COMPLEX RICCATI DIFFERENTIAL EQUATIONS REVISITED

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Abstract. We utilise a new approach via the so-called re-scaling method to derive a thorough theory for polynomial Riccati differential equations in the complex domain.

1. Introduction

The basic features concerning the value distribution of the solutions to Riccati differential equations

\[ w' = a_0(z) + a_1(z)w + a_2(z)w^2 \]

with polynomial coefficients are well understood due to the pioneering work of Wittich (see his book [15], Chapter V, pp. 73–80). The solutions are meromorphic in the complex plane, and every non-rational solution has order of growth

\[ \rho = \limsup_{r \to \infty} \frac{\log T(r, w)}{\log r} = 1 + n/2 \]

mean type, where the non-negative integer \( n \) depends on the coefficients \( a_\nu \) only. The aim of this paper is to refine the results of Wittich and others (Bank [1], Gundersen [5], Hellerstein and Rossi [7, 8]; see also Laine’s book [9], Chapter 5) on equation (1) and the associated linear differential equation (set \( a_2w = -u'/u \))

\[ u'' - \left( \frac{a_2'(z)}{a_2(z)} + a_1(z) \right)u' + a_0(z)a_2(z)u = 0 \]

by a new approach which has been developed earlier to investigate the solutions of Painlevé differential equations (see [12]). By a simple change of variables (retaining the original notation \( z, w \)) we obtain

\[ (R) \quad w' = a(z) - w^2 \]

with

\[ a(z) = z^n + O(|z|^{n-1}) \quad (z \to \infty). \]

Up to finitely many, all poles are simple with residue 1; \( w \) has counting function

\[ n(r, w) = O(r^\rho). \]

Our proofs are solely based on the estimate (4), a new existence proof for asymptotic expansions, and the method of re-scaling.

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2. Re-scaling and the distribution of poles

Throughout the whole paper \( w \) denotes any non-rational solution to the Riccati equation (R). For \( h \neq 0 \) we set
\[
w_h(z) = h^{-n/2}w(h + h^{-n/2}z),
\]
where \( h^{-n/2} \) denotes any branch, the same at every occurrence (\( h^{-n/2}h^{-n/2} = h^{-n} \)).

**Theorem 1.** The re-scaled family \( \{w_h|_{|h|>1}\} \) is normal in the sense of Montel, and every limit function \( w = \lim_{h \to \infty} w_h \) satisfies the differential equation
\[
w' = 1 - w^2.
\]

We note that the solution \( w = \coth z \) with pole at the origin has the poles \( k\pi i, \ k \in \mathbb{Z} \), and no others. Any sequence \( \sigma = (p_k) \) satisfying the approximate recursion
\[
p_{k+1} = p_k + \omega p_k^{-n/2} + o(|p_k|^{-n/2})
\]
with \( \omega = \pm i\pi \) fixed is called a string.

**Theorem 2.** Let \( w \) be any solution to (R). Then the set of poles on \( |z| > r_0 \) consists of finitely many strings of poles. Each string \( \sigma \) accumulates at some Stokes ray
\[
s_\nu: \arg z = \theta_\nu = \frac{(2\nu + 1)\pi}{n + 2}
\]
and has counting function
\[
n(r, \sigma) = \frac{\nu^\sigma}{\pi q} + o(r^q).
\]

**Remark.** We note that \( w \) has **Nevanlinna characteristic** \( T(r, w) = \ell \frac{r^q}{\pi q^2} + o(r^q) \),
where \( \ell = \ell(w) \) denotes the number of strings of poles.

3. Stokes sectors and asymptotic expansions

The open sectors
\[
S_\nu: \left| \arg z - \frac{2\nu\pi}{n + 2} \right| < \frac{\pi}{n + 2}
\]
are called **Stokes sectors**. They are bounded by the Stokes rays \( s_\nu \) and \( s_{\nu-1} \), and will be enumerated as follows:

(a) \( 0 \leq \nu \leq n + 1 \) if \( n \) is even, and
(b) \( -m - 1 \leq \nu \leq m + 1 \) if \( n = 2m + 1 \) is odd.

