On Painlevé's Equations I, II and IV

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Abstract—We give a new proof of the fact that the solutions of Painlevé's differential equations I, II and IV are meromorphic functions in the complex plane. The method of proof is based on differential inequality techniques.

1—Introduction—In a recent paper, Aimo Hinkkanen and Ilpo Laine [HL] have given a proof of the widely accepted fact that the solutions of Painlevé's equations

$$w'' = z + 6w^2 \tag{I}$$

and

$$w'' = \alpha + zw + 2w^3 \tag{II}$$

are meromorphic in the complex plane[1]. The necessity for such a proof is explained in detail in that paper, the main reason being that the *proofs* presented in several textbooks such as Bieberbach [B], Golubew [Go], Hille [H], Ince [I] and some others seem to be incomplete, or have gaps, to say the least.

In the present paper the corresponding result for Painlevé's fourth equation will be proved.

Theorem 1 *The solutions of Painlevé's differential equation*

$$w'' = \frac{w'^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w} \tag{IV}$$

are meromorphic functions in $\mathbb{C}$.

The proof differs in many aspects from that in [HL], and also from the other *proofs* mentioned above. It uses *differential inequality methods,* which not only shorten the proof substantially, but also have still another advantage: discussion of several cases and subcases can be reduced—apart from the natural cases *zero or pole?*—, and the proof becomes more transparent.

The methods also apply to Painlevé's equations (I) and (II), and thus lead to a proof of Theorem 2 below. However, Theorem 2 is also an immediate consequence

[1] I learned about this at the Oberwolfach Conference *Funktionentheorie,* February 14–20, 1999.

of Theorem 1. This follows from the fact that equations (I) and (II) can be derived from (IV) by the *process of coalescence*—see section 7. Computer algebra software may help to verify some of the calculations, but is *not* necessary.

Theorem 2 [HL] *The solutions of Painlevé's equations (I) and (II) are meromorphic in the plane.*

Remark Equations (I), (II) and (III), see section 8, are due to P. Painlevé [P], [PP], while (IV) and (V) were discovered by B. Gambier [Ga]; R. Fuchs [F] completed the list by adding the master type (VI).[2]

2—Proof of Theorem 1: Prologue—Let w be any local solution of (IV), determined by $w(z_0) = w_0$ and $w'(z_0) = w_0'$, say, in a neighbourhood of z_0, and let $\Gamma : t \mapsto z_t,\ \ 0 \le t \le 1$, be a rectifiable curve starting at z_0 and ending at z_1. We assume that w admits analytic continuation along every closed subarc of $\Gamma_1 : t \mapsto z_t,\ \ 0 \le t < 1$; it is thus implicitely assumed that w has no poles on Γ_1. We have to show that w has a *meromorphic* continuation along Γ, and then apply the *Monodromy Theorem.*

If we assume, as we may, that z_1 is a singularity, then

$$|w(z)| + |w'(z)| + |w(z)|^{-1} \to +\infty \quad \text{as} \quad z \to z_1 \quad \text{along} \quad \Gamma. \tag{1}$$

This is a consequence of Painlevé's Theorem (for systems), see Bieberbach [B], p. 20, which expresses the fact that, as $t \to 1-$, the curve $t \mapsto (z_t, w(z_t), w'(z_t))$ leaves every compact subset of $\mathbb{C} \times \mathbb{C}^* \times \mathbb{C}$, the domain of definition of the right hand side of (IV); here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Since we expect a pole at $z = z_1$, we will have a closer look at the behaviour of solutions at poles. An easy computation gives

$$w(z) = \frac{\pm 1}{z - z_1} - z_1 + \frac{1}{3}(\pm z_1^2 \pm 2\alpha - 4)(z - z_1) + h(z - z_1)^2 + a_3(z - z_1)^3 + \cdots$$

($h \in \mathbb{C}$ is a free parameter) and so

$$w(z) + z = \frac{\pm 1}{z - z_1} + \frac{1}{3}(\pm z_1^2 \pm 2\alpha - 1)(z - z_1) + \cdots$$

Hence, $w^2 + 2zw$ has a local primitive at z_1,

$$V(z) = -\int (w^2(z) + 2zw(z))dz = \frac{1}{z - z_1} + \cdots, \tag{2}$$

and using (IV) we obtain

$$w'^2 = w^4 + 4zw^3 + 4(z^2 - \alpha)w^2 - 2\beta + 4wV, \tag{3}$$

[2] For this fact I rely on the books of Golubew and Hille; the paper [F] was not available to me.

