Exceptional Solutions of n-th Order
Periodic Linear Differential Equations

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Dedicated to the memory of Steven Bank

Let
\[ L := D^n + p_{n-1}(z)D^{n-1} + \cdots + p_1(z)D + p_0(z), \quad D = d/dz, \]
be a linear differential operator, whose coefficients are (constants or) \(2\pi\)i-periodic entire functions of order one, mean type. We will prove that any exceptional solution of \(L[w] = 0\), i.e., any solution satisfying \(\log N(r, 1/w) = o(r)\), has the form
\[ w(z) = e^{S(e^{z/q}, e^{-z/q})} \sum_{j=1}^{m} e^{cz} P_j(z, e^{z/q}), \]
where \(q \geq 1\) is an integer, the \(c_j\)'s are complex constants and \(S\) and the \(P_j\)'s are polynomials. We give also a new proof of a result due to Steinbart, who classified the so-called subnormal solutions – solutions satisfying \(\log T(r, w) = o(r)\).

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1 Introduction

In [?] Bank and Langley considered the differential equation
\[ w^{(n)} + A_{n-1}w^{(n-1)} + \cdots + A_1w' + A_0(e^z, e^{-z})w = 0, \]
with \(A_\nu\) constant for \(1 \leq \nu \leq n-1\) and \(A_0\) a non-constant polynomial \(^1\). They proved that any solution satisfying \(\log N(r, 1/w) = o(r)\) \(^2\) has the form
\[ w(z) = P(e^{z/q}) \exp(cz + S(e^{z/q}, e^{-z/q})), \]

\(^1\)That means \(A_0(x, y) = B(x) + C(y)\), \(B\) and \(C\) polynomials, not both constant.
\(^2\)For terminology in Nevanlinna theory see Hayman [?], Jank-Volkmann [?] or Nevanlinna [?].
where $P$ and $S$ are polynomials, $q \geq 1$ is an integer, and $c$ is a complex number. The case $n = 2$ is due to Bank and Laine [?].

We consider a linear differential equation

$$L[w] := w^{(n)} + p_{n-1}(z)w^{(n-1)} + \cdots + p_0(z)w = 0$$

with entire periodic coefficients

$$p_\nu(z) = A_\nu(e^z, e^{-z}),$$

where the $A_\nu$'s are polynomials, at least one being non-constant. Then every solution $w$ is an entire function which satisfies the growth condition $\log \log M(r, w) = O(r)$, as an easy application of Gronwall’s Lemma shows, and hence

$$\log N(r, \frac{1}{w}) \leq \log T(r, w) = O(r).$$

We call any solution of (1),(2) exceptional, if it satisfies

$$\log N(r, \frac{1}{w}) = o(r),$$

and subnormal, if even

$$\log T(r, w) = o(r)$$

is true.

**Example 1** The functions $e^z, e^{-z} + e^z, e^{-z} - 2e^z$ have constant non-zero Wronskian determinant, and hence they form a zero-free, and so an exceptional fundamental set of solutions of some equation

$$w''' + A(e^z)w' + B(e^z)w = 0.$$ 

Every non-trivial linear combination $a_1 w_1 + a_2 w_2$ is exceptional, but no non-trivial solution is subnormal.

**Example 2** The Wronskian determinant of the entire functions $e^{-\frac{z}{2}}, e^{\frac{z}{4}}, e^{\frac{z}{2}} + e^z, e^{-\frac{z}{2}} - e^z$

is a non-zero constant, and hence we have

$$w^{(4)} + A(e^z)w'' + B(e^z)w' + C(e^z)w = 0$$

with polynomials $A, B, C$ and $w = w_j, j = 1, 2, 3, 4$. Every non-trivial linear combination $a_1 w_1 + a_2 w_2$ is subnormal, but no solution different from these and $w = \text{const} w_{3,4}$ is exceptional.