In the second case \( s_{-m-2} = s_{m+1} \) coincides with the negative real axis.

Let \( f \) be meromorphic on some sector \( S: \phi_1 < \arg z < \phi_2 \). Then \( f \) is said to have the **asymptotic expansion** \( f \sim \sum_{k=0}^{\infty} c_k z^{-k/q} \) for some \( q \in \mathbb{N} \), if for every \( \delta > 0 \) and every \( n \in \mathbb{N} \)
\[
f(z) - \sum_{k=0}^{n} c_k z^{-k/q} = o(|z|^{-n/q}) \quad (z \to \infty)
\]
is valid, uniformly on every sub-sector \( S(\delta): \phi_1 + \delta < \arg z < \phi_2 - \delta \). Obviously, the sector \( S \) is ‘pole-free’ for \( f \) in the following sense: to every \( \delta > 0 \) there exists \( r(\delta) > 0 \), such that \( f \) has no poles on \( S(\delta), \ |z| > r(\delta) \). It follows from Theorem 2 that the Stokes sectors \( S_\nu \) are ‘pole-free’ for every solution to equation (R). By \( \sqrt{z} \)
we denote the branch of the square root with \( \text{Re} \sqrt{z} > 0 \) on \( |\text{arg} z| < \pi \), and set \( z^{n/2} = (\sqrt{z})^n \) if \( n \) is odd.

**Theorem 3.** The function \( z^{-n/2}w(z) \) has an asymptotic expansion

(a) \( \varepsilon + \sum_{k=1}^{\infty} c_k z^{-k} \) if \( n \) is even, and

(b) \( \varepsilon + \sum_{k=1}^{\infty} c_k z^{-k/2} \) if \( n \) is odd

on every ‘pole-free’ sector \( S \), with \( \varepsilon = \varepsilon(w) \in \{-1, 1\} \) and coefficients \( c_k \) only depending on \( \varepsilon \), but neither on \( w \) nor the sector \( S \). The solution \( w \) is uniquely determined by its asymptotic expansion if \( S \) contains some sub-sector \( S' \) such that

\[ \varepsilon \text{ Re } z^\theta < 0 \quad \text{on } S'. \]

**Remark.** In particular, Theorem 3 holds on Stokes sectors \( S_\nu \) with \( \varepsilon = \varepsilon_\nu = \varepsilon(w) \). If (8) is valid on \( S_\nu \), then the corresponding solution is uniquely determined and is denoted by \( w_\nu \). With every solution \( w \) we associate its symbol

(a) \( \Sigma = \Sigma(w) = [\varepsilon_0, \ldots, \varepsilon_n] \) if \( n \) is even, and

(b) \( \Sigma = \Sigma(w) = [\varepsilon_{-m-1}, \ldots, \varepsilon_m] \) if \( n = 2m + 1 \) is odd.

Solutions having the symbol \( \Sigma(w) \) with entries \( \varepsilon_\nu = (-1)^\nu \) are called generic. Noting that \((-1)^\nu \text{Re } z^\theta > 0\) holds on \( S_\nu \), we obtain from Theorem 3:

**Theorem 4.** Any generic solution \( w \) has counting function of poles

\[ n(r, w) = \frac{2r^\theta}{\pi} + o(r^\theta). \]

**Theorem 5.** Suppose \( w \) has symbol \( \Sigma \). Then \( w \) has

(a) no string of poles asymptotic to the Stokes ray \( s_\nu \) if \( \varepsilon_\nu = \varepsilon_{\nu+1} \),

(b) exactly one such string if \((-1)^\nu(\varepsilon_\nu - \varepsilon_{\nu+1}) = 2 \), while

(c) \((-1)^\nu(\varepsilon_\nu - \varepsilon_{\nu+1}) = -2 \) is impossible.

