The Formula of Riemann-Hurwitz and Iteration of Rational Functions

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Abstract
An elementary proof of the Riemann-Hurwitz Formula for plane domains is given, avoiding the concept of Euler-characteristic.

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1 Introduction

The formula of Riemann-Hurwitz (see [1] or [5]) plays an important role in iteration theory of rational functions, perhaps the most important one besides Montel’s normality criterion. It is a relation between the Euler-characteristics of two compact Riemann surfaces and two integers associated with a proper analytic map \( f : V \to W \).

However, the Euler-characteristic is, even in the case of a spherical domain, (which is, with its natural one-point-compactification, a compact Riemann surface), a rather complicated concept, and in iteration theory the Riemann-Hurwitz-formula is always applied either to compute some connectivity numbers from given data (see [3]) or else to prove the existence of critical points. Thus it is desirable to have a proof of the Riemann-Hurwitz-formula avoiding the concept of Euler-characteristic. Since the Euler-characteristic of a domain \( V \) of connectivity \( n \) is \( 2 - n \), the Riemann-Hurwitz-formula may be stated as follows:

Riemann-Hurwitz-Formula: Let \( V \) and \( W \) be domains on the Riemann sphere of finite connectivity \( m \) and \( n \), respectively, and let \( f : V \to W \) be a \( k \)-sheeted (ramified) proper map having \( r \) critical points (counted by multiplicity). Then

\[
m - 2 = k(n - 2) + r. \tag{RH}
\]
Here, as usual, proper map means that preimages of compact subsets of $W$ are compact. Then $f$ assumes every value exactly $k$-times, for some finite integer $k$.

2 The Main Lemma

The proof of (RH) will be based on the following

**Lemma:** Let $V$ be a domain of connectivity $m$, which is divided by $k$ cross-cuts $c_1, \ldots, c_k$ (disjoint in $V$) into $l$ domains $V_1, \ldots, V_l$, of connectivity $m_1, \ldots, m_l$, respectively. Then

$$\sum_{j=1}^{l} (m_j - 2) = m - 2 - k.$$  \hfill (*)

**Remark:** A cross-cut $c$ is a Jordan arc lying in $V$ except for its end points, which belong to $\partial V$. It is well known (see [4]) that either $V \setminus c$ is a domain of connectivity $m - 1$ or else consists of two domains $V^*$ and $V^{**}$ of connectivity $m^*$ and $m^{**}$, respectively, such that $m^* + m^{**} = m + 1$.

**Proof** of eq. (*) We proceed by induction. The fundamental property of a cross-cut mentioned above settles the case $k = 1$.

For $k > 1$ we first assume that $V \setminus c_1$, say, is not a domain. Then it consists of domains $V^*$ and $V^{**}$ of connectivity $m^*$ and $m^{**}$, respectively, which are divided by $k^*$ and $k^{**}$ cross-cuts, $k^* + k^{**} = k - 1$, into domains $V_1, \ldots, V_l$ and $V_{l+1}, \ldots, V_l$. Thus (*) applies and gives

$$\sum_{j=1}^{l} (m_j - 2) = m^* - 2 - k^*$$

and

$$\sum_{j=l+1}^{l} (m_j - 2) = m^{**} - 2 - k^{**}.$$  

Adding up gives the desired result.

If, however, $V \setminus c_1$ is a domain, then it is $(m - 1)$ ply connected and is divided by $c_2, \ldots, c_k$ into the domains $V_2, \ldots, V_m$, thus

$$\sum_{j=1}^{l} (m_j - 2) = (m - 1) - 2 - (k - 1) = m - 2 - k.$$  

This completes the proof of the lemma.

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1This is an old result, which has recently been rediscovered for several times.
3 Proof of the Riemann-Hurwitz-formula

First, by applying the Riemann-mapping-theorem \( m \) and \( n \) times, respectively, we map the domains \( V \) and \( W \) conformally onto domains \( V^* \) and \( W^* \) which are bounded by analytic Jordan curves and/or singletons. Thus \( f \) induces a \( k \)-sheeted proper map \( f^* : V^* \rightarrow W^* \) which is analytic in \( V^* \). Writing \( (f, V, W) \) instead of \( (f^*, V^*, W^*) \), we may assume that this situation holds a priori.

