

Sierpiński and non-Sierpiński curve Julia sets in families of rational maps

Norbert Steinmetz

ABSTRACT

We discuss the dynamics as well as the structure of the parameter plane of certain families of rational maps with few critical orbits. Our paradigm is the family $R_t(z) = t(1 + (4/27)z^3/(1-z))$, with dynamics governed by the behaviour of the postcritical orbit $(R_t^n(t))_{n \in \mathbb{N}}$. In particular, it is shown that if t escapes (that is, $R_t^n(t)$ tends to infinity), then the Julia set of R_t is a Cantor set, or a Sierpiński curve, or a curve with one or else infinitely many cut-points; each of these cases actually occurs.

1. Sierpiński curve Julia sets

Sierpiński curve Julia sets of rational maps were discovered by Milnor and Lei [11] in the family of degree-2 maps $z \mapsto a(z + 1/z) + b$. Most of the work on Sierpiński curve Julia sets, however, has been done by Devaney and his group in the so-called McMullen family $F_\lambda(z) = z^n + \lambda/z^n$ (see [1, 4, 5] out of a myriad of papers). In that family the occurrence of Sierpiński curve Julia sets is the rule rather than the exception. To be more precise: for any *escape parameter* λ (such that the critical values $\pm\sqrt{\lambda}$ escape to infinity under iteration) the Julia set of F_λ is either a Cantor set or a Sierpiński curve.

The family (R_t) under consideration displays several features not found in the family F_λ . For example, for escape parameters t it turns out that the Julia set \mathcal{J}_t is a Cantor set, or a Sierpiński curve, or else a curve that has one or else infinitely many *cut-points* and hence is not a Sierpiński curve.

The paper is organised as follows. In Section 2 the family (R_t) and its closely related families are introduced. In Section 3 we describe the decomposition of the parameter plane into the *bifurcation locus*, the *escape locus* and the *hyperbolic locus*, while in Section 4 we collect some auxiliary results. Sections 5–7 deal with the components of the escape locus, the approximation (kernel convergence in the sense of Carathéodory) of the bifurcation locus by Julia sets of associated rational functions, and the question of whether or not the corresponding Julia sets are Sierpiński curves. Section 8 is devoted to the components of the hyperbolic locus.

2. The Morosawa–Pilgrim family

In [13] Pilgrim showed that

either any critically finite hyperbolic rational map with two critical points is conjugate to some polynomial, or else every Fatou component is bounded by a Jordan curve.

As an example, showing that his result was sharp, he considered that the map

$$z \mapsto 2(z+2)(z-1)^2/(3z-2),$$

which is not conjugate to any polynomial, has three critical points with critical orbits

$$1 \xrightarrow{(2)} 0 \xrightarrow{(3)} -2 \rightarrow 0 \quad \text{and} \quad \infty \xrightarrow{(2)} \infty,$$

and the boundary curve of the basin of attraction at infinity is a figure-eight curve. In particular, the Julia set is not a Sierpiński curve. On the other hand, Morosawa [12] proved that

any simply connected invariant Fatou component B of some sub-hyperbolic rational map is bounded by a Jordan curve, provided that there exists some backward invariant complementary component of \overline{B} .

As an example he considered the map $z \mapsto 27z^2(z - 1)/((3z - 2)^2(3z + 1))$, with critical orbit

$$\frac{2}{3} \xrightarrow{(2)} \infty \xrightarrow{(3)} 1 \rightarrow 0 \xrightarrow{(2)} 0.$$

The Fatou set consists of the simply connected super-attracting basin about $z = 0$, and its preimages of any order. The Julia set is a Sierpiński curve; for the convenience of the reader we will recall its definition.

A *Sierpiński curve* is a non-empty, compact, connected, locally connected, and nowhere dense subset of the complex plane having complementary domains bounded by mutually disjoint Jordan curves.

