Exceptional Solutions of n-th Order Periodic Linear Differential Equations

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Dedicated to the memory of Steven Bank

Let $L := D^n + p_{n-1}(z)D^{n-1} + \cdots + p_1(z)D + p_0(z)$, $D = d/dz$, be a linear differential operator, whose coefficients are (constants or) $2\pi i$-periodic entire functions of order one, mean type. We will prove that any exceptional solution of $L[w] = 0$, i.e., any solution satisfying $\log N(r, 1/w) = o(r)$, has the form

$$w(z) = e^{S(e^{z/q}, e^{-z/q})} \sum_{j=1}^{m} e^{c_j z} P_j(z, e^{z/q}) ,$$

where $q \geq 1$ is an integer, the c_j's are complex constants and S and the P_j's are polynomials. We give also a new proof of a result due to Steinbart, who classified the so-called subnormal solutions – solutions satisfying $\log T(r, w) = o(r)$.

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1 Introduction

In [**?**] Bank and Langley considered the differential equation

$$w^{(n)} + A_{n-1}w^{(n-1)} + \cdots + A_1 w' + A_0(e^z, e^{-z})w = 0 ,$$

with A_ν constant for $1 \leq \nu \leq n-1$ and A_0 a non-constant polynomial [1]. They proved that any solution satisfying $\log N(r, 1/w) = o(r)$ [2] has the form

$$w(z) = P(e^{z/q}) \exp(cz + S(e^{z/q}, e^{-z/q})) ,$$

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[1] That means $A_0(x, y) = B(x) + C(y)$, B and C polynomials, not both constant.

[2] For terminology in Nevanlinna theory see Hayman [**?**], Jank-Volkmann [**?**] or Nevanlinna [**?**].

where P and S are polynomials, $q \geq 1$ is an integer, and c is a complex number. The case $n = 2$ is due to Bank and Laine [**?**].
We consider a linear differential equation

$$L[w] := w^{(n)} + p_{n-1}(z)w^{(n-1)} + \cdots + p_0(z)w = 0 \tag{1}$$

with entire periodic coefficients

$$p_\nu(z) = A_\nu(e^z, e^{-z}), \tag{2}$$

where the A_ν's are polynomials, at least one being non-constant. Then every solution w is an entire function which satisfies the growth condition $\log\log M(r,w) = O(r)$, as an easy application of Gronwall's Lemma shows, and hence

$$\log N\Big(r, \frac{1}{w}\Big) \leq \log T(r,w) = O(r).$$

We call any solution of (**??**),(**??**) *exceptional,* if it satisfies

$$\log N\Big(r, \frac{1}{w}\Big) = o(r),$$

and *subnormal,* if even

$$\log T(r,w) = o(r)$$

is true.

Example 1 The functions $e^{e^z}, e^{-z+e^z}, e^{-z-2e^z}$ have constant non-zero Wronskian determinant, and hence they form a zero-free, and so an exceptional fundamental set of solutions of some equation

$$w''' + A(e^z)w' + B(e^z)w = 0.$$

Every non-trivial linear combination $a_1w_1 + a_2w_2$ is exceptional, but no non-trivial solution is subnormal.

Example 2 The Wronskian determinant of the entire functions

$$e^{-\frac{5}{4}z}, e^{-\frac{1}{4}z}, e^{-\frac{5}{4}z+e^z}, e^{-\frac{5}{4}z-e^z}$$

is a non-zero constant, and hence we have

$$w^{(4)} + A(e^z)w'' + B(e^z)w' + C(e^z)w = 0$$

with polynomials A, B, C and $w = w_j$, $j = 1,2,3,4$. Every non-trivial linear combination $a_1w_1 + a_2w_2$ is subnormal, but no solution different from these and $w = \text{const } w_{3,4}$ is exceptional.

Example 3 Consider $w(z) = e^{\eta e^z}$, where η is any n-th root of unity, $n \geq 2$. Then a simple proof by induction shows that $w^{(\nu)}(z) = P_\nu(\eta e^z)w(z)$ holds, where $P_\nu(x) = x^\nu + \cdots + x$ is a polynomial, independent of η. Hence there exist complex constants λ_ν such that $P_n(x)+\lambda_{n-1}P_{n-1}(x)+\cdots+\lambda_1P_1(x)=x^n$, and $\eta^n=1$ yields

$$w^{(n)}+\lambda_{n-1}w^{(n-1)}+\cdots+\lambda_1w'-e^{nz}w=0\,.$$

It is obvious that no solution $w(z)\neq \text{const.}e^{\eta e^z}$ is exceptional, and no non-trivial solution is subnormal.