If \( n = 2m + 1 \) is odd and \( \nu = m + 1 \), the term \( \varepsilon_{\nu+1} \) has to be replaced by \(-\varepsilon_{-m-1} \). In case (a), \( w \) has an asymptotic expansion on \( \theta_{\nu-1} < \text{arg } z < \theta_{\nu+1} \). Generic solutions have exactly one string of poles along every Stokes ray, and in any case we have

\[ n(r, w) = \frac{r^\theta}{\pi \theta} \sum_\nu (-1)^\nu \varepsilon_\nu + o(r^\theta). \]

**4. Exceptional solutions**

The non-generic solutions are called exceptional. Exceptional solutions \( w_\nu \) have the ‘false’ asymptotics

\[ w_\nu \approx (-1)^\nu z^{n/2} \quad \text{on } S_\nu \]

and are uniquely determined by that condition.

**Example 1.** The Riccati equation \( w' = z^2 + a_0 - w^2 \) is closely related to the Weber–Hermite equation

\[ y' = y^2 + 2zy - 2 - 2\alpha \quad (w = -y - z, \ a_0 = 1 + 2\alpha). \]

There are four exceptional solutions which may be described by their respective symbols \([-1, -1, 1, -1], [1, 1, 1, -1], [1, -1, -1, 1], \text{ and } [1, -1, 1, 1] \). The poles are
distributed along two rays: \(|\arg z - \frac{\pi}{2}| = \frac{\pi}{4}\), \(|\arg z + \frac{\pi}{2}| = \frac{3\pi}{4}\), and \(|\arg z - \frac{\pi}{2}| = \frac{3\pi}{4}\), respectively.

**Example 2.** The Riccati equation \(w' = z + a_0 - w^2\) is closely related to the Airy equation \(y' = z + y^2\). It has three exceptional solutions with symbols \([-1, -1, -1]\), \([1, 1, -1]\), and \([-1, 1, 1]\), and strings of poles asymptotic to (actually: on) \(\arg z = \pi\), \(\arg z = \frac{\pi}{2}\), and \(\arg z = -\pi/3\), respectively.

**Theorem 6.** To every Stokes sector \(S_\nu\) there exists a unique exceptional solution \(w_\nu\). It has the asymptotic expansion (9) also on the Stokes sectors adjacent to \(S_\nu\), and no strings of poles along the Stokes rays that form the boundary of \(S_\nu\). The number \(d_\nu = n - \ell_\nu\), where \(\ell_\nu\) denotes the number of strings of poles of \(w_\nu\), is even.

**Remark.** The exceptional solutions \(w_\nu\) correspond to those solutions to the linear differential equation \(y'' = a(z)y\) that are sub-dominant on \(S_\nu\); \(y_\nu = \exp \int w(z) \, dz\) is called sub-dominant on \(S_\nu\), if \(y_\nu\) tends to zero exponentially as \(z \to \infty\) on \(S_\nu\).

**Example 3.** Gundersen and Steinbart [6] considered the linear differential equation \(f'' - z^2 f = 0\). They proved among others that certain contour integrals

\[
f_\nu(z) = \frac{1}{2\pi i} \int_{C_\nu} e^{P(z,w)} \, dw
\]

represent solutions having no zeros along given Stokes rays \(s_{\nu-1}\) and \(s_{\nu}\). These solutions give rise to exceptional solutions \(w_\nu = f'_\nu / f_\nu\) to the special Riccati equation \(w' = z^2 - w^2\), which is invariant under the transformations \(w(z) \mapsto \eta w(\eta z), \eta^{n+2} = 1\). There are exactly two solutions that are invariant under these transformations, namely those which either have a pole or else a zero at the origin. These solutions are generic, hence there are \(n + 2\) mutually distinct exceptional solutions. They are obtained from a single one, \(w_0\), say, by rotating the plane:

\[
w_\nu(z) = e^{\frac{2\nu \pi i}{n+2}} w_0(e^{\frac{2\nu \pi}{n+2}} z);
\]

\(w_\nu\) has a single string of poles along every Stokes ray \(s_\mu\) except those that bound the Stokes sector \(S_\nu\).