**Example 3** Consider $w(z) = e^{\eta e^z}$, where $\eta$ is any $n$-th root of unity, $n \geq 2$. Then a simple proof by induction shows that $w^{(\nu)}(z) = P_\nu(\eta e^z)w(z)$ holds, where $P_\nu(x) = x^\nu + \cdots + x$ is a polynomial, independent of $\eta$. Hence there exist complex
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constants \( \lambda \) such that \( P_n(x) + \lambda_{n-1} P_{n-1}(x) + \cdots + \lambda_1 P_1(x) = x^n \), and \( \eta^n = 1 \) yields
\[
w^{(n)} + \lambda_{n-1} w^{(n-1)} + \cdots + \lambda_1 w' - e^{nz} w = 0 .
\]
It is obvious that no solution \( w(z) \neq \text{const.} e^{\eta z} \) is exceptional, and no non-trivial solution is subnormal.

2 Results

We will prove the following representation theorem for exceptional solutions:

**Theorem 1** Any exceptional solution of eq. (??), (??) is a linear combination of linearly independent exceptional solutions
\[
w_j(z) = P_j(z, e^{z/q}) \exp \left( c_j z + S(e^{z/q}, e^{-z/q}) \right),
\]
(3) \( 1 \leq j \leq k \leq n \), where the \( P_j \)'s are polynomials, \( q \geq 1 \) is an integer, the \( c_j \)'s are complex numbers, mutually distinct mod 1, and \( S \) is a polynomial independent of \( j \).

A closer examination of the proof of Theorem 1 shows that the following is true:

**Theorem 2** Let \( w(z) = P(z, e^{z/q}) \exp \left( cz + S(e^{z/q}, e^{-z/q}) \right) \) be an exceptional solution of eq. (??), (??). Then either
\[
w(z) = P(e^{z/q}) \exp \left( cz + S(e^{z/q}, e^{-z/q}) \right)
\]
holds, or else
\[
w(z) = P(z, e^z) \exp \left( cz + S(e^z, e^{-z}) \right).
\]

Theorem 1 corresponds to a recent result due to Steinbart [?], see Theorem 3 below, which we need for a proof of Theorem 1. For the sake of completeness we will give an independent proof in section 5. For related results see [?], [?], [?], [?], [?], [?].

**Theorem 3** The linear space of subnormal solutions of (??), (??) has a basis consisting of functions \( w_{\mu j}(z) = P_{\mu j}(z, e^z)e^{cz} \), where
\[
P_{\mu j}(z, x) = \ell_{j-1}(z) Q_{\mu 1}(x) + \cdots + \ell_1(z) Q_{\mu j-1}(x) + Q_{\mu j}(x),
\]
(6) \( 1 \leq j \leq k_\mu \), \( 1 \leq \mu \leq m \), say. The \( Q_{\mu \kappa} \)'s are polynomials, \( c_\mu \) is a complex number, and \( \ell_\kappa \) is given by \( \kappa! (2\pi i)^{\kappa} \ell_\kappa(z) = \prod_{\nu=0}^{\kappa-1} (z - 2\nu \pi i) \).

**Remark** Theorems 1 and 3 – as far as they concern the form of subnormal and exceptional solutions – apply also to inhomogeneous equations
\[
L[w] = A(e^z, e^{-z}),
\]
since the solutions also solve a homogeneous equation \( L_0L[w] = 0 \), where \( L_0 \) is any linear differential operator with constant coefficients which annihilates \( A(e^z, e^{-z}) \). Note however, that in this case we have \( m(r, 1/w) = O(r) \) for every solution, and so every exceptional solution is also subnormal.

It follows from Theorem 3 that, if there is any subnormal solution, then there exists also a subnormal solution which is \( 2\pi i \)-periodic up to a factor \( e^{cz} \), in contrast to the case of exceptional solutions, where the period may have any value \( 2\pi iq \), \( 1 \leq q \leq n \).

In case \( q > 1 \), however, there exist at least \( q \) linearly independent exceptional solutions. By modifying Example 3 it is easily seen that every \( q \), \( 1 \leq q \leq n \), may occur.