2 Results

We will prove the following representation theorem for exceptional solutions:

Theorem 1 *Any exceptional solution of eq.* (**??**),(**??**) *is a linear combination of linearly independent exceptional solutions*

$$w_j(z)=P_j(z,e^{z/q})\exp\Big(c_jz+S(e^{z/q},e^{-z/q})\Big)\,, \tag{3}$$

$1\le j\le k\le n$, *where the* P_j*'s are polynomials,* $q\ge 1$ *is an integer, the* c_j*'s are complex numbers, mutually distinct mod* 1 , *and* S *is a polynomial independent of* j .

A closer examination of the proof of Theorem 1 shows that the following is true:

Theorem 2 *Let* $w(z)=P(z,e^{z/q})\exp\Big(cz+S(e^{z/q},e^{-z/q})\Big)$ *be an exceptional solution of eq.* (**??**),(**??**). *Then either*

$$w(z)=P(e^{z/q})\exp\Big(cz+S(e^{z/q},e^{-z/q})\Big) \tag{4}$$

holds, or else

$$w(z)=P(z,e^{z})\exp\Big(cz+S(e^{z},e^{-z})\Big)\,. \tag{5}$$

Theorem 1 corresponds to a recent result due to Steinbart [**?**], see Theorem 3 below, which we need for a proof of Theorem 1. For the sake of completeness we will give an independent proof in section 5. For related results see [**?**],[**?**],[**?**],[**?**],[**?**],[**?**].

Theorem 3 *The linear space of subnormal solutions of* (**??**),(**??**) *has a basis consisting of functions* $w_{\mu j}(z)=P_{\mu j}(z,e^z)e^{c_\mu z}$, *where*

$$P_{\mu j}(z,x)=\ell_{j-1}(z)Q_{\mu 1}(x)+\cdots+\ell_1(z)Q_{\mu j-1}(x)+Q_{\mu j}(x)\,, \tag{6}$$

$1\le j\le k_\mu$, $1\le \mu\le m$, *say. The* $Q_{\mu\kappa}$*'s are polynomials,* c_μ *is a complex number, and* ℓ_κ *is given by* $\kappa!(2\pi i)^\kappa\ell_\kappa(z)=\prod_{\nu=0}^{\kappa-1}(z-2\nu\pi i)$.

Remark Theorems 1 and 3 – as far as they concern the form of subnormal and exceptional solutions – apply also to inhomogeneous equations

$$L[w]=A(e^z,e^{-z})\,,$$

since the solutions also solve a homogeneous equation $L_0L[w]=0$, where L_0 is any linear differential operator with constant coefficients which annihilates $A(e^z,e^{-z})$. Note however, that in this case we have $m(r,1/w)=O(r)$ for every solution, and so every exceptional solution is also subnormal.
It follows from Theorem 3 that, if there is any subnormal solution, then there exists also a subnormal solution which is $2\pi i$-periodic up to a factor e^{cz}, in contrast to the case of exceptional solutions, where the period may have any value $2\pi iq$, $1\le q\le n$. In case $q>1$, however, there exist at least q linearly independent exceptional solutions. By modifying Example 3 it is easily seen that every q, $1\le q\le n$, may occur.

Example 4 Let $1\le q\le n$ be an integer and set $w_j(z)=e^{e^{(2\pi ij+z)/q}}$. If L_* denotes the operator of order q which annihilates each w_j (Example 3) and L_0 is any operator of order $n-q$ with constant coefficients, then $L=L_0L_*$ has order n, coefficients of type (**??**), and $L[w_j]=0$, $1\le j\le q$.

For one step in the proof of Theorem 1 we need a theorem of Nevanlinna [**?**], sometimes called Nevanlinna's Third Main Theorem:

Nevanlinna's Theorem *Let* Ψ_j, $1\le j\le n$, *be linearly independent meromorphic functions in the plane satisfying* $\Psi_1+\Psi_2+\cdots+\Psi_n=1$. *Then*

$$T(r,\Psi_j)\le\sum_{k=1}^{n}\Big(N\Big(r,\frac{1}{\Psi_k}\Big)-N\big(r,\Psi_k\big)\Big)+N(r,\Psi_j)+N(r,W)-N\Big(r,\frac{1}{W}\Big)+S(r)$$

holds, where W *is the Wronskian determinant of* $\Psi_1,\ldots,\Psi_n$, *and* $S(r)$ *is the usual remainder term,* $S(r)=O(\log\max_{1\le k\le n}rT(r,\Psi_k))$ *outside a set of finite measure.*

We will also frequently make use of the following more or less well-known facts, without further reference:

(a) $\log T(r,e^h)=o(r)$ and $\log T(r,e^h)=O(r)$ (h entire) imply $T(r,h)=o(r)$ and $T(r,h)=O(r)$, respectively;

(b) $\log N\big(r,f\big)+\log N\Big(r,\frac{1}{f}\Big)=o(r)$ (f meromorphic in the plane) implies $f(z)=g(z)e^{h(z)}$ for appropriately chosen functions h and g, with $\log T(r,g)=o(r)$ (see Ahmad [**?**], and also Jank-Volkmann [**?**]).