In the general case (R) the solutions \(w_\nu\) need not be mutually distinct.

**Example 4.** The eigenvalue problem \(f'' + (z^4 - \lambda)f = 0\), \(f \in L^2(\mathbb{R})\), has infinitely many solutions \((\lambda_k, f_k)\) \((0 < \lambda_k \to \infty)\), see Titchmarsh [13]. The eigenfunctions \(f_k\) have only finitely many non-real zeros. For every eigenpair \((\lambda, f) = (\lambda_k, f_k)\), \(u(z) = f(e^{-i \pi /6} z)\) satisfies \(u'' - (z^4 + e^{-i \pi /3} \lambda) u = 0\), and \(w = u'/u\) solves

\[
w' = z^4 + e^{-i \pi /3} \lambda - w^2.
\]

Up to finitely many the poles of the exceptional solution \(w = w_2 = w_3\) belong to the rays \(\arg z = \frac{\pi}{6}\) and \(\arg z = \frac{7\pi}{6}\), hence \(w\) has the symbol \([1, -1, -1, -1, 1, 1]\).

**Example 5.** Eremenko and Gabrielov [2] considered the linear equation

\[
y'' - (z^3 - az + \lambda)y = 0.
\]

For certain real parameters \(a\) and \(\lambda\) it has solutions with infinitely many zeros, only finitely many of them are non-real or real and positive. Thus \(w' = z^3 - az + \lambda - w^2\) has a solution \(w\) with symbol \([1, 1, 1, 1]\), hence \(w = w_1 = w_{-1}\), and mutually distinct solutions \(w_0, w_{-2}\), and \(w_2\) with symbols \([1, -1, -1, 1, 1, -1, 1, -1, 1, 1]\), and \([1, -1, 1, -1, -1, 1, -1, 1, -1, 1]\), respectively, each having three strings of poles.
5. Poles close to a single line

Several papers (Eremenko and Merenkov [3], Eremenko and Gabrielov [2], Gundersen [4, 5], Shin [11]) are devoted to the question whether or not the linear differential equation

\[ y'' - P(z)y = 0 \quad (P(z) = a_n z^n + \cdots \text{ a polynomial of degree } n, \ |a_n| = 1) \]

has solutions with all but finitely many zeros on the real axis. From Theorem 5 we obtain (see also [3, 4]):

**Theorem 7.** Suppose that equation (10) has a solution whose zeros are asymptotic to the real axis. Then the following is true:

- If \( n \) is even, then either
  - \( y \) has only finitely many zeros, or else
  - \( n \equiv 0 \mod 4, a_n = -1, y \) has exactly one string of zeros asymptotic to the negative and positive real axis, and \( y'/y \approx \pm iz^{n/2} \) holds on the upper and lower half-plane, respectively.

- If \( n = 2m + 1 \) is odd, then either
  - \( a_n = 1, y \) has exactly one string of poles asymptotic to the negative real axis with asymptotics \( y'/y \approx (-1)^{m+1}z^{n/2} \) on \( |\arg z| < \pi \), or else
  - \( a_n = -1, y \) has exactly one string of poles asymptotic to the positive real axis with asymptotics \( y'/y \approx (-1)^{m+1}(-z)^{n/2} \) on \( |\arg(-z)| < \pi \).

If \( P \) is real, then in each case all but finitely many zeros are real and \( y \) is a (multiple of a) real entire function.

