

The Formula of RIEMANN-HURWITZ and Iteration of Rational Functions

NORBERT STEINMETZ

Fachbereich Mathematik
Universität Dortmund
Postfach 50 05 00
4600 Dortmund 50

Abstract

An elementary proof of the Riemann-Hurwitz Formula for plane domains is given, avoiding the concept of Euler-characteristic.

MOS-Classification: 30F10

1 Introduction

The formula of RIEMANN-HURWITZ (see [1] or [5]) plays an important role in iteration theory of rational functions, perhaps the most important one besides MONTEL's normality criterion. It is a relation between the EULER-characteristics of two compact RIEMANN surfaces and two integers associated with a proper analytic map $f : V \rightarrow W$.

However, the EULER-characteristic is, even in the case of a spherical domain, (which is, with its natural one-point-compactification, a compact RIEMANN surface), a rather complicated concept, and in iteration theory the RIEMANN-HURWITZ-formula is always applied either to compute some connectivity numbers from given data (see [3]) or else to prove the existence of critical points. Thus it is desirable to have a proof of the RIEMANN-HURWITZ-formula avoiding the concept of EULER-characteristic. Since the EULER-characteristic of a domain V of connectivity n is $2 - n$, the RIEMANN-HURWITZ-formula may be stated as follows:

RIEMANN-HURWITZ-FORMULA: *Let V and W be domains on the RIEMANN sphere of finite connectivity m and n , respectively, and let $f : V \rightarrow W$ be a k -sheeted (ramified) proper map having r critical points (counted by multiplicity). Then*

$$m - 2 = k(n - 2) + r. \tag{RH}$$

Here, as usual, *proper map* means that preimages of compact subsets of W are compact. Then f assumes every value exactly k -times, for some finite integer k .¹

2 The Main Lemma

The proof of (RH) will be based on the following

LEMMA: Let V be a domain of connectivity m , which is divided by k cross-cuts c_1, \dots, c_k (disjoint in V) into l domains V_1, \dots, V_l , of connectivity m_1, \dots, m_l , respectively. Then

$$\sum_{j=1}^l (m_j - 2) = m - 2 - k. \quad (*)$$

REMARK: A cross-cut c is a JORDAN arc lying in V except for its end points, which belong to ∂V . It is well known (see [4]) that either $V \setminus c$ is a domain of connectivity $m - 1$ or else consists of two domains V^* and V^{**} of connectivity m^* and m^{**} , respectively, such that $m^* + m^{**} = m + 1$.

PROOF of eq. (*): We proceed by induction. The fundamental property of a cross-cut mentioned above settles the case $k = 1$.

For $k > 1$ we first assume that $V \setminus c_1$, say, is not a domain. Then it consists of domains V^* and V^{**} of connectivity m^* and m^{**} , respectively, which are divided by k^* and k^{**} cross-cuts, $k^* + k^{**} = k - 1$, into domains V_1, \dots, V_{l^*} and V_{l^*+1}, \dots, V_l . Thus (*) applies and gives

$$\sum_{j=1}^{l^*} (m_j - 2) = m^* - 2 - k^*$$

and

$$\sum_{j=l^*+1}^l (m_j - 2) = m^{**} - 2 - k^{**}.$$

Adding up gives the desired result.

If, however, $V \setminus c_1$ is a domain, then it is $(m - 1)$ ply connected and is divided by c_2, \dots, c_k into the domains V_2, \dots, V_m , thus

$$\sum_{j=1}^l (m_j - 2) = (m - 1) - 2 - (k - 1) = m - 2 - k.$$

This completes the proof of the lemma.

¹This is an old result, which has recently been rediscovered for several times.

3 Proof of the RIEMANN-HURWITZ-formula

First, by applying the RIEMANN-mapping-theorem m and n times, respectively, we map the domains V and W conformally onto domains V^* and W^* which are bounded by analytic JORDAN curves and/or singletons. Thus f induces a k -sheeted proper map $f^* : V^* \rightarrow W^*$ which is analytic in $\overline{V^*}$. Writing (f, V, W) instead of (f^*, V^*, W^*) , we may assume that this situation holds a priori.

We will first discuss the case where f is unramified. Then any local branch of f^{-1} may be continued along any curve in W . If W is simply connected, then, by the monodromy theorem, f^{-1} is single-valued in W and thus is a conformal mapping. This proves that $n = 1$ (and $r = 0$) implies $m = k = 1$ and so (RH).² We proceed by induction. In case $m > 1$, we take a cross-cut c in W , which diminishes the connectivity number: $W^* = W \setminus c$ is $(m - 1)$ -connected. Since c lifts to k cross-cuts c_1, \dots, c_k of V (here we need that f is analytic in \overline{V}), the preimage $f^{-1}(W^*)$ consists of l domains V_1, \dots, V_l of connectivity m_1, \dots, m_l , respectively, and f is a k_j -sheeted proper map $V_j \rightarrow W^*$, where $k_1 + \dots + k_l = k$. Adding up the RIEMANN-HURWITZ-formulae

$$m_j - 2 = k_j((m - 1) - 2),$$

and using (*) we get the desired result. Thus the proof is complete.

