of modular functions: it relates the *absolute invariant* $J(\tau) = \dfrac{g_2^3}{g_2^3 - 27g_3^2}$, a modular function invariant under the *modular group*, to the *elliptic modular function* $\lambda(\tau) = (e_1, e_2, e_3, \infty)$, which is invariant under the *congruence subgroup modulo 2* via $J = R_{\rm mod} \circ \lambda$. The map $R_{\rm mod}$ has the critical orbit relation

$$\begin{array}{cccccc} & 2 & & c_+ & & \\ & \downarrow\downarrow & & \downarrow\downarrow\downarrow & & \\ -1 \Rightarrow & 1 & \Rightarrow \infty \Leftrightarrow \infty \Leftarrow & 0 & & \text{with } c_\pm = (1 \pm i\sqrt{3})/2, \\ & \uparrow\uparrow & & \uparrow\uparrow\uparrow & & \\ & 1/2 & & c_- & & \end{array}$$

from which it is easily seen that Lemma 4 (with A the immediate basin about ∞, D the Fatou component about 0 (or else 1), and E the component of $\widehat{\mathbb{C}} - A$ containing 0 (or else 1)), and Lemma 5 apply. We note that $R_{\rm mod}$ is invariant under the *factor group* $\Gamma/\Gamma(2)$, so that $\mathcal{J}$ has a lot of symmetries.

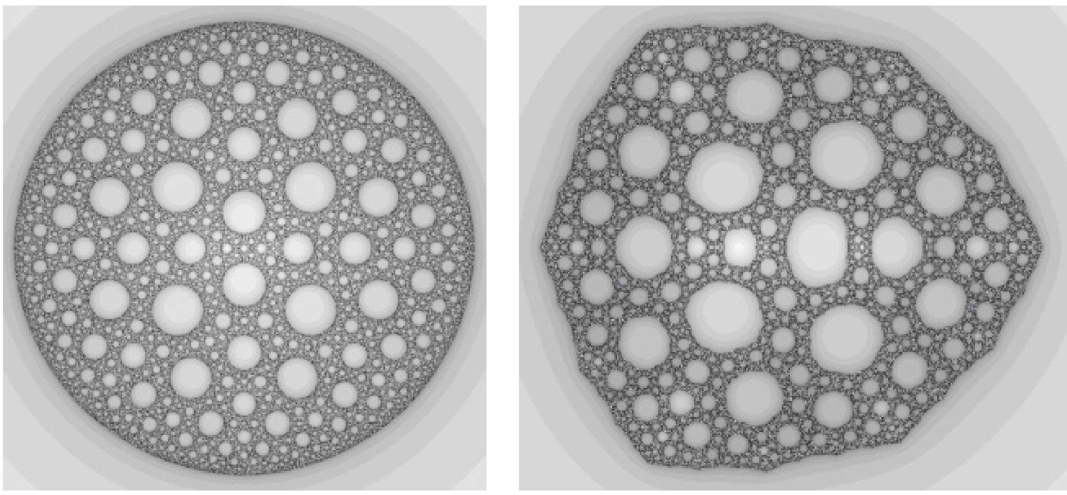


Figure 3 The dynamical planes for the maps $R_{\rm mod}$ (left) and $R_{Morosawa}$ (right). The Julia sets are Sierpiński curves.

Lemma 5 *Let R be rational and D any domain bounded by a simple closed curve. If ∂D is free of critical values of R, then any two different pre-images of D have disjoint closures.*

Proof. Since ∂D is locally connected, every boundary point of D is *accessible,* i.e., to every $z_0 \in \partial D$ there exists an arc $\gamma : [0,1) \to D$, such that $\lim_{t\to 1} \gamma(t) = z_0$ exists. If D' and D'' are pre-images of D under R with common boundary point z_0, let γ be any such arc in D ending at $R(z_0)$. Then γ has two lifts under R, $\gamma' \subset D'$ and $\gamma'' \subset D''$, both ending at z_0 (this is not necessarily true if ∂D is just a closed curve). Since $R'(z_0) \neq 0$, this contradicts $R(\gamma') = R(\gamma'') = \gamma$. ■

References

1. P. Blanchard, R. L. Devaney, D. M. Look, P. Seal, and Y. Shapiro, *Sierpiński curve Julia sets and singular perturbations of complex polynomials*, preprint 2003.