if the constant of integration is suitably chosen. Now, for each complex number c, the logarithmic derivative $w'/(w-c)$ has a simple pole at z_1 with residue -1, and so

$$U = V + \frac{w'}{w-c} \tag{4}$$

is regular at z_1. We prove now as a first step

Lemma 1 *U is bounded on Γ_1 provided $|w-c|$ is bounded away from zero.*

Remark In the next section we will show that the additional assumption, for the constant $c = 1$ and, in fact, for every $c \neq 0$, can be fulfilled by deforming Γ appropriately. The idea to introduce the auxiliary function U is not new–at least in cases (I) and (II). What is new is that its boundedness is proved *a priori,* by differential inequality methods. The method gives much more: it turns out that U' is also bounded, and hence U is Lipschitz-continuous on Γ_1.

Proof Differentiating (4) and using (IV) and (3) to eliminate w'' and w' yields

$$U' = Q_1 U + Q_2 + (Q_3 U + Q_4)^{1/2}, \tag{5}$$

where Q_1, Q_2, Q_3, Q_4 are polynomials in z and $(w-c)^{-1}$, and hence are bounded functions on Γ_1 (this computation was *first* made by hand and *then* verified by using computer algebra software). Thus

$$\frac{d}{ds}|U| \leq |U'| \leq K_1|U| + K_2$$

holds on Γ_1, where s denotes arc-length and K_1 and K_2 are positive constants. Since Γ_1 is rectifiable, this yields boundedness of U on Γ_1, and Lemma 1 is proved.

3—Another differential inequality technique—Boundedness of U holds true if one can assure that, for some constant c, $|w-c|$ is bounded away from zero, or if one can *deform* Γ into an equivalent $\tilde{\Gamma}$, such that this is true for $\tilde{\Gamma}$. We will prove that this is always possible:

Lemma 2 *Under the hypothesis (1), there exists a curve $\tilde{\Gamma}$ with the following properties:*

1° *$\tilde{\Gamma}$ is rectifiable and has endpoints z_0 and z_1.*
2° *Analytic continuation of w along $\tilde{\Gamma}_1$ is possible and leads to the same type of singularity at z_1, actually, to the same result.*
3° *On $\tilde{\Gamma}_1$, $|w-1|$ is bounded away from zero.*

To this end we first prove Lemma 3 below, which is also the central result in [HL] for Painlevé's equations I and II:

Lemma 3 *Let $P(z,w)$ be a polynomial in both variables and let $z = z(w)$ be the solution of the initial value problem*

$$z'' = -\frac{z' + P(z,w)z'^3}{2w}, \quad z(w_0) = z_0, \ z'(w_0) = \eta, \tag{6}$$

where $w_0 \neq 0$ and $0 < |\eta| < 1$. Then there exists $\delta > 0$ such that $z = z(w)$ exists in $|w - w_0| \leq |w_0|/2$ and satisfies $\frac{1}{2}|\eta| \leq |z'(w)| \leq 2|\eta|$, whenever $0 < |\eta| < \delta$. If z_0 and w_0 are restricted to $|z_0| \leq K$ and $1/K \leq |w_0| \leq K$, then δ may be chosen to depend on K (and P) only.

Proof Let $\varrho > 0$ be the largest radius with the following properties:

$$\varrho \leq |w_0|/2, \ z(w) \text{ exists in } |w - w_0| < \varrho \text{ and satisfies } |z(w) - z_0| < 1.$$

Set $M = \sup |P(z,w)|$, where the variables are constrained to $|w - w_0| \leq |w_0|/2$, $|z - z_0| \leq 1$, $|z_0| \leq K$ and $1/K \leq |w_0| \leq K$. Then $M = M(K)$ obviously depends on K only, and in $|w - w_0| < \varrho$ the following estimate holds:

$$|z''(w)| \leq \frac{|z'(w)| + M|z'(w)|^3}{|w_0|}.$$

We set $v(r) = |z'(w_0 + re^{i\alpha})|$, $\alpha \in \mathbb{R}$ fixed, and observe that

$$|v'(r)| = \left| \frac{d}{dr} \left| z'(w_0 + re^{i\alpha}) \right| \right| \leq |z''(w_0 + re^{i\alpha})|$$

holds, and so

$$-\frac{v + Mv^3}{|w_0|} \leq v' \leq \frac{v + Mv^3}{|w_0|}, \quad v(0) = |\eta|. \tag{7}$$