If f is ramified, it has finitely many critical values $w_1, \dots, w_s \in W$. Then $W^* = W \setminus \{w_1, \dots, w_s\}$ has connectivity $n + s$, and it is easily seen that $V^* = f^{-1}(W^*)$ has connectivity $m + ks - r$: any w_j has p_j preimages with multiplicities q_j^ν such that $\sum_{\nu=1}^{p_j} q_j^\nu = k$, and $\sum_{j=1}^s \sum_{\nu=1}^{p_j} (q_j^\nu - 1) = r$, thus $\sum_{j=1}^s p_j = sk - r$. Since $f : V^* \rightarrow W^*$ is k -sheeted and unramified,

$$m + ks - r - 2 = k(m + s - 2)$$

and so (RH) holds true also in this case. This completes the proof.

4 A Result of Mueller and Rudin

In a recent paper, MUELLER and RUDIN [3] considered the group $\text{PRH}(V)$ of proper self-maps of a domain V of finite connectivity $m > 2$. It is clear that this group either must be infinite or else consists only of conformal maps. The RIEMANN-HURWITZ-formula now gives $m - 2 = k(m - 2) + r$ and so, since $m > 2$, $k = 1$ and $r = 0$, thus any such map is conformal, and since any conformal self-map induces a permutation of the boundary components, it has to be shown that

²This is also true if $n = 0$, i.e. if $W = \widehat{\mathbb{C}}$. Then f^{-1} is a MÖBIUS-transform and so $V = \widehat{\mathbb{C}}$, $m = 0$ and $k = 1$.

different conformal self-maps induce different permutations. This can be done by considering a circular slits region, as is done in [3].

We are interested in the set $\text{PRH}(V, W)$ of proper maps of some domain V onto some domain W , where V and W have finite connectivity m and n , respectively. We will only discuss the nondegenerate case, where none of the boundary components is a singletons, and will consider first some special cases:

(a) $m = 1$: Then also $n = 1$ and $\text{PRH}(V, W)$ is essentially the set of finite BLASCHKEproducts.

(b) $m = n = 2$: Then V and W are conformally equivalent to annuli

$$\mathcal{A}_r = \{z : r < |z| < 1\}$$

and \mathcal{A}_R , and $\text{PRH}(\mathcal{A}_r, \mathcal{A}_R)$ is empty if $R \neq r^k$ for any positive integer, and otherwise consists of the mappings $z \mapsto e^{i\alpha} z^k$ and $z \mapsto r e^{i\alpha} / z^k$.

(c) $m \geq 2$ and $n = 2$: In this case, $\text{PRH}(V, W)$ in general is infinite. To prove this we consider a BLASCHKEproduct f of degree m having distinct zeros, and let W be the unit disk from which a small disk $|w| \leq \varepsilon$ is removed. Then $e^{i\alpha} f$ is a m -sheeted proper map of $V = f^{-1}(W)$ onto W .

(d) $m = 2$ and $n = 1$: Here, too, $\text{PRH}(V, W)$ in general is infinite. For a proof consider the JOUKOWSKYfunction $f(z) = z + 1/z$. Then f is a proper $2 : 1$ -map of the annulus $\mathcal{A} = \{z : r < |z| < 1/r\}$ onto the interior \mathcal{E} of an ellipse, and if \mathcal{E} is mapped conformally onto the unit disk (by ϕ , say), then $e^{i\alpha} \phi \circ f$ is a $2 : 1$ -map of \mathcal{A} onto the unit disk.

By considering circular slits regions, it is easy to prove that

if $n > 2$, then $\text{PRH}(V, W)$ is finite.

REMARK: It would be interesting to study the cases $m > n = 1$ and $m > n = 2$ in more detail, and also to give a nontrivial upper bound if $m \geq n > 2$.

References

- [1] O. FORSTER, *Vorlesungen über RIEMANNsche Flächen*, Springer-Verlag 1971.
- [2] M. Yu. LYUBICH, *The dynamics of rational transforms: the topological picture*, Russ. Math. Surveys 41 (1986), 43 - 117.
- [3] C. MUELLER and W. RUDIN, *Proper holomorphic self-maps of plane regions*, Complex Variables **17** (1991), 113 - 121.
- [4] M. H. A. NEWMAN, *Elements of the topology of plane sets of points*, Cambridge University Press 1964.
- [5] S. STOILOW, *Lecons sur les principes topologiques de la théorie des fonctions analytiques*, Gauthier-Villars, Paris 1956.