6. The Schwarzian derivative

In [10] Nevanlinna considered the locally univalent meromorphic functions \( f \) of finite order. They are characterised by the fact that their Schwarzian derivative

\[ S_f = (f''/f')' - \frac{1}{2}(f''/f')^2 \]

is a polynomial \( 2P \), say. Moreover, \( f \) is the quotient \( y(z;0)/y(z;\infty) \) of two linearly independent solutions to the linear differential equation

\[ y'' + P(z)y = 0, \]

which is equivalent to the Riccati equation \( w' = -P(z) - w^2 \) via \( w = y'/y \). The generic solutions have counting function of poles and Nevanlinna characteristic \( T(r, w) \sim \text{Cr}^\varrho \) with \( \varrho = 1 + \frac{1}{2} \text{deg} P; C > 0 \) is some known constant. Every exceptional solution \( w_\nu \), however, has counting function and Nevanlinna characteristic \( T(r, w_\nu) \sim C \frac{n+2-2\delta}{n+2} r^\varrho \), where \( d_\nu \) is some positive integer such that \( \sum_\nu d_\nu = n + 2 \).

Since the zeros of \( f - a \) are the same as the zeros of \( y(z; a) = y(z;0) - ay(z;\infty) \), hence coincide with the poles of \( w(z; a) = y'(z; a)/y(z; a) \), it follows that \( f \) has Nevanlinna deficiencies \( \delta(a_\nu) = \frac{2d_\nu}{n+2} (w_\nu(z) = w(z; a_\nu)) \) with \( \sum_\nu \delta(a_\nu) = 2 \).

7. Proof of Theorem 1 and Theorem 2

**Proof of Theorem 1.** From

\[ w_h'(3) = h^{-n}a(h + h^{-n/2}3) + w_h(3)^2 \]

and \( z^{-n}a(z) \to 1 \) as \( z \to \infty \) it follows that

\[ |w_h'(3)| \leq 2 + |w_h(3)|^2 \]
holds on \(|z| < R, |h| > \eta R\). Thus the family \((w^p_h)|_{|h|\geq 1}\) of spherical derivatives
\[
u^p_h = \frac{|w^p_h|}{1 + |w^p_h|^2}
\]
is bounded on \(|z| < R\) by \(M(R) = \sup \{w^p_h(\zeta) : |z| < R, 1 < |h| < \eta R\} + 2\), say. The limit function \(w = \lim_{h_k \to \infty} w_{h_k} \equiv \infty\) does not occur since otherwise \(w_{h_k} = 1/w_{h_k}\) would tend to zero, this contradicting \(w_{h_k} = 1 - h_k^{-n}a(h_k + h_k^{-n/2})^2 w_{h_k}^2 \to 1\). Thus every limit function \(w\) satisfies (5) outside the set \(\mathcal{P}\) of poles of \(w\).

**Proof of Theorem 2.** From Theorem 1 and Hurwitz’ Theorem it follows that given \(\epsilon > 0\) and \(R > 0\) there exists some \(r_0 > 0\), such that the disc
\[
\triangle_R(p) = \{z : |z - p| < R|p|^{-n/2}\}
\]
about any pole \(p\) with \(|p| > r_0\) contains the poles \(\tilde{p}_k\) with
\[
|\tilde{p}_k - (p + k\pi i p^{-n/2})| < \epsilon|p|^{-n/2} \quad (-k_1(p) \leq k \leq k_2(p)),
\]
and no others; the numbers \(k_1\) and \(k_2\) are bounded by a number only depending on \(R\) (for example, \(k_1 = k_2 = 318\) if \(R = 1000\) and \(r_0\) is sufficiently large). Thus up to finitely many every pole is contained in a unique string of poles \((p_k)\) satisfying (6). Then \(z_k = p_k^\theta (\theta = n/2 + 1)\) satisfies
\[
z_{k+1} = z_k + \omega \theta + o(1)
\]
with \(\omega = \pm \pi i\) fixed, hence \(z_k = \omega \theta k + o(k), \quad p_k = (\omega \theta k)^{1/\nu}(1 + o(1))\), and
\[
\frac{n + 2}{2} \arg p_k = \arg \omega + o(1) = \pm \frac{\pi}{2} + o(1) \mod 2\pi,
\]
that is, \(\arg p_k = \theta + o(1) = \frac{2\nu + 1}{n + 2} + o(1)\) holds for some \(\nu\). The counting function of \(\sigma\) equals \(n(r, \sigma) = \frac{r^\nu}{\pi \theta} + o(r^\theta)\), and from \(n(r, w) = O(r^\theta)\) it follows that there are only finitely many strings of poles.