By usual differential inequality techniques— a good reference is Walter's monograph [W]—we thus have

$$y_-(r) \leq v(r) \leq y_+(r),$$

where $y = y_\pm$ have to be computed from

$$y' = \pm \frac{y + My^3}{|w_0|}, \ y(0) = |\eta|.$$

For y_+ it is easy to find the upper bound $\dfrac{|\eta| e^{r/|w_0|}}{(1 - Me|\eta|^2)^{1/2}}$ for $0 \leq r \leq |w_0|/2$. Thus if we restrict η to $0 < |\eta| < (4eM)^{-1/2}$, we obtain

$$|z'(w)| \leq \frac{|\eta| e^{1/2}}{\sqrt{3/4}} < 1.91|\eta| < 2|\eta|, \quad w = w_0 + re^{i\alpha},$$

this inequality being true for $0 \leq r < \varrho \leq |w_0|/2$. It follows that

$$|z(w) - z_0| < 2|\eta|\varrho \leq 2|\eta||w_0|/2 \leq |\eta|K$$

for $|w - w_0| < \varrho$. If we restrict η again, this time to $|\eta|K < 1$, we obtain

$$|z'(w)| < 2|\eta| \quad \text{and} \quad |z(w) - z_0| < 1 \;\text{ in }\; |w - w_0| < \varrho.$$

Thus $z(w)$ cannot have a singularity on $|w - w_0| = \varrho$, again by Painlevé's Theorem, and so, by definition of ϱ this implies $\varrho = |w_0|/2$, under the restriction that

$$0 < |\eta| < \min\{K^{-1}, (4eM(K))^{-1/2}\} = \delta(K).$$

The lower estimate for $|z'(w)|$ follows from

$$\begin{aligned}|z'(w_0 + re^{i\alpha})| &\geq y_-(r) > \frac{y_-(r)}{(1 + My_-^2(r))^{1/2}} = \frac{|\eta|e^{-r/|w_0|}}{(1 + M|\eta|^2)^{1/2}} \\ &\geq \frac{|\eta|}{(e + 1/4)^{1/2}} > 0.58|\eta| > \frac{1}{2}|\eta|.\end{aligned}$$

4—Deformation of Γ—We will prove now that a curve $\tilde{\Gamma}$ with properties $1^\circ - 3^\circ$ in Lemma 2 always exists. We may assume that Γ_1 contains no zeros, poles and 1-points of w. If $\liminf |w - 1| > 0$ (all limits are taken with respect to $z \to z_1$ along Γ_1) we are done: $|w - 1|$ is bounded away from zero on $\tilde{\Gamma}_1 = \Gamma_1$. On the other hand, if $\lim |w - 1| = 0$, it follows from its definition (2) that V is bounded, and hence is w' by (3). This, however, contradicts (1). We thus may assume $\liminf |w - 1| = 0$ and $\limsup |w - 1| > \varepsilon > 0$. Then for ε sufficiently small—we need $\varepsilon \leq 1/5$ later on—, the set

$$\Lambda = \{z \in \Gamma_1 : |w(z) - 1| < \varepsilon\}$$

consists of countably many arcs λ_j, which will be replaced by arcs $\tilde{\lambda}_j$ of length $(\tilde{\lambda}_j) \leq 4\pi \cdot$ length (λ_j), and such that $|w - 1| = \varepsilon$ on $\tilde{\lambda}_j$.

By condition (1) we have $\lim |w'(z)| = +\infty$ as $z \to z_1$ on Λ. Consider some arc $\lambda = \lambda_j$ ($j \geq j_0$ so that $|w - 1| = \varepsilon$ at both end points of λ_j and $|w'|$ is sufficiently large on λ_j). Let z_0 be any point on λ and set $w_0 = w(z_0)$ and $w'(z_0) = 1/\eta$. Then the map $z \mapsto w$ has a local inverse $w \mapsto z$ which satisfies

$$z'' = -\frac{1}{2w}\Big(z' + \big(3w^4 + 8zw^3 + 4(z^2 - \alpha)w^2 + 2\beta\big)z'^3\Big),$$
$$z(w_0) = z_0, \quad z'(w_0) = \eta.$$