**Example 4** Let \( 1 \leq q \leq n \) be an integer and set \( w_j(z) = e^{c(2\pi ij + z)/q} \). If \( L_0 \) denotes the operator of order \( q \) which annihilates each \( w_j \) (Example 3) and \( L_0 \) is any operator of order \( n - q \) with constant coefficients, then \( L = L_0L_* \) has order \( n \), coefficients of type (??), and \( L[w_j] = 0, 1 \leq j \leq q \).

For one step in the proof of Theorem 1 we need a theorem of Nevanlinna [?], sometimes called Nevanlinna’s Third Main Theorem:

**Nevanlinna’s Theorem** Let \( \Psi_j, 1 \leq j \leq n \), be linearly independent meromorphic functions in the plane satisfying \( \Psi_1 + \Psi_2 + \cdots + \Psi_n = 1 \). Then

\[
T(r, \Psi_j) \leq \sum_{k=1}^{n} \left( N \left( r, \frac{1}{\Psi_k} \right) - N(r, \Psi_k) \right) + N(r, \Psi_j) + N(r, W) - N \left( r, \frac{1}{W} \right) + S(r)
\]

holds, where \( W \) is the Wronskian determinant of \( \Psi_1, \ldots, \Psi_n \), and \( S(r) \) is the usual remainder term, \( S(r) = O(\log \max_{1 \leq k \leq n} rT(r, \Psi_k)) \) outside a set of finite measure.

We will also frequently make use of the following more or less well-known facts, without further reference:

(a) \( \log T(r, e^h) = o(r) \) and \( \log T(r, e^h) = O(r) \) (\( h \) entire) imply \( T(r, h) = o(r) \) and \( T(r, h) = O(r) \), respectively;

(b) \( \log N(r, f) + \log N \left( r, \frac{1}{f} \right) = o(r) \) (\( f \) meromorphic in the plane) implies \( f(z) = g(z)e^{h(z)} \) for appropriately chosen functions \( h \) and \( g \), with \( \log T(r, g) = o(r) \) (see Ahmad [?], and also Jank-Volkmann [?]).

(c) \( T(r, f) = O(r) \) (\( f \) entire and \( 2\pi i \)-periodic) implies \( f(z) = P(e^z, e^{-z}) \), \( P \) a polynomial, and \( T(r, f) \sim \text{const.} r \).

**3 Proof of Theorem 1**

Let \( w \) be a exceptional solution of (??),(??). As everyone would do, we consider the exceptional solutions \( f_j(z) = w(z + 2\pi ij) \), \( j = 0, 1, 2, \ldots \), and denote by \( q \) the largest integer such that the functions \( f_0, f_1, \ldots, f_{q-1} \) are linearly independent. Then there exist complex numbers \( \lambda_j \), not all zero, such that \( f_q = \sum_{j=0}^{q-1} \lambda_j f_j \) holds, and so \( \sum_{\lambda_j \neq 0} \Phi_j = 1 \) with \( \Phi_j = \lambda_j f_j / f_q \). Note that \( \lambda_0 \neq 0 \), since otherwise \( f_1, \ldots, f_q \)
would be linearly dependent. Nevanlinna’s Theorem then gives \( \log T(r, \Phi_0) = o(r) \), and hence \( w(z + 2\pi iq) = K(z)w(z) \), where \( K \) is meromorphic and satisfies \( \log T(r, K) = o(r) \).

Without loss of generality we may assume \( q = 1 \), for otherwise we could introduce the new independent variable \( z' = z/q \). By remark (b), \( w \) may be written as

\[
w(z) = g(z)e^{H(z)},
\]

where \( g \) is entire with \( \log M(r, g) = o(r) \), and \( H \) is entire and has order one, mean type, at most. This gives

\[
\frac{g(z + 2\pi i)}{g(z)} = K(z)e^{-H(z+2\pi i) + H(z)},
\]

and so

\[
H(z + 2\pi i) - H(z) = Q(z), \quad T(r, Q) = o(r)
\]

by remark (a). If we set \( h(z) = H(2\pi iz) \) and \( q(z) = Q(2\pi iz) \), then the following lemma applies to the functional equation

\[
h(z + 1) - h(z) = q(z).
\]  

**Lemma 1** Let \( q \) be an entire function, restricted to the growth condition \( \log M(r, q) = o(r) \). Then eq. (??) has exactly one solution in this class satisfying \( h(0) = 0 \).