2. P. Blanchard, R. L. Devaney, D. M. Look, M. Morena Rocha, P. Seal, S. Siegmund, and D. Uminsky, *Sierpiński carpets and gaskets as Julia sets of rational maps*, preprint 2003.

3. R. L. Devaney, *A myriad of Sierpiński curve Julia sets*, preprint 2005.

4. R. L. Devaney, *Structure of the McMullen domain in the parameter planes for rational maps,* Fundamenta Math. **185** (2005), 267-285.

5. R. L. Devaney, *Baby Mandelbrot sets adorned with halos in families of rational maps*, preprint 2005.

6. R. L. Devaney, *The McMullen domain: Satellite Mandelbrot Sets and Sierpiński holes*, preprint 2005.

7. R. L. Devaney and D. M. Look, *Symbolic dynamics for a Sierpiński curve Julia set*, preprint 2004.

8. R. L. Devaney and D. M. Look, *Buried Sierpiński curve Julia sets*, preprint 2005.

9. R. L. Devaney and D. M. Look, *A criterion for Sierpiński curve Julia sets for rational maps*, preprint 2005.

10. R. L. Devaney, D. M. Look, and D. Uminsky, *The escape trichotomy for singularly pertubed rational maps*, preprint 2005.

11. R. L. Devaney, M. Morena Rocha, and S. Siegmund, *Rational maps with generalized Sierpiński gasket Julia sets*, preprint 2004.

12. R. L. Devaney and S. M. Moretta, *The McMullen domain: rings around the boundary*, preprint 2005.

13. A. Douady and J. H. Hubbard, *Itération des polynômes quadratiques*, C. R. Acad. Sci. Paris **294** (1982), 123-126.

14. R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. Éc. Norm. Sup. **16** (1983), 193-217.

15. C. McMullen, *Automorphisms of rational maps*, in *Holomorphic Functions and Moduli I,* Math. Sci. Res. Inst. Publ. 10, Springer 1988.

16. C. McMullen, *Complex Dynamics and Renormalization*, Princeton University Press, Princeton 1994.

17. C. McMullen, *The Mandelbrot set is universal*, in *The Mandelbrot set, theme and variations*, Cambridge University Press, Lond. Math. Soc. Lect. Note **274** (2000).

18. J. Milnor, *Dynamics in One Complex Variable*, Vieweg Verlag, Braunschweig/Wiesbaden 1999.

19. J. Milnor and Tan Lei, *A "Sierpiński carpet" as Julia set*, Appendix F in *Geometry and dynamics of quadratic rational maps*, Experiment. Math. **2** (1993), 37-83.

20. S. Morosawa, *Julia sets of subhyperbolic rational functions,* Complex Variables **41** (2000), 151-162.

21. K. M. Pilgrim, *Cylinders for iterated rational maps,* Thesis, Berkeley 1994.

22. P. Roesch, *On captures for the family* $f_\lambda(z) = z^2 + \lambda/z^2$, to appear.

23. M. Shishikura, *On the quasiconformal surgery of the rational functions,* Ann. Éc. Norm. Sup. **20** (1987), 1-29.

24. N. Steinmetz, *Rational iteration. Complex Analytic Dynamical Systems,* Verlag Walter de Gruyter, Berlin 1993.

25. N. Steinmetz, *On the dynamics of the McMullen Family* $R(z) = z^m + \lambda/z^\ell$, preprint 2005.

26. G. T. Whyburn, *Topological characterization of the Sierpiński curve*, Fund. Math. **45** (1958), 320-324.

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