By Lemma 3, $z(w)$ exists in $|w - w_0| \leq |w_0|/2$ and so in $|w - 1| \leq \varepsilon$, since $|w_0|/2 \geq 2\varepsilon$ (note that $|w_0 - 1| \leq \varepsilon \leq 1/5$). If we replace, in the w-plane, the arc $\mu = w(\lambda)$ by $\tilde{\mu}$, the shorter arc on $|w - 1| = \varepsilon$ joining the end points of μ, then length $(\tilde{\mu}) \leq \pi \cdot$ length (μ) holds. We set $\tilde{\lambda} = z(\tilde{\mu})$; then from $|z'(w_1)/z'(w_2)| \leq 4$ for $|w_j - 1| \leq \varepsilon$, see Lemma 3, it follows that length $(\tilde{\lambda}) \leq 4\pi \cdot$ length (λ). It is obvious that analytic continuation along $\tilde{\lambda}_j = \tilde{\lambda}$ is possible and yields the same result as does analytic continuation along λ_j. Thus the curve $\tilde{\Gamma}$, which is obtained by replacing the arcs λ_j by the arcs $\tilde{\lambda}_j$, has the required properties. Note that $\tilde{\Gamma}_1$ is rectifiable and so has an end-point, namely z_1.

5—Proof of Theorem 1: Epilogue—We know from Lemma 1 and Lemma 2 that $U = V + w'/(w-1)$ and $1/(w-1)$ are bounded on Γ_1. Then (3) may be written as

$$\left(w' + \frac{2w}{w-1}\right)^2 = w^4 + 4zw^3 + 4(z^2-\alpha)w^2 - 2\beta + 4Uw + \frac{4w^2}{(w-1)^2}. \tag{8}$$

We distinguish two cases as follows:

Case 1 $w(\xi_n) \to 0$ on some sequence $(\xi_n) \subset \Gamma_1$, $\xi_n \to z_1$.

For the moment we consider z, U (bounded!) and w as independent variables and expand the right hand side of (8) in a power series about $w = 0$, the result being

$$-2\beta + 4Uw + 4(z^2 - \alpha + 1)w^2 + \cdots \tag{9}$$

Now suppose first that we are in

Subcase 1(i) $-2\beta = \gamma^2 \neq 0$. Then, with the right choice of γ, we have, at least on a subsequence of (ξ_n), denoted again (ξ_n),

$$w' = \gamma\Big(1 + \frac{2}{\gamma^2}(U+\gamma)w + c_2(z,U)w^2 + \cdots\Big), \quad z = \xi_n,$$

where $c_2(z,U)$ and the following coefficients are polynomials in z, U. We take this relation to define $u = u(z)$ by

$$w' = \gamma(1 + uw). \tag{10}$$

Actually we do not define a *function* u*—where should it be defined? A function needs a domain of definition, and this could only be the domain of* w! *Note, however, that* w *has no domain of definition, since* Γ_1 *could have self-intersections*—but a germ near $z = z_0$, say, defined by (10), which admits analytic continuation along Γ_1; this remark remains valid at other occasions, too. We note that u is bounded on (ξ_n), and hence may assume that $\lim_{n\to\infty} u(\xi_n) = u_0$ exists.

Differentiating (10) and using (IV) and (10) to eliminate w'' and w' we obtain

$$u' = -\frac{\gamma}{2}u^2 + \frac{1}{2\gamma}\left(3w^2 + 8zw + 4(z^2-\alpha)\right). \tag{11}$$

Now observe that equations (10), (11) constitute a first-order system of ordinary differential equations. Thus, the desired continuation of w to z_1 is obtained by solving that system with initial values $w(z_1) = 0$, $u(z_1) = u_0$.

We now come to

Subcase 1(ii) $\beta = 0$. Since we expect a zero of order two, we set $w = v^2$, where the same remark as above applies to v —a germ which admits analytic continuation along Γ_1. From (8) and (9) we obtain

$$\left(v' + \frac{v}{v^2-1}\right)^2 = U + (z^2 - \alpha + 1)v^2 + \cdots,$$

whenever $|v|$ is small. Thus v' is bounded on (ξ_n), and again we may assume that $\lim_{n\to\infty} v'(\xi_n) = v_0'$ exists. Then the solution of the initial value problem

$$v'' = \frac{3}{4}v^5 + 2zv^3 + (z^2 - \alpha)v, \quad v(z_1) = 0, \; v'(z_1) = v_0',$$

provides analytic continuation of v, and hence of $w = v^2$, to $z = z_1$.

Case 2 $w(\xi_n) \to \infty$ on some sequence $(\xi_n) \subset \Gamma_1$, $\xi_n \to z_1$.