The proof will be given in section 4.

By that lemma and remarks (a) and (c), \( H \) may be written as \( H(z) = p(z) + S(e^z, e^{-z}) \), where \( p \) satisfies \( T(r, p) = o(r) \) and \( S \) is a polynomial. This yields the representation

\[
w(z) = g(z)e^{p(z)}e^{S(e^z, e^{-z})}.
\]

Now \( v(z) = g(z)e^{p(z)} = w(z)e^{-S(e^z, e^{-z})} \) is a subnormal solution of some equation of type (?),(?), and hence, by Theorem 3, has the representation \( v(z) = \sum_{j=1}^{k} e^{c_j z}P_j(z, e^z) \), with \( c_j \) mutually distinct mod 1.

It is obvious that, for \( w_j(z) = P_j(z, e^z) \exp(c_j z + S(e^z, e^{-z})) \),

\[
L[w_j] = Q_j(z, e^z, e^{-z}) \exp(c_j z + S(e^z, e^{-z})
\]

holds with polynomials \( Q_j \), and hence

\[
e^{-S(e^z, e^{-z})}L[w] = \sum_{j=1}^{k} Q_j(z, e^z, e^{-z})e^{c_j z}
\]

is an exponential polynomial with frequencies \( \equiv c_j 1 \); any such function vanishes identically if and only if \( Q_j = 0 \) for \( 1 \leq j \leq k \), i.e., if and only if \( L[w_j] = 0 \). This finishes the proof of Theorem 1.

**4 Proof of Lemma 1**
We first remark, that the condition \( \log M(r, q) = o(r) \) is equivalent to \( q = Q \ast \exp \), where \( \ast \) is the Hadamard convolution, and \( Q \) is an entire function. Hence we may write \( q(z) = \sum_{k=0}^{\infty} \frac{y_k}{k!} z^k \) and set \( h(z) = \sum_{k=1}^{\infty} \frac{x_k}{k!} z^k \). Then eq. (10) is equivalent to
\[ x_{k+1} + \sum_{n=2}^{\infty} \frac{x_{k+n}}{n!} = y_k, \quad k = 0, 1, 2, \ldots \] (8)

We consider that equation in the space \( \ell^\infty \) of bounded complex sequences \( x = (x_0, x_1, x_2, \ldots) \), endowed with the supremum-norm \( ||x||_{\infty} = \sup_{k \geq 0} x_k \). Then (10) may be written as
\[ \sigma x + T \sigma^2 x = y, \] (9)
where \( \sigma \) is the shift operator \( (x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, x_3, \ldots) \), and the linear map \( T : \ell^\infty \longrightarrow \ell^\infty \) is represented by the infinite matrix
\[
\begin{pmatrix}
\frac{1}{\pi^2} & \frac{1}{\pi^4} & \frac{1}{\pi^6} & \cdots \\
0 & \frac{1}{\pi^2} & \frac{1}{\pi^4} & \cdots \\
0 & 0 & \frac{1}{\pi^2} & \frac{1}{\pi^4} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]
Note that \( T \sigma \) has contraction constant \( e - 2 < 1 \), and that the growth condition \( \log M(r, q) = o(r) \) is equivalent to \( \lim_{n \to \infty} \sqrt[n]{||\sigma^n y||_{\infty}} = 0 \).

By the contraction principle, the corresponding equation \( u + T \sigma u = y \) has a unique solution \( u \in \ell^\infty \), and hence eq. (10) has a unique solution \( x = (0, x_1, x_2, \ldots) \in \ell^\infty \).