(c) $T(r,f)=O(r)$ (f entire and $2\pi i$-periodic) implies $f(z)=P(e^z,e^{-z})$, P a polynomial, and $T(r,f)\sim$ const. r.

3 Proof of Theorem 1

Let w be a exceptional solution of (**??**),(**??**). As everyone would do, we consider the exceptional solutions $f_j(z)=w(z+2\pi ij)$, $j=0,1,2,\ldots$, and denote by q the largest integer such that the functions $f_0,f_1,\ldots,f_{q-1}$ are linearly independent. Then there exist complex numbers λ_j, not all zero, such that $f_q=\sum_{j=0}^{q-1}\lambda_jf_j$ holds, and so $\sum\limits_{\lambda_j\ne0}\Phi_j=1$ with $\Phi_j=\lambda_jf_j/f_q$. Note that $\lambda_0\ne0$, since otherwise $f_1,\ldots,f_q$

would be linearly dependent. Nevanlinna's Theorem then gives $\log T(r,\Phi_0) = o(r)$, and hence $w(z+2\pi iq) = K(z)w(z)$, where K is meromorphic and satisfies $\log T(r,K) = o(r)$.
Without loss of generality we may assume $q = 1$, for otherwise we could introduce the new independent variable $z' = z/q$. By remark (b), w may be written as

$$w(z) = g(z)e^{H(z)},$$

where g is entire with $\log T(r,g) = o(r)$, and H is entire and has order one, mean type, at most. This gives

$$\frac{g(z+2\pi i)}{g(z)} = K(z)e^{-H(z+2\pi i)+H(z)},$$

and so

$$H(z+2\pi i) - H(z) = Q(z)\,, \quad T(r,Q) = o(r)$$

by remark (a). If we set $h(z) = H(2\pi iz)$ and $q(z) = Q(2\pi iz)$, then the following lemma applies to the functional equation

$$h(z+1) - h(z) = q(z)\,. \tag{7}$$

Lemma 1 *Let q be an entire function, restricted to the growth condition $\log M(r,q) = o(r)$. Then eq. (**??**) has exactly one solution in this class satisfying $h(0) = 0$.*

The proof will be given in section 4.
By that lemma and remarks (a) and (c), H may be written as $H(z) = p(z) + S\big(e^z, e^{-z}\big)$, where p satisfies $T(r,p) = o(r)$ and S is a polynomial. This yields the representation

$$w(z) = g(z)e^{p(z)}e^{S(e^z,e^{-z})}.$$

Now $v(z) = g(z)e^{p(z)} = w(z)e^{-S(e^z,e^{-z})}$ is a subnormal solution of some equation of type (**??**),(**??**), and hence, by Theorem 3, has the representation $v(z) = \sum_{j=1}^{k} e^{c_j z}P_j(z,e^z)$, with c_j mutually distinct mod 1.
It is obvious that, for $w_j(z) = P_j\,(z,e^z)\exp\left(c_j z + S(e^z,e^{-z})\right)$,

$$L[w_j] = Q_j(z,e^z,e^{-z})\exp\big(c_j z + S(e^z,e^{-z})\big)$$

holds with polynomials Q_j, and hence

$$e^{-S(e^z,e^{-z})}L[w] = \sum_{j=1}^{k} Q_j(z,e^z,e^{-z})e^{c_j z}$$

is an exponential polynomial with frequencies $\equiv c_j\ 1$; any such function vanishes identically if and only if $Q_j = 0$ for $1 \le j \le k$, i.e., if and only if $L[w_j] = 0$. This finishes the proof of Theorem 1.

