The Polynomials Associated With A Julia Set

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Abstract

We prove that, with two exceptions, the set of polynomials with Julia set $\mathcal{J}$ has the form $\{\sigma p^n : n \in \mathbb{N}, \sigma \in \Sigma\}$, where p is one of these polynomials and Σ is the symmetry group of $\mathcal{J}$. The exceptions occur when $\mathcal{J}$ is a circle or a straight line segment.

Several papers [1, 2, 3, 5] have appeared dealing with the relation between polynomials having the same Julia set $\mathcal{J}$ (for notation the reader is referred to [8]). This relation is very simple and by no means surprising:

Theorem *To any* Julia *set (of a polynomial)* $\mathcal{J}$, *which is not a circle or a straight line segment, there exists a polynomial* p *such that any polynomial with* Julia *set* $\mathcal{J}$ *can be written in the form* σp^n, *where* σ *is a rotation mapping* $\mathcal{J}$ *onto itself, and* n *is a positive integer.*

PROOF: We denote by $\Sigma = \Sigma(\mathcal{J})$ the symmetry group of $\mathcal{J}$, i.e. the group of Möbius transformations mapping $\mathcal{J}$ onto itself. There exists a polynomial p of lowest degree associated with $\mathcal{J}$. Without loss of generality we may assume that p is given by

$$p(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0 \,,$$

since otherwise we could consider the JULIA set $M(\mathcal{J})$, the conjugate group $M\Sigma M^{-1}$ and the conjugate polynomial $M \circ p \circ M^{-1}$, where $M(z) = \alpha z + \beta$ is a suitably chosen Möbius transformation. BAKER and EREMENKO [1] have shown that p may be written as

$$p(z) = z^\mu p_0(z^m) \,,$$

where m is the order of the symmetry group of $\mathcal{J}$, namely

$$\Sigma = \{z \mapsto \delta z : \delta^m = 1\} \,,$$

and p_0 is a polynomial. In a similar way, any polynomial q associated with $\mathcal{J}$ can be written in this form, say $q(z) = z^\nu q_0(z^m)$. It is easily seen that $p \circ q = \delta q \circ p$ holds, and also that the polynomials

$$\hat{p}(z) = z^\mu (p_0(z))^m \quad \text{and} \quad \hat{q}(z) = z^\nu (q_0(z))^m$$

commute; this can also be found in [1]. According to results due to JULIA [6], FATOU [4] and RITT [7] we have to consider three different cases:

(a) $\hat{p}(z) = z^d$ is a monomial. Then p is a monomial, too, and the JULIA set is the unit circle.

(b) $\hat{p}$ is conjugate to some polynomial T, such that T or $-T$ is a TCHEBYCHEV polynomial for the interval $[-2, 2]$. Then from

$$T(\alpha z + \beta) = \alpha z^\mu (p_0(z))^m + \beta$$

and the fact, that T' has only simple zeros, it follows that $m \le 2$, and hence $m = 2$. Also we have $\beta = \pm 2$, since only the values ± 2 are critical for T, and so

$$T(\alpha z^2 \pm 2) = \alpha (p(z))^2 \pm 2$$

holds. We consider the respective conjugates

$$h(i\sqrt{\alpha}\,z) = i\sqrt{\alpha}\,p(z)\,, \quad h(\sqrt{\alpha}\,z) = \sqrt{\alpha}\,p(z)\,,$$

and obtain

$$(h(z))^2 = 2 - T(2 - z^2)\,, \quad (h(z))^2 = 2 + T(z^2 - 2)\,,$$

respectively. In both cases the interval $[-2, 2]$ is completely invariant under h, and so the JULIA set of h coincides with $[-2, 2]$ and the JULIA set of p is a straight line segment.

(c) There exist integers k and l such that $\hat{p}^k = \hat{q}^l$. In particular it follows that $(\deg \hat{p})^k = (\deg \hat{q})^l$ and so d divides $\deg q$, i.e., $q(z) = cz^{ds} + \cdots$. Now we use a device that can be found in RITT's paper [7]. Let

$$B(z) = z\left(1 + \frac{\alpha_1}{z^m} + \cdots\right)$$

be the normalized solution of BÖTTCHER's functional equation

$$B(p(z)) = (B(z))^d\,;$$

it depends only on the JULIA set $\mathcal{J}$, since, near infinity, $\log|B(z)|$ represents the Green's function of the outer domain of $\mathcal{J}$ with pole at ∞. In a neighbourhood of this point we define a function r by the relation

$$B(r(z)) = c(B(z))^s\,.$$

Then $B(r(p(z))) = c(B(z))^{ds} = B(q(z))$ and hence

$$r \circ p = q$$

follows. Writing $r(z) = r_0(z) + O(1/z)$ as $z \to \infty$, where r_0, a polynomial, denotes the principal part of r at ∞, we obtain

$$q(z) - r_0(p(z)) = O(z^{-d})\,,$$

and hence $r = r_0$ is a polynomial. It is obvious that r maps each of the sets $\mathcal{J}$ and $\mathcal{F}$ – the common FATOU set – onto itself, and hence either r has JULIA set $\mathcal{J}$ or else $r(z) = \delta z$, $\delta^m = 1$, holds. Repetition of this argument, if necessary, leads to the conclusion that $q = \delta p^n$. Conversely, each polynomial δp^n has JULIA set $\mathcal{J}$. This proves our theorem.

REMARK: It is clear that, in the non-exceptional case, δp^k and εp^n are permutable if and only if $\delta^{\nu k-1} = \varepsilon^{\mu n-1}$ holds. This was already observed by RITT.

REFERENCES

1. I. N. BAKER, A. EREMENKO, *A problem on Julia sets,* Ann. Acad. Sci. Fenn. Series A **12** (1987), 229-236.
2. A. F. BEARDON, *Symmetries of Julia sets,* Bull. London Math. Soc. **22** (1990), 576-582.
3. A. F. BEARDON, *Polynomials with identical Julia sets,* Complex Variables **17** (1992), 195-200.
4. P. FATOU *Sur l'iteration analytique et les substitutions permutables,* J. Math. **2** (1923), 343-362.
5. J. L. FERNÁNDEZ, *A note on the Julia set of polynomials,* Complex Variables **12** (1989), 83-85.
6. G. JULIA, *Mémoire sur la permutabilité des fractions rationelles,* Ann. Sci. Éc. Norm. Sup. **39** (1922), 131-215.
7. J. F. RITT *On the iteration of rational functions,* Trans. Amer. Math. Soc. **21** (1920), 348-356.
8. N. STEINMETZ, *Rational iteration,* de Gruyter 1993.