From
\[ ||x||_{\infty} \leq \frac{1}{3 - e} ||y||_{\infty} \quad \text{and} \quad \sigma^{n+2} x + T \sigma^{n+2} x = \sigma^n y \]
it then follows that \( \lim_{n \to \infty} \sqrt[n]{||\sigma^n x||_{\infty}} = 0 \). Thus \( h(z) = \sum_{k=1}^{\infty} \frac{x_k}{k!} z^k \) is the unique solution of eq. (10) with \( h(0) = 0 \) and \( \log M(r, h) = o(r) \), and this proves Lemma 1.

**5 Proof of Theorem 2**

To prove Theorem 2, we recall the proof of Theorem 1. With the notation used there we have either \( \lambda_j = 0 \) for \( 1 \leq j < q \) or else \( \lambda_j \neq 0 \) for some \( j \in \{1, \ldots, q-1\} \).

In the first case \( f_q = \lambda_0 f_0 \) holds, and hence (11). In the second case it follows from \( T(r, f_j/f_q) = O(r) \) that the difference \( S(e^{(z+2\pi i j)/q}, e^{-(z+2\pi i j)/q}) - S(e^{z/q}, e^{-z/q}) \) is a constant, and hence \( S(x, y) = U(x^t, y^t) \) holds, where \( U \) is a polynomial, \( t = q/r \) and \( r \) is the greatest common divisor of \( j \) and \( q \); note that \( t > 1 \). Since \( v(z) = w(z)e^{-U(e^{z/r}, e^{-z/r})} \) is a subnormal solution of some eq. (11), where now the coefficients are merely \( 2\pi r \)-periodic, we find that
\[ w(z) = Q(z, e^{z/r})e^{U(e^{z/r}, e^{-z/r})} \] (10)

holds, and also that
\[
\text{the functions } f_0, f_1, \ldots, f_r \text{ are linearly dependent.} \] (11)
Now let \( r \geq 1 \) be the smallest divisor of \( q \) such that (\ref{eq:1}) and (\ref{eq:2}) hold with \( q = rt \). Repetition of the proof of Theorem 1, but now starting with the functions \( f_0, f_t, \ldots, f_{rt} \), instead of \( f_0, f_1, \ldots, f_q \), shows that, by minimality of \( r \), either \( r = 1 \) holds or else \( f_{rt} = \lambda_0 f_0 \). By (\ref{eq:2}), the first case leads to (\ref{eq:3}), and the second case to (\ref{eq:4}), with \( q \) replaced by \( r \), as was already observed in the beginning of the proof. This finishes the proof of Theorem 2.

### 6 Subnormal Solutions and Floquet’s Theory: Proof of Theorem 3

Let \( w_1, w_2, \ldots, w_m \) form a base of the linear space of subnormal solutions. Since the functions \( w_1(z + 2\pi i), w_2(z + 2\pi i), \ldots, w_m(z + 2\pi i) \) form also a base, there exists a regular \((m, m)\)-matrix \( \Lambda \) with

\[
w(z + 2\pi i) = \Lambda w(z),
\]

where \( w \) denotes the column vector \((w_1, \ldots, w_m)^t\).

There is no loss of generality to assume that \( \Lambda \) has Jordan canonical form. Thus \( \{w_1, w_2, \ldots, w_m\} \) splits off into subsets which belong to different Jordan blocks. If \( \{w_1, w_2, \ldots, w_k\} \) is one of these subsets, then

\[
\begin{align*}
w_1(z + 2\pi i) &= \lambda w_1(z) \\
w_2(z + 2\pi i) &= \lambda w_2(z) + w_1(z) \\
\vdots &= \vdots \\
w_k(z + 2\pi i) &= \lambda w_k(z) + w_{k-1}(z),
\end{align*}
\]

holds, where \( \lambda \) is an eigenvalue of \( \Lambda \). An easy calculation then gives

\[
\begin{align*}
w_1(z) &= e^{\pi i} \phi_1(z) \\
w_2(z) &= e^{2\pi i} (\ell_1(z) \phi_1(z) + \phi_2(z)), \\
\vdots &= \vdots \\
w_k(z) &= e^{k\pi i} (\ell_{k-1}(z) \phi_1(z) + \cdots + \ell_1(z) \phi_{k-1}(z) + \phi_k(z))
\end{align*}
\]

with \( 2\pi i c = \log \lambda, \ 2\pi i \text{-periodic entire functions } \phi_j \text{ and polynomials } \ell_j \text{ defined by} \)

\[
j!(2\pi i)^j \ell_j(z) = \prod_{\nu=0}^{j-1} (z - 2\nu \pi i).
\]