4 Proof of Lemma 1

We first remark, that the condition $\log M(r,q)=o(r)$ is equivalent to $q=Q*\exp$, where $*$ is the Hadamard convolution, and Q is an entire function. Hence we may write $q(z)=\sum\limits_{k=0}^{\infty}\frac{y_k}{k!}z^k$ and set $h(z)=\sum\limits_{k=1}^{\infty}\frac{x_k}{k!}z^k$. Then eq. (**??**) is equivalent to

$$x_{k+1}+\sum_{n=2}^{\infty}\frac{x_{k+n}}{n!}=y_k\,,\quad k=0,1,2,\ldots\,. \tag{8}$$

We consider that equation in the space ℓ^∞ of bounded complex sequences $x=(x_0,x_1,x_2,\ldots)$, endowed with the supremum-norm $||x||_\infty=\sup\limits_{k\geq 0}x_k$. Then (**??**) may be written as

$$\sigma x+T\sigma^2x=y\,, \tag{9}$$

where σ is the shift operator $(x_0,x_1,x_2,\ldots)\mapsto(x_1,x_2,x_3,\ldots)$, and the linear map $T:\ell^\infty\longrightarrow\ell^\infty$ is represented by the infinite matrix

$$\begin{pmatrix} \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \cdots \\ 0 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots \\ 0 & 0 & \frac{1}{2!} & \frac{1}{3!} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Note that $T\sigma$ has contraction constant $e-2<1$, and that the growth condition $\log M(r,q)=o(r)$ is equivalent to $\lim\limits_{n\to\infty}\sqrt[n]{||\sigma^n y||_\infty}=0$.
By the contraction principle, the corresponding equation $u+T\sigma u=y$ has a unique solution $u\in\ell^\infty$, and hence eq. (**??**) has a unique solution $x=(0,x_1,x_2,\ldots)\in\ell^\infty$. From

$$||x||_\infty\leq\frac{1}{3-e}||y||_\infty \quad\text{and}\quad \sigma^{n+1}x+T\sigma^{n+2}x=\sigma^n y$$

it then follows that $\lim\limits_{n\to\infty}\sqrt[n]{||\sigma^n x||_\infty}=0$. Thus $h(z)=\sum\limits_{k=1}^{\infty}\frac{x_k}{k!}z^k$ is the unique solution of eq. (**??**) with $h(0)=0$ and $\log M(r,h)=o(r)$, and this proves Lemma 1.

5 Proof of Theorem 2

To prove Theorem 2, we recall the proof of Theorem 1. With the notation used there we have either $\lambda_j=0$ for $1\leq j<q$ or else $\lambda_j\neq 0$ for some $j\in\{1,\ldots,q-1\}$. In the first case $f_q=\lambda_0 f_0$ holds, and hence (**??**). In the second case it follows from $T(r,f_j/f_q)=O(r)$ that the difference $S(e^{(z+2\pi ij)/q},e^{-(z+2\pi ij)/q})-S(e^{z/q},e^{-z/q})$ is a constant, and hence $S(x,y)=U(x^t,y^t)$ holds, where U is a polynomial, $t=q/r$ and r is the greatest common divisor of j and q; note that $t>1$. Since $v(z)=w(z)e^{-U(e^{z/r},e^{-z/r})}$ is a subnormal solution of some eq. (**??**), where now the coefficients are merely $2\pi r$-periodic, we find that

$$w(z)=Q(z,e^{z/r})e^{U(e^{z/r},e^{-z/r})} \tag{10}$$

holds, and also that

$$\textit{the functions } f_0,f_t,\ldots,f_{rt} \textit{ are linearly dependent.} \tag{11}$$

Now let $r \geq 1$ be the smallest divisor of q such that (**??**) and (**??**) hold with $q = rt$. Repetition of the proof of Theorem 1, but now starting with the functions $f_0, f_t, \dots, f_{rt}$, $q = rt$, instead of $f_0, f_1, \dots, f_q$, shows that, by minimality of r, either $r = 1$ holds or else $f_{rt} = \lambda_0 f_0$. By (**??**), the first case leads to (**??**), and the second case to (**??**), with q replaced by r, as was already observed in the beginning of the proof. This finishes the proof of Theorem 2.

6 Subnormal Solutions and Floquet's Theory: Proof of Theorem 3

Let $w_1, w_2, \dots, w_m$ form a base of the linear space of subnormal solutions. Since the functions $w_1(z+2\pi i), w_2(z+2\pi i), \dots, w_m(z+2\pi i)$ form also a base, there exists a regular (m,m)-matrix Λ with

$$\mathbf{w}(z+2\pi i) = \Lambda \mathbf{w}(z),$$

where $\mathbf{w}$ denotes the column vector $(w_1, \dots, w_m)^t$.
There is no loss of generality to assume that Λ has Jordan canonical form. Thus $\{w_1, w_2, \dots, w_m\}$ splits off into subsets which belong to different Jordan blocks. If $\{w_1, w_2, \dots, w_k\}$ is one of these subsets, then