**8. Proof of Theorem 3**

Let \(w\) be any solution to \((R)\) and \(S\): \(|\arg z - \phi_0| < \eta\) any sector that is ‘pole-free’ for \(w\). From Theorem 1 then it follows that \(w(z)z^{-n/2}\) tends to either \(+1\) or \(-1\) as \(z \to \infty\); the convergence to \(+1\), say, is uniform on each closed sub-sector \(S(\delta)\): \(|\arg z - \phi_0| \leq \eta - \delta\) (take any sequence \(h_k \to \infty\) in \(S(\delta)\) such that \(\lim_{h_k \to \infty} |w(h_k)h_k^{-n/2} - 1| = \lim_{z \to \infty} |w(z)z^{-n/2} - 1|\) on \(S(\delta)\)). If \(n = 2m\) is even we set \(v(z) = z^{-m}w(z)\) to obtain
\[
z^{-m}v' + mz^{-m-1}v = a(z)z^{-2m} - v^2.
\]
If, however, \(n = 2m + 1\) is odd set \(v(z) = z^{-n}w(z^2)\) to obtain
\[
z^{-n-1}v' + nz^{-n-2}v = 2a(z^2)z^{-2n} - v^2.
\]
From (11) resp. (12) and the fact that \(v(z) \to \pm 1\) on some sector \(S\) we have to conclude \(v \sim \pm 1 + \sum_{k=1}^{\infty} c_k z^{-k}\) on \(S\). For definiteness we will consider equation (11) with \(v(z) \to 1\) on \(S\). If we assume that
\[
v(z) = 1 + \sum_{k=1}^{n} c_k z^{-k} + o(|z|^{-n}) = \psi_n(z) + o(|z|^{-n})
\]
has already been proved (this is true for \( n = 0 \)) we obtain from
\[
v'(z) = \psi_n'(z) + o(|z|^{-n-1})
\]
and (11)
\[
a(z)z^{-2m} - v^2 = z^{-m}\psi'_n(z) + mz^{-m-1}\psi_n(z) + o(|z|^{-n-m-1}).
\]
The algebraic equation
\[
a(z)z^{-2m} - y^2 = z^{-m}\psi'_n(z) + mz^{-m-1}\psi_n(z)
\]
has a unique solution \( y = 1 + \sum_{k=1}^{\infty} c_k z^{-k} \) about \( z = \infty \), and from \( v + y = 2 + o(1) \)
and \( (v - y)(v + y) = v^2 - y^2 = o(|z|^{-n-m-1}) \) it follows that
\[
v = y + o(|z|^{-n-m-1}) = 1 + \sum_{k=1}^{n+1} c_k z^{-k} + o(|z|^{-n-1}) = \psi_{n+1}(z) + o(|z|^{-n-1}).
\]
It is obvious that \( c_k = c'_k \) holds for \( 0 \leq k \leq n \), and this proves the existence part.
The proof is the same in all other cases.

To prove the uniqueness part of Theorem 3 we assume that \( w_1 \) and \( w_2 \) have the same asymptotic expansion on the sector \( S \).
Then \( u = w_1 - w_2 \) solves
\[
u' = - (w_1(z) + w_2(z))u = -2\varepsilon z^{n/2}(1 + O(|z|^{-\frac{1}{2}}))u,
\]
hence \( u = C \exp(-\frac{2\varepsilon}{\delta}z^\delta + O(|z|^\delta)) \) holds. Our hypothesis \( \varepsilon \text{Re} z^\delta < 0 \) and \( u \to 0 \)
on \( S' \subset S \) then gives \( u = C = 0 \), and this proves Theorem 3 completely.