By a theorem of Whyburn [18] any two Sierpiński curves are homeomorphic. We shall also briefly describe a dynamical construction yielding the standard example: let \mathcal{H} be the space of non-empty compact subsets of the plane, endowed with the Hausdorff metric, and define a map $\Phi : \mathcal{H} \rightarrow \mathcal{H}$ by the following procedure: set $\phi_{\mu\nu} = \frac{1}{3}(z + \mu + i\nu)$, $\mu, \nu \in \{0, 1, 2\}$, and

$$\Phi(C) = \bigcup_{(\mu,\nu) \neq (1,1)} \phi_{\mu\nu}(C) \quad \text{for } C \in \mathcal{H}.$$

Then Φ is a contraction on the complete metric space \mathcal{H} , and its unique fixed point is a (the standard) Sierpiński curve. We note that the usual construction of the *Sierpiński triangle* does not lead to a Sierpiński curve, since different complementary domains of this may and will have common boundary points.

Conjugating Morosawa's example by $z \mapsto 1/z$ gives the map

$$z \mapsto \frac{(3 - 2z)^2(3 + z)}{27(1 - z)} = 1 + \frac{(4/27)z^3}{1 - z},$$

with critical orbit $\frac{3}{2} \xrightarrow{(2)} 0 \xrightarrow{(3)} 1 \rightarrow \infty \xrightarrow{(2)} \infty$. Similarly, Pilgrim's example is conjugate to $z \mapsto -3(1 + (4/27)z^3/(1 - z))$, with critical orbit

$$\frac{3}{2} \xrightarrow{(2)} 0 \xrightarrow{(3)} -3 \rightarrow 0 \quad \text{and} \quad \infty \xrightarrow{(2)} \infty.$$

Thus both examples are embedded in a natural way into the degree-3 family

$$R_t(z) = t \left(1 + \frac{(4/27)z^3}{1 - z} \right), \tag{1}$$

which will be called the *Morosawa–Pilgrim family*. This family is the case $d = 3$ in the sequence of families

$$(z, t) \mapsto t \left(1 + \frac{((d - 1)^{d-1}/d^d)z^d}{1 - z} \right), \quad d \geq 3, \tag{2}$$

which itself has natural generalisations

$$(z, t) \mapsto -t \frac{\sum_{j=\ell+1}^d \binom{d}{j} (z-1)^j}{\sum_{j=0}^{\ell} \binom{d}{j} (z-1)^j}, \quad d \geq 2 + \ell \geq 3. \tag{3}$$

The critical orbits for (2) and (3) are

$$\frac{d}{d-1} \xrightarrow{(2)} 0 \xrightarrow{(d)} t, \quad \infty \xrightarrow{(d-1)} \infty, \quad \text{and} \quad 1 \xrightarrow{(\ell+1)} 0 \xrightarrow{(d)} t, \quad \infty \xrightarrow{(d-\ell)} \infty,$$

respectively. The focus of this paper is on family (1), and it is left to the reader to detect common and perhaps also different features of the above families.

3. Decomposition of the parameter plane

The map R_t has a super-attracting fixed point at $z = \infty$; its super-attracting basin will be denoted by \mathcal{A}_t . Böttcher’s function Φ_t is the solution of Böttcher’s functional equation

$$\Phi_t \circ R_t(z) = -\frac{4}{27} t \Phi_t(z)^2,$$

normalised by $\Phi_t(z) \sim z$ at $z = \infty$. From standard facts (for notation and results in complex dynamics the reader is referred to [10, 16]) it follows that \mathcal{A}_t is simply connected if and only if $t \notin \mathcal{A}_t$, while for $t \in \mathcal{A}_t$ the Fatou set coincides with \mathcal{A}_t and is infinitely connected. In the first case Φ_t maps \mathcal{A}_t conformally onto $|w| > 27/(4|t|)$.

The bifurcation locus \mathfrak{B} of the family (R_t) is defined as the set of $t \in \mathbb{C}^*$ such that the Julia set does not move continuously over any neighbourhood of t ; see McMullen [9]. By the central theorem in [9], small copies of the bifurcation loci of the families $(z, c) \mapsto z^3 + c$ and $(z, c) \mapsto z^6 + c$ are dense in \mathfrak{B} .

The escape locus \mathfrak{E} of the Morosawa–Pilgrim family is the set of those t such that $R_t^n(t) \rightarrow \infty$ as $n \rightarrow \infty$; \mathfrak{E} is divided into two parts:

- the Cantor locus \mathfrak{E}_0 containing all parameters such that $t \in \mathcal{A}_t$;
- the escape-connectedness locus $\mathfrak{E} \setminus \mathfrak{E}_0$, which itself is divided into countably many sets $\mathfrak{E}_n = \{t : R_t^n(t) \in \mathcal{A}_t, R_t^{n-1}(t) \notin \mathcal{A}_t\}$.