We proceed in the same manner as in Case 1, just replace w with $1/v$, and obtain the following first-order system

$$\begin{aligned} v' &= \pm(1 + 2zv - 2\alpha v^2 + uv^3) + 2v^2, & v(z_1) = 0; \\ u' &= 4(\alpha - uv) \mp (2 + \beta + 2\alpha^2 + 2zu - 4\alpha vu + \frac{3}{2}u^2v^2), & u(z_1) = u_0. \end{aligned}$$

Similar to subcase 1(i), u is defined by the first equation and is bounded, at least on a subsequence of (ξ_n), provided the sign is chosen appropriately. We may assume that $\lim_{n\to\infty} u(\xi_n) = u_0$ exists. Hence, again, the solution of that system provides analytic continuation of w to $z = z_1$, where now $w = v^{-1}$ has a simple pole at $z = z_1$. Thus, Theorem 1 is completely proved.

6—Painlevé's equations I and II—The method of proof also applies to equations (I) and (II). We just indicate the changes to be made in the **Proof** of Theorem 2; the details are left to the reader. First of all, things become easier, instead of (1) we have

$$|w(z)| + |w'(z)| \to +\infty \quad \text{as} \quad z \to z_1 \quad \text{along} \quad \Gamma_1 \tag{I/II 1}$$

(for notation see section 2), and (3) has to be replaced with

$$w'^2 = 4w^3 + 2zw + V, \quad V = -2\int w(z)dz, \tag{I 3}$$

and

$$w'^2 = w^4 + zw^2 + 2\alpha w + V, \quad V = -\int w^2(z)dz, \tag{II 3}$$

respectively. Also, Lemma 1 remains valid for

$$U = V + \frac{w'}{w} \tag{I/II 4}$$

(in both cases the same expression!), provided $|w|$ is bounded away from zero on Γ_1, or Γ can be deformed as is described in Lemma 2, with $|w - 1|$ replaced with $|w|$. This is always possible, the deformation now being based on

Lemma I/II 3 *Let $P(z, w)$ be a polynomial with $\deg_w P = m \geq 2$, and let $z = z(w)$ be the solution of the initial value problem*

$$z'' = -P(z, w)z'^3, \quad z(w_0) = z_0, \quad z'(w_0) = \eta, \quad 0 < |\eta| < 1.$$

Then there exist $\varrho > 0$ such that $z(w)$ exists in $|w - w_0| \le \varrho$ and satisfies there $\frac{1}{2}|\eta| \le |z'(w)| \le 2|\eta|$. If z_0 and w_0 are restricted to $|z_0| \le K$ and $|w_0| \le K$, then one may choose $\varrho \ge \delta|\eta|^{-\frac{2}{m+1}}$, where $\delta > 0$ depends on K (and P) only.

Remark Lemma I/II 3 is proved in the paper of Hinkkanen and Laine [HL] separately for $P(z, w) = z + 6w^2$ and $P(z, w) = \alpha + zw + 2w^3$. The authors deduce, from the recursion formula for the coefficients in the power series expansion $z(w) = z_0 + \eta(w - w_0) + \sum_{k=2}^{\infty} c_k(w - w_0)^k$, inductively an appropriate estimate for these coefficients. Here, the estimate of the coefficients is replaced with an estimate of the right hand side of the differential equation, which is somewhat easier. The conclusions agree, the lower bounds *const* $|\eta|^{-\frac{2}{3}}$ in case (I), and *const* $|\eta|^{-\frac{1}{2}}$ in case (II), obtained for the radius of convergence, are the same as in Lemma I/II 3.

Proof For $\sigma > 0$ set $M(\sigma) = \max |P(z, w)|$ under the constraints

$$|z - z_0| \le 1, \quad |z_0| \le K, \quad |w - w_0| \le \sigma \quad \text{and} \quad |w_0| \le K.$$

It is obvious that $M(\sigma)$ is independent of z_0 and w_0, and also that $M(\sigma) \sim const(K) \cdot \sigma^m$ holds as $\sigma \to +\infty$. Since $\sigma \mapsto \sigma M(\sigma)$ is strictly increasing and unbounded, there is a unique σ satisfying $2\sigma M(\sigma)|\eta|^2 = 5/8$. It is obvious, too, that $\sigma \sim \delta(K)\, |\eta|^{-\frac{2}{m+1}}$ as $|\eta| \to 0$.