9. Proof of Theorem 5

Since all but finitely many poles of \( w \) are simple with residue 1, the Residue Theorem gives
\[
n(r, w) = \frac{1}{2\pi i} \int_{\Gamma_r} w(z) \, dz + o(1),
\]
where the simple closed curve \( \Gamma_r \) is obtained from the circle \( C_r : |z| = r \) by replacing the intersection of \( C_r \) with any disc \( \Delta_r(p) = \{ z : |z - p| < \epsilon |p|^{-n/2} \} \) \( (\epsilon > 0 \) sufficiently small, \( p \) any pole of \( w \) ) by an appropriate sub-arc of \( \partial \Delta_r(p) \).
From \( w = O(|z|^{n/2}) = O(|z|^{\delta_m}) \) on \( \Gamma_r \) (this following from the normality of the family \( w_{h_j}(z) = h^{n/2}w(h + h^{-n/2}j) \) and the fact that \( \Gamma_r \cap \{ z : |\arg z - \theta_\nu| < \delta \} \) has length at most \( 2\pi \delta r \) as \( \delta \to 0 \), it follows that the contribution of the Stokes sector \( S_\nu \) to the counting function of poles equals
\[
(-1)^\nu \varepsilon_{\nu} \frac{r^\delta}{\pi \delta} + o(r^\delta) \quad (\nu = n/2 + 1).
\]
In particular, \( w \) has \( \sum_{\nu} (-1)^\nu \varepsilon_{\nu} \) strings of poles. Integrating \( w \) along the line segment \( \sigma \) from \( r_0 e^{i(\theta_\nu - \delta)} \) \( (\delta > 0 \) small, \( r_0 > 0 \) large \) to \( r e^{i(\theta_\nu + \delta)} \) gives
\[
\frac{1}{2\pi i} \int_{\sigma} w(z) \, dz = \varepsilon_{\nu} \frac{r^\delta}{\pi \delta} e^{i\delta e^{i(\theta_\nu - \delta)}} + o(r^\delta) = (-1)^\nu \varepsilon_{\nu} \frac{r^\delta}{\pi \delta} e^{-i\delta e^{i(\theta_\nu - \delta)}} + o(r^\delta).
\]
Thus, if \( \gamma_{\nu} \) denotes the simple closed curve which consists of the line segment \( \sigma \), the part of \( \Gamma_r \) from \( r e^{i(\theta_\nu - \delta)} \) to \( r e^{i(\theta_\nu + \delta)} \), the line segment from \( r e^{i(\theta_\nu + \delta)} \) to \( r_0 e^{i(\theta_\nu - \delta)} \), and the circular arc on \( |z| = r_0 \) from \( r_0 e^{i(\theta_\nu + \delta)} \) to \( r_0 e^{i(\theta_\nu - \delta)} \) we obtain
\[
\frac{1}{2\pi i} \int_{\gamma_{\nu}} w(z) \, dz = (-1)^\nu \varepsilon_{\nu} \frac{r^\delta}{\pi \delta} [\varepsilon_{\nu} - \varepsilon_{\nu+1}] + O(\delta r^\delta) + o(r^\delta)
\]
(r → ∞, δ → 0). Now the integral on the left hand side equals the number of poles inside γω, while \((-1)^{\nu} \frac{1}{2} [\varepsilon_0 - \varepsilon_{\nu+1}]\) coincides with the number of strings of poles along the Stokes ray s_ρ: arg z = \theta_\nu. From this the assertions (a), (b), and (c) in Theorem 5 immediately follow.

\[ \square \]