Finally, the interior of $\mathbb{C}^* \setminus \mathfrak{E}$ is divided into the hyperbolic locus \mathfrak{H} and the exotic locus \mathfrak{X} , which is conjecturally empty. The hyperbolic locus itself consists of countably many sets

- \mathfrak{H}_n , such that, for $t \in \mathfrak{H}_n$, R_t has a finite (super)-attracting cycle of exact period n .

The components of \mathfrak{H} and \mathfrak{E} are called hyperbolic and escape components, respectively. (Note, however, that R_t is hyperbolic for $t \in \mathfrak{E}$, too.) It is convenient to introduce the rational maps

$$Q_n(t) = R_t^n(t). \tag{4}$$

The sequence (Q_n) tends to infinity in \mathfrak{E} , while it is bounded (and hence also normal) in $\mathfrak{H} \cup \mathfrak{X}$. In \mathfrak{H}_n the sequence $(Q_{nk})_{k \in \mathbb{N}}$ converges locally uniformly to a (super)-attracting fixed point z_t of R_t^n , while in (the components of) \mathfrak{X} convergent subsequences (Q_{n_k}) will be irregular in the sense that $n_{k+1} - n_k \rightarrow \infty$. We note some simple facts on the maps Q_n , leaving proofs to the reader:

- $\deg Q_n = (3^{n+1} - 1)/2$;
- $Q_n(t) = t + (4/27)t^4 + (4/27)t^5 + \dots$ about $t = 0$;
- $Q_n(t) = \mu_n t^{d_n} + O(t^{d_n-1})$ about $t = \infty$, with $d_n = 2^{n+1} - 1$ and $\mu_n = -(4/27)^{d_n-1}$.

A (super)-attracting cycle $\mathfrak{U} = U_0 \cup U_1 \cup \dots \cup U_{n-1}$ of Fatou components is of the first kind, if the critical point $3/2$ is not in \mathfrak{U} , and of the second kind otherwise. In the first case

$$R_t^n : U_j \xrightarrow{(3)} U_j$$

has degree 3 with one critical point of order 2, while the degree of

$$R_t^n : U_j \xrightarrow{(6)} U_j$$

is 6 with one critical point of order 5 in the case $3/2 \in U_j$. In any case, for $n \geq 2$ the components U_j are simply connected. It is easy to show that

- the multiplier map $\lambda_t = \frac{d}{dz} R_t^n(z)|_{z=z_t}$ is analytic in any connected component H_n of \mathfrak{H}_n and maps H_n properly onto the unit disc \mathbb{D} .

Due to the theorem of Mañé, Sad, and Sullivan [8] we may speak of hyperbolic components of the first and the second kind, respectively. For the same reason, the sets \mathfrak{E}_0 , \mathfrak{E}_n , \mathfrak{H}_n , and \mathfrak{X} are open and mutually disjoint. The following statements are also easily proved:

- every escape component of order $n \geq 1$ contains a root of the equation $Q_{n-1}(t) = 1$, called its centre;
- every zero of Q_{n-1} , $n \geq 2$, is the centre of some hyperbolic component of period n ; more precisely, every hyperbolic component of period $n \geq 2$ of the first or second kind contains a root of the equation $Q_{n-2}(t) = -3$ or $Q_{n-2}(t) = 3/2$, respectively.

It turns out that the major result is that each escape and hyperbolic component has exactly one centre.

4. Auxiliary results

For $t \notin \mathfrak{E}_0$, and hence $t \notin \mathcal{A}_t$, the Böttcher function maps \mathcal{A}_t conformally onto $|w| > 27/(4|t|)$, while for $t \in \mathfrak{E}_0$ neither Φ_t nor $\log |\Phi_t|$ admits unrestricted (analytic or harmonic) continuation throughout \mathcal{A}_t , in contrast to the polynomial case, where $\log |\Phi_t|$ is Green's function for the basin about infinity. Nevertheless it is possible to define an analytic function $\Phi_t(t)$ in \mathfrak{E}_0 by using the already classical Douady–Hubbard technique [7]. To this end we first prove the following lemma.