Let $\varrho > 0$ be the largest radius such that

$$\varrho \le \sigma, \quad z(w) \;\; \text{exists in} \;\; |w - w_0| < \varrho \;\; \text{and satisfies} \;\; |z'(w)| < 2|\eta|,$$

and hence $|z(w) - z_0| < 2|\eta|\varrho \le 2|\eta|\sigma < 1$ holds, at least for sufficiently small $|\eta|$. We remark that $z(w)$ has no singularities on $|w - w_0| = \varrho$, again by Painlevé's Theorem. Similar to the proof of Lemma 3 we find, setting $v(r) = |z'(w_0 + re^{i\alpha})|$, that

$$-Mv^3 \le v' \le Mv^3, \quad v(0) = |\eta|,$$

and so

$$|\eta|\Big(1 + 2M(\sigma)|\eta|^2 r\Big)^{-\frac{1}{2}} \le v(r) \le |\eta|\Big(1 - 2M(\sigma)|\eta|^2 r\Big)^{-\frac{1}{2}}.$$

Our choice of σ now gives

$$|\eta|/2 < \sqrt{8/13}\,|\eta| < |z'(w_0 + re^{i\alpha})| < \sqrt{8/3}\,|\eta| < 2|\eta|$$

and $|z(w_0 + re^{i\alpha}) - z_0| < 2|\eta|\varrho < 1$ for $0 \le r < \varrho \le \sigma$. As in the proof of Lemma 3 we may conclude that $\varrho = \sigma$, and the proof of Lemma I/II 3 is finished.

The rest of the story can be found in various textbooks, e.g., in [Go], [H], [I], and, of course in [HL]. However, we have the advantage to know *a priori* that U is bounded on Γ_1, and so we will give the complete proof for the convenience of the reader.

It follows from boundedness of U, (I/II 3) and (I/II 4) that w' is bounded on each sequence where w is bounded. Since, by Painlevé's Theorem, this leads to analytic continuation along Γ, we may assume that $w(z) \to \infty$ as $z \to z_1$ along Γ. Set $v = w^{-1/2}$ and $v = w^{-1}$, respectively (note that this means: v is defined near the starting point z_0 and admits analytic continuation along Γ_1). From (I/II 3) we obtain $(v' - v^5/4)^2 = 1 + zv^4/2 + Uv^6/4 + v^{10}/16$ and $(v' - v^3/2)^2 = 1 + zv^2 + 2\alpha v^3 + Uv^4 + v^6/4$, and so, as $z \to z_1$ along Γ_1,

$$v' = v^5/4 + \varepsilon\left(1 + zv^4/4 + Uv^6/8\right) + O\left(|v|^8\right)$$

and

$$v' = v^3/2 + \varepsilon\left(1 + zv^2/2 + \alpha v^3 + (4U - z^2)v^4/8\right) + O\left(|v|^5\right),$$

respectively; since $v \to 0$ as $z \to z_1$ along Γ_1, the sign $\varepsilon = \pm 1$ is fixed for z close to z_1, for the sake of simplicity we assume $\varepsilon = 1$, say. We proceed now as in the proof of Theorem 1: define, in the respective cases, germs u by

$$v' = 1 + \frac{z}{4}v^4 + \frac{v^5}{4} + uv^6 \tag{Ia}$$

and

$$v' = 1 + \frac{z}{2}v^2 + \alpha_* v^3 + uv^4, \quad \alpha_* = \alpha + \frac{1}{2}. \tag{IIa}$$

Since u is bounded we may assume that $\lim_{n\to\infty} u(\xi_n) = u_0$ on some sequence $\xi_n \to z_1$. Differentiating (Ia) and (IIa) and using the respective differential equation (I) and (II) we obtain

$$u' = -\frac{z^2}{16}v - \frac{3z}{16}v^2 - \frac{1}{8}v^3 - zv^3u - \frac{5}{4}v^4u - 3v^5u^2 \tag{Ib}$$

and

$$u' = -\frac{\alpha_* z}{2} - \alpha_*^2 v - zvu - 3\alpha_* v^2 u - 2v^3u^2. \tag{IIb}$$

Hence, the solutions of the first-order systems (Ia)–(Ib) and (IIa)–(IIb), respectively, with initial values $v(z_1) = 0$, $u(z_1) = u_0$, provide analytic continuation to z_1. The proof of Theorem 2 is now complete.

7—The process of coalescence—There is still another method to prove that the solutions of some Painlevé's equation (X), say, are meromorphic in the plane: suppose it were known that the solutions of Painlevé's equation (Y) are meromorphic in the plane, and that (X) can be derived from (Y) by the *process of coalescence*. Then the solutions of (X) are also meromorphic in $\mathbb{C}$.