$$\begin{array}{rcl} w_1(z+2\pi i) & = & \lambda w_1(z) \\ w_2(z+2\pi i) & = & \lambda w_2(z) + w_1(z) \\ \cdots\cdots & = & \cdots\cdots \\ w_k(z+2\pi i) & = & \lambda w_k(z) + w_{k-1}(z), \end{array}$$

holds, where λ is an eigenvalue of Λ. An easy calculation then gives

$$\begin{array}{rcl} w_1(z) & = & e^{cz}\phi_1(z) \\ w_2(z) & = & e^{cz}(\ell_1(z)\phi_1(z) + \phi_2(z)), \\ \cdots & = & \cdots\cdots \\ w_k(z) & = & e^{cz}(\ell_{k-1}(z)\phi_1(z) + \cdots \ell_1(z)\phi_{k-1}(z) + \phi_k(z)) \end{array} \tag{12}$$

with $2\pi i c = \log\lambda$, $2\pi i$-periodic entire functions ϕ_j and polynomials ℓ_j defined by $j!(2\pi i)^j \ell_j(z) = \prod_{\nu=0}^{j-1}(z - 2\nu\pi i)$.

Remark If the coefficients in (**??**) are arbitrary entire and $2\pi i$-periodic functions, then Floquet's Theory (see, e.g., [**?**]), or Fuchs' Theory, applied to equation (**??**) after the substitution $x = e^z$, leads to a distinguished fundamental set of solutions, which consists of blocks (**??**), where now the ϕ_j's are merely $2\pi i$-periodic entire functions.

It remains to show, that, in our case, each function ϕ_j has the form

$$\phi_j(z) = Q_j(e^z, e^{-z}), \quad Q_j \text{ a polynomial.} \tag{13}$$

For a proof we need the following

Lemma 2 *Let ϕ be a $2\pi i$-periodic entire function with $\log T(r,\phi) = o(r)$ and $L[\phi] = A(e^z, e^{-z})$, where L is given by (**??**),(**??**), and A is a polynomial. Then ϕ itself is a polynomial in e^z and e^{-z}.*

The proof will be given in section 7.
We may assume that $c = 0$ holds, for otherwise we could replace $w_j(z)$ by $w_j(z)e^{-cz}$ and L by $L^*[v] = e^{-cz}L\,[ve^{cz}]$. Now Lemma 2 applies to $\phi = \phi_1$ and $A = 0$, and hence (**??**) holds for $j = 1$. If this is true for $1 \le j < \kappa \le k$, then it follows from (**??**) and (**??**) that

$$L[\phi_\kappa] = -\big(L[\ell_1\phi_{\kappa-1} + \cdots + \ell_{\kappa-1}\phi_1]\big) = A(z, e^z, e^{-z})\,,$$

A a polynomial. But ϕ_κ and the coefficients of L are $2\pi i$-periodic, and hence $A(z, e^z, e^{-z}) = A(e^z, e^{-z})$ is also $2\pi i$-periodic. Thus Lemma 2 applies with $\phi = \phi_\kappa$ and gives (**??**) for $j = \kappa$. This finishes the proof of Theorem 3.

7 Proof of Lemma 2

We write $\phi(z) = f(e^z)$, where f is analytic in $0 < |x| < \infty$, and so $f(x) = g(x) + h(x^{-1})$ holds with g and h entire functions, $h(0) = 0$. Now f satisfies some (inhomogeneous) linear differential equation with singularities only at $x = 0$ and $x = \infty$,

$$x^s f^{(n)}(x) + q_{n-1}(x)f^{(n-1)}(x) + \cdots + q_0(x)f(x) = q(x)\,, \tag{14}$$

where $s \ge n$ is an integer and the q's are polynomials. Since $f(x) = g(x) + O(x^{-1})$ as $x \to \infty$, we obtain a similar relation for g ,

$$x^s g^{(n)}(x) + q_{n-1}(x)g^{(n-1)}(x) + \cdots + q_0(x)g(x) = p(x) + O(x^{-1})\,,$$

where p is also a polynomial. The method of central index (see, e.g., Jank-Volkmann [**?**]) then yields either $\log M(r, g) \sim$ const. r^α , $\alpha > 0$, or else g is a polynomial. It is, however, obvious that

$$T(r, w) = T(r, g \circ \exp) + T(r, h \circ \exp) + O(1)$$

holds, and so g has to be a polynomial. In the same manner (consider the limit $x \to 0$) it is proved that h is a polynomial. This proves Lemma 2.

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