**Remark** If the coefficients in (\ref{eq:5}) are arbitrary entire and \( 2\pi i \)-periodic functions, then Floquet’s Theory (see, e.g., [\?]), or Fuchs’ Theory, applied to equation (\ref{eq:6}) after the substitution \( x = e^z \), leads to a distinguished fundamental set of solutions, which consists of blocks (\ref{eq:7}), where now the \( \phi_j \)’s are merely \( 2\pi i \)-periodic entire functions.

It remains to show, that, in our case, each function \( \phi_j \) has the form

\[
\phi_j(z) = Q_j(e^z, e^{-z}), \quad Q_j \text{ a polynomial.}
\]

For a proof we need the following

**Lemma 2** Let \( \phi \) be a \( 2\pi i \)-periodic entire function with \( \log T(r, \phi) = o(r) \) and \( L[\phi] = A(e^z, e^{-z}) \), where \( L \) is given by (\ref{eq:8}), (\ref{eq:9}), and \( A \) is a polynomial. Then \( \phi \) itself is a polynomial in \( e^z \) and \( e^{-z} \).
The proof will be given in section 7. We may assume that \( c = 0 \) holds, for otherwise we could replace \( w_j(z) \) by \( w_j(z)e^{-cz} \) and \( L \) by \( L^*[v] = e^{-cz}L[ve^{cz}] \). Now Lemma 2 applies to \( \phi = \phi_1 \) and \( A = 0 \), and hence (??) holds for \( j = 1 \). If this is true for \( 1 \leq j < \kappa \leq k \), then it follows from (??) and (??) that

\[
L[\phi_\kappa] = -(L[\ell_1\phi_{\kappa-1} + \cdots + \ell_{\kappa-1}\phi_1]) = A(z, e^z, e^{-z}),
\]

A a polynomial. But \( \phi_\kappa \) and the coefficients of \( L \) are \( 2\pi i \)-periodic, and hence \( A(z, e^z, e^{-z}) = A(e^z, e^{-z}) \) is also \( 2\pi i \)-periodic. Thus Lemma 2 applies with \( \phi = \phi_\kappa \) and gives (??) for \( j = \kappa \). This finishes the proof of Theorem 3.

7 Proof of Lemma 2

We write \( \phi(z) = f(e^z) \), where \( f \) is analytic in \( 0 < |x| < \infty \), and so \( f(x) = g(x) + h(x^{-1}) \) holds with \( g \) and \( h \) entire functions, \( h(0) = 0 \). Now \( f \) satisfies some (inhomogeneous) linear differential equation with singularities only at \( x = 0 \) and \( x = \infty \),

\[
x^s f^{(n)}(x) + q_{n-1}(x)f^{(n-1)}(x) + \cdots + q_0(x)f(x) = q(x), \quad (14)
\]

where \( s \geq n \) is an integer and the \( q \)'s are polynomials. Since \( f(x) = g(x) + O(x^{-1}) \) as \( x \to \infty \), we obtain a similar relation for \( g \),

\[
x^s g^{(n)}(x) + q_{n-1}(x)g^{(n-1)}(x) + \cdots + q_0(x)g(x) = p(x) + O(x^{-1}),
\]

where \( p \) is also a polynomial. The method of central index (see, e.g., Jank-Volkmann [?]) then yields either \( \log M(r, g) \sim \text{const.} \ r^n, \alpha > 0 \), or else \( g \) is a polynomial. It is, however, obvious that

\[
T(r, w) = T(r, g \circ \exp) + T(r, h \circ \exp) + O(1)
\]

holds, and so \( g \) has to be a polynomial. In the same manner (consider the limit \( x \to 0 \)) it is proved that \( h \) is a polynomial. This proves Lemma 2.

References