The true meaning of this statement will be illustrated by two examples:

Example 1 (from II to I) Let w be a local solution of (I) with initial values

$$w(z_0) = w_0, \quad w'(z_0) = w_0'. \tag{12}$$

Let ε be any non-zero complex number and let y be the solution of

$$y'' = 4\varepsilon^{-15} + xy + 2y^3, \quad y(x_0) = \varepsilon w_0 + \varepsilon^{-5}, \quad y'(x_0) = \varepsilon^{-1} w_0', \tag{II$_\varepsilon$}$$

where $x_0 = \varepsilon^2 z_0 - 6\varepsilon^{-10}$. Clearly y depends on ε, $y(x) = y(x;\varepsilon)$, and

$$\omega = w(z;\varepsilon) = \varepsilon^{-1} y(x;\varepsilon) - \varepsilon^{-6}, \quad z = \varepsilon^{-2}x + 6\varepsilon^{-12},$$

solves

$$\omega'' = z + 6\omega^2 + \varepsilon^6 \left(z\omega + 2\omega^3\right) \tag{13}$$

and has the same initial values as w at $z = z_0$. By analytic dependence on parameters (see, Bieberbach [B], p. 12), it follows that $w(z) = \lim_{\varepsilon\to 0} w(z;\varepsilon)$, more precisely,

$$w(z;\varepsilon) = w(z) + \varepsilon^6 w_1(z) + \varepsilon^{12} w_2(z) + \ldots,$$

locally uniformly, as $\varepsilon \to 0$, and so w is meromorphic in $\mathbb{C}$, provided all solutions of $(\mathrm{II})_\varepsilon$ are.

Remark The poles of the solutions of (13) are simple, while $w(z)$ has poles of order two. Thus it happens always that two simple poles of $w(z;\varepsilon)$ merge to a doubly pole in the limit $\varepsilon \to 0$.

Example 2 (from IV to II) Now let w be a local solution of (II) with initial values (12). Set

$$\omega = w(z;\varepsilon) = \varepsilon y(x) - \frac{1}{4}\varepsilon^{-2}, \quad z = \varepsilon^{-1}x + \frac{1}{4}\varepsilon^{-4}, \tag{14}$$

where y solves Painlevé's equation

$$y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha(\varepsilon))y + \frac{\beta(\varepsilon)}{y} \tag{IV$_\varepsilon$}$$

with parameters $\alpha(\varepsilon) = -2\alpha - \varepsilon^{-6}/32$ and $\beta(\varepsilon) = -\varepsilon^{-12}/512$, and initial values

$$y(x_0) = \varepsilon^{-1} w_0 + \frac{1}{4}\varepsilon^{-3}, \quad y'(x_0) = \varepsilon^{-2} w_0', \quad x_0 = \varepsilon z_0 - \frac{1}{4}\varepsilon^{-3}.$$

Then this transformation leads to the equation

$$\omega'' = \alpha + z\omega + 2\omega^3 + \varepsilon^2 \left(2\omega'^2 - 2\omega^4 + 4z\omega^2 + 4\alpha\omega + \frac{z^2}{4}\right) + O\left(\varepsilon^4\right),$$

as $\varepsilon \to 0$. Thus, by the same reason, the solutions of (II) are meromorphic since the solutions of $(\mathrm{IV})_\varepsilon$ are.

Together both examples show that Theorem 1 implies Theorem 2. Equation (IV) is the master type for those Painlevé's equations *without fixed singularities,* namely (I), (II) and (IV).

Remark The transformation in Example 1 can be found in Ince's book [I]. However, the transformation given there to transform (IV) into (II) leads merely to some linear second-order equation. I would like to thank Lukas Geyer, who found experimentally the transformation (14).

8—A general approach—We now describe a general method, which might apply to Painlevé's equations with fixed singularities. Although the computations could prove to be too complicated to be made *by hand*, they certainly can be performed by using computer algebra software.

We consider Painlevé's equations

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}, \tag{III}$$

$$\begin{aligned} w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right) w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2 w}(\alpha w^2 + \beta) \\ + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \end{aligned} \tag{V}$$

and

$$\begin{aligned} w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right) w'^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right) w' \\ + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right). \end{aligned} \tag{VI}$$

The solutions have or have not movable poles. If they do, the poles are of order one or two, this depending on parameters. If there are no movable poles one can apply a transformation of type

$$y = \frac{1}{w}, \quad y = \frac{1}{w-1} \text{ (only in (V) and (VI))}, \quad y = \frac{1}{w-z} \text{ (only in (VI))}.$$

We now assume that the solutions have movable poles of order $k = 1$ or $k = 2$. We write

$$w'' = S(z, w, w') \tag{P}$$

to denote the equation under consideration; S is rational in (z, w), and a quadratic polynomial with respect to w'.