10. Proof of Theorem 6

It is easily seen that equation (11) resp. (12), written as

\[ z^{-q} v' = f(z, v) \quad (q = m \text{ resp. } q = n + 1) \]

has a formal solution \( \varepsilon_\nu + \sum_{\nu=1}^{\infty} c_\nu z^{-\nu} \) with \( \varepsilon_\nu = -(-1)^\nu. \) Since \( \lim_{z \to \infty} f_v(z, \varepsilon_\nu) = -2\varepsilon_\nu \neq 0, \) Theorem 12.1 in Wasow’s monograph [14] applies to the corresponding equation for \( v - \varepsilon_\nu. \) Hence to every sector \( |\arg z - \theta_0| < \frac{\pi}{2} \) there exists a solution to equation (14) with asymptotic expansion \( v \sim \varepsilon_\nu + \sum_{\nu=1}^{\infty} c_\nu z^{-\nu}. \) In particular, for every \( \nu \) we obtain a (unique) solution \( w = w_\nu \) to (R) with the desired asymptotic expansion (9) on the Stokes sector \( S_\nu. \)

\[ \square \]

11. Proof of Theorem 7

If \( y(z) = P_1(z) \) has only finitely many zeros, then \( n = 2 \deg P_2 - 2 \) is even, and not much more can be said (of course, \( P \) can be computed explicitly from \( P_1 \) and \( P_2 \)). From now on we assume that \( y \) has infinitely many zeros. The change of variables \( w(z) = \frac{\eta y(\eta z)}{y(\eta z)} \) with \( \eta^{n+2} a_n = 1 \) transforms equation (10) into equation (R) with \( a(z) = \eta^{2} P(\eta z) = z^n + \cdots, \) hence the question whether or not there are solutions \( y \) to (10) having infinitely many zeros, ‘most’ of them close to the real axis is transformed into the question for solutions \( w \) to (R) having just one string of poles asymptotic to some Stokes ray \( s_\nu: \arg z = \theta_\nu \) if \( n \) is odd, and asymptotic to the Stokes rays \( s_\nu \) and \( s_{\nu+1} \) if \( n = 2m \) is even, respectively. This yields \( \eta = e^{i\nu} \) up to an arbitrary root of unity of order \( n + 2, \) and we are free to choose \( \eta = e^{-i\nu} \) if \( n \) is even, and \( \eta = \pm 1 \) if \( n = m + 1 \) if \( n = 2m + 1 \) is odd. In the first case we obtain \( a_n = -1, \) and from Theorem 5 it follows that \( \epsilon_0 - \epsilon_1 = 2 \) and \( (-1)^{m+1}(\epsilon_{m+1} - \epsilon_{m+2}) = 2, \) hence \( \epsilon_0 = 1 \) and \( \epsilon_1 = -1, \) this implying \( \epsilon_2 = \cdots = \epsilon_{m+1} = \epsilon_1 = -1, \epsilon_{m+2} = \cdots = \epsilon_{2m+1} = \epsilon_0 = 1, \) \( m = 2k \) and \( n = 4k. \) This proves the first part of Theorem 6.

In the second case we have \( a_n = +1 \) and \( a_n = -1 \) with zeros asymptotic to the negative and positive real axis, respectively, and asymptotic expansions \( y'/y \approx (-1)^{m+1} z^{-n/2} \) on \( |\arg z| < \pi \) resp. \( y'/y \approx (-1)^{m+1} (-z)^{-n/2} \) on \( |\arg(-z)| < \pi \) (note that \( z^{-n/2} \) means \( (\sqrt{z})^n). \)

Now \( y \) is uniquely determined up to a constant factor. Thus if \( P \) is a real polynomial, then the zeros of \( y^*(z) = y(\overline{z}) \) are also asymptotic to the real axis, hence \( y \) and \( y^* \) are linearly dependent, and \( y \) is a multiple of a real function with all but finitely many zeros real.

\[ \square \]

References


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