We will first discuss the case where \( f \) is unramified. Then any local branch of \( f^{-1} \) may be continued along any curve in \( W \). If \( W \) is simply connected, then, by the monodromy theorem, \( f^{-1} \) is single-valued in \( W \) and thus is a conformal mapping. This proves that \( n = 1 \) (and \( r = 0 \)) implies \( m = k = 1 \) and so (RH). We proceed by induction. In case \( m > 1 \), we take a cross-cut \( c \) in \( W \), which diminishes the connectivity number: \( W^* = W \setminus c \) is \( (m - 1) \)-connected. Since \( c \) lifts to \( k \) cross-cut in \( V \), we get \( m_1 = m - 1 \) and \( k_1 = k \). Adding up the Riemann-Hurwitz-formulae

\[ m_j - 2 = k_j((m - 1) - 2), \]

and using (*) we get the desired result. Thus the proof is complete.

If \( f \) is ramified, it has finitely many critical values \( w_1, \ldots, w_s \in W \). Then \( W^* = W \setminus \{w_1, \ldots, w_s\} \) has connectivity \( n + s \), and it is easily seen that \( V^* = f(W^*) \) has connectivity \( m + ks - r \); any \( w_j \) has \( p_j \) preimages with multiplicities \( q_j^\nu \) such that

\[ \sum_{\nu=1}^{p_j} q_j^\nu = k, \text{ and } \sum_{j=1}^{s} \sum_{\nu=1}^{p_j} (q_j^\nu - 1) = r, \text{ thus } \sum_{j=1}^{s} p_j = sk - r. \]

Since \( f : V^* \rightarrow W^* \) is \( k \)-sheeted and unramified,

\[ m + ks - r - 2 = k(m + s - 2) \]

and so (RH) holds true also in this case. This completes the proof.

4 A Result of Mueller and Rudin

In a recent paper, Mueller and Rudin [3] considered the group \( \text{PRH}(V) \) of proper self-maps of a domain \( V \) of finite connectivity \( m > 2 \). It is clear that this group either must be infinite or else consists only of conformal maps. The Riemann-Hurwitz-formula now gives \( m - 2 = k(m - 2) + r \) and so, since \( m > 2 \), \( k = 1 \) and \( r = 0 \), thus any such map is conformal, and since any conformal self-map induces a permutation of the boundary components, it has to be shown that

\[ ^2 \text{This is also true if } n = 0, \text{ i.e. if } W = \hat{\mathbb{C}}. \text{ Then } f^{-1} \text{ is a Möbius transform and so } V = \hat{\mathbb{C}}; m = 0 \text{ and } k = 1. \]
different conformal self-maps induce different permutations. This can be done by considering a circular slits region, as is done in [3]. We are interested in the set $PRH(V, W)$ of proper maps of some domain $V$ onto some domain $W$, where $V$ and $W$ have finite connectivity $m$ and $n$, respectively. We will only discuss the nondegenerate case, where none of the boundary components is a singletons, and will consider first some special cases:

(a) $m = 1$: Then also $n = 1$ and $PRH(V, W)$ is essentially the set of finite BLASCHKE products.

(b) $m = n = 2$: Then $V$ and $W$ are conformally equivalent to annuli

$$\mathcal{A}_r = \{ z : r < |z| < 1 \}$$

and $\mathcal{A}_R$, and $PRH(\mathcal{A}_r, \mathcal{A}_R)$ is empty if $R \neq r^k$ for any positive integer, and otherwise consists of the mappings $z \mapsto e^{i\alpha}z^k$ and $z \mapsto re^{i\alpha}/z^k$.

(c) $m \geq 2$ and $n = 2$: In this case, $PRH(V, W)$ in general is infinite. To prove this we consider a BLASCHKE product $f$ of degree $m$ having distinct zeros, and let $W$ be the unit disk from which a small disk $|w| \leq \varepsilon$ is removed. Then $e^{i\alpha}f$ is a $m$-sheeted proper map of $V = f^{-1}(W)$ onto $W$.

(d) $m = 2$ and $n = 1$: Here, too, $PRH(V, W)$ in general is infinite. For a proof consider the JOUKOWSKY function $f(z) = z + 1/z$. Then $f$ is a proper $2 : 1$ - map of the annulus $\mathcal{A} = \{ z : r < |z| < 1/r \}$ onto the interior $\mathcal{E}$ of an ellipse, and if $\mathcal{E}$ is mapped conformally onto the unit disk (by $\phi$, say), then $e^{i\alpha}\phi \circ f$ is a $2 : 1$ - map of $\mathcal{A}$ onto the unit disk.

By considering circular slits regions, it is easy to prove that

if $n > 2$, then $PRH(V, W)$ is finite.

**Remark:** It would be interesting to study the cases $m > n = 1$ and $m > n = 2$ in more detail, and also to give a nontrivial upper bound if $m \geq n > 2$.

**References**