One possible method to continue w analytically along Γ might be as follows. We assume, as before, that Γ is rectifiable and does not pass through one of the fixed singularities, and also that $w \neq 0, \infty$ (in case (III)), $w \neq 0, 1, \infty$ (in case (V)), and $w \neq 0, 1, z, \infty$ (in case (VI)), on Γ_1.

1. Try to find a degree-two polynomial with respect to w', $R(z, w, w')$, say, with rational coefficients in (z, w), such that $V = R(z, w, w')$ has simple poles with residue 1 at movable poles of w. This requires knowledge of the first coefficients in the Laurent expansion

$$w(z) = (z - z_1)^{-k}\big[c_0(z_1) + c_1(z_1)(z - z_1) + \cdots\big],$$

which can be found by computer algebra methods.

2. Set $U = V + k^{-1}w'/(w-\zeta)$, so that U is regular at movable poles (note that $k \in \{1,2\}$ is the multiplicity of the poles of w). Then

$$\big(w' - a(z,w)\big)^2 = b(z,w)U + c(z,w) \tag{15}$$

holds, where a, b, c are rational. Differentiating (15) and using (P) and (15) one obtains

$$U' = Q_1 U + Q_2 + (Q_3 U + Q_4)^{1/2},$$

where Q_1, Q_2, Q_3, Q_4 are rational in z and, this being the crucial point, *polynomials* in $(w-\zeta)^{-1}$. In that case one can deduce that U is *bounded* on Γ_1, provided $(w-\zeta)^{-1}$ is.

3. For arbitrary $\zeta \notin \{0,1\} \cup \Gamma$ we can prove that $(w-\zeta)^{-1}$ is bounded, by deforming Γ equivalently. This is true since the analogue to Lemma 3 holds for equations of type

$$z'' = -\frac{z' S_1(z,w,z')}{S_2(z,w)}, \quad z(w_0) = w_0, \quad z'(w_0) = \eta \neq 0, \tag{16}$$

S_1 and S_2 polynomials. In the present situation we have

$$z'' = -z'^3 S(z, w, 1/z').$$

Lemma 3' *Given (16) and $K > 1$, there exist positive numbers $\varepsilon = \varepsilon(K)$ and $\varrho = \varrho(K)$ with the following properties: the solution of (16) exists in $|w - w_0| < \varrho$ and satisfies $|\eta|/2 < |z'(w)| < 2|\eta|$, whenever $0 < |\eta| < \varepsilon$, and provided $|z_0| \le K$, $|w_0| \le K$ and* $\operatorname{dist}\big\{(z_0, w_0), Z\big\} \ge 1/K$; *here Z is the variety $\{(z,w) : S_2(z,w) = 0\}$.*

4. Suppose now that we have been able to prove that U is bounded on Γ_1. Then several cases may occur on some sequence $(\xi_n) \subset \Gamma_1$, $\xi_n \to z_1$: $w(\xi_n) \to \infty$, $w(\xi_n) \to 0$, $w(\xi_n) \to 1$ (only in (V) and (VI)) or $w(\xi_n) \to z_1$ (only in (VI). We then set $v = 1/w$, $v = w$, $v = w - 1$, or $v = w - z$, respectively. Then $v(\xi_n) \to 0$, and from (15) we obtain

$$\big(v' - a^*(z,v)\big)^2 = b^*(z,v)U + c^*(z,v), \tag{17}$$

a^*, b^*, c^* rational and regular at $(z_1, 0)$. It may become necessary to replace v with v^2, this depends on the fact whether the movable poles, zeros, one-points or fix-points of w have order one or two. Extracting the square root in (17) we obtain, for some $p \ge 1$,

$$v' = A(z,v) + \big(a(z) + b(z)U\big)v^p + O(|v|^{p+1}),$$

as $z \to z_1$, at least on some subsequence of (ξ_n). Here A is a polynomial in v, of degree less than p; its coefficients are algebraic, and regular at z_1, as are a and b. Thus, if u is defined by

$$v' = A(z,v) + v^p u = P(z,v,u), \tag{18}$$

it is bounded on (ξ_n), so that we may assume $u_0 = \lim_{n\to\infty} u(\xi_n)$ to exist. Differentiation of (18) and using (P) and (18) then may lead to some equation

$$u' = Q(z, v, u), \tag{19}$$

Q a polynomial in v and u with algebraic coefficients, regular at z_1.

5. Finally, if this all works, the solution of (18) and (19) with initial values $v(z_1) = 0$ and $u(z_1) = u_0$ then will give the continuation of w into $z = z_1$.

I strongly believe that this program can be carried out, supported by good computer algebra software. The *mathematics* behind is contained in Lemma 3', the proof of which is left to the reader.

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