LEMMA 1. *Given $t \in \mathfrak{E}_0$, there exists some simply connected and forward-invariant domain $D \subset \mathcal{A}_t$ containing the points ∞ and t , but neither of the points 1 and 0. Böttcher's function has an analytic continuation to D , ramified only at ∞ .*

Proof. Let $D_0 = \{z : |z| > \rho\}$ satisfy $R_t(\overline{D_0}) \subset D_0$, and let D_k denote the connected component of $R_t^{-1}(D_{k-1})$ containing D_{k-1} . Then $R_t : D_k \rightarrow D_{k-1}$ is a proper map of degree 2 with one critical point, and D_k is simply connected, as long as $z = 0$ (the critical point of R_t) is not contained in D_k ; this is the same as saying that $z = t$ (the critical value of R_t) is not contained in D_{k-1} . Let $k = k(t)$ be the first integer such that $t \in D_k$. Then D_k is simply connected while D_{k+1} is multiply connected and contains $z = 0$ and also $z = 1$, since $R_t : D_{k+1} \rightarrow D_k$ has degree 3. We may thus choose $D = D_k$, which is simply connected and contains the critical value $z = t$, but contains neither the pole $z = 1$ nor the critical points $z = 3/2$ and $z = 0$. The last assertion of Lemma 1 follows from the fact that analytic continuation of Φ_t in the simply connected domain D is given by $\Phi_t(z) = -27/(4t) \lim_{k \rightarrow \infty} \sqrt[2^k]{t R_t^k(z)}$ and Hurwitz's theorem. \square

REMARK. It is obvious that, for any $t \in \mathbb{C}^*$, the condition $R_s(\overline{D_0}) \subset D_0$ holds in some disc $|\mathfrak{s} - t| < \delta(t)$. Thus the exhaustion (D_k) may be constructed locally uniformly with respect to t , and consequently the number $k = k(t)$ tends to infinity as $t \rightarrow \partial \mathfrak{E}_0 \cap \mathbb{C}^*$ from \mathfrak{E}_0 . Otherwise there would be some $t_0 \in \partial \mathfrak{E}_0 \cap \mathbb{C}^*$ and some $k_0 \in \mathbb{N}$ such that $R_{t_0}^{k_0}(t_0) \in D_0 \subset \mathcal{A}_{t_0}$. This would imply that $t_0 \in \mathfrak{E}_{k_0}$, contradicting the fact that the sets \mathfrak{E}_k are open and mutually disjoint.

Since the map $t \mapsto \Phi_t$ is analytic, we may define analytic functions

$$\Xi_n(t) = -\frac{4}{27}t\Phi_t(Q_n(t)) \quad \text{on } \mathfrak{E}_n. \tag{5}$$

We note that $Q_n(t) \in \mathcal{A}_t$, but $Q_{n-1}(t) \notin \mathcal{A}_t$ for $t \in \mathfrak{E}_n$. The following result is crucial if we are to proceed further. The idea of proof of the main part is not new (see, for example [14]) and is based on *quasiconformal surgery*.

LEMMA 2. *Let E denote any connected component of \mathfrak{E}_n , $n \geq 0$. Then the map Ξ_n is analytic and locally univalent and satisfies*

$$|\Xi_n(t)| > 1 \quad \text{on } E \quad \text{and} \quad |\Xi_n(t)| \rightarrow 1 \quad \text{as } t \rightarrow \partial E \cap \mathbb{C}^*.$$

Proof. To prove local univalence we fix some $t_* \in E$ and set $R_* = R_{t_*}$. We assume that the closed disc $|t - t_*| \leq 3\epsilon$ belongs to the Fatou component of R_* containing t_* , and denote by (η_t) any family of diffeomorphisms $\eta_t : \mathbb{C} \rightarrow \mathbb{C}$, depending analytically on t in the disc $|t - t_*| < \epsilon$, and such that

$$\eta_t(w) = \begin{cases} w & \text{in } |w - t_*| > 3\epsilon \\ w - t_* + t & \text{in } |w - t_*| < \epsilon. \end{cases}$$

Then

$$g_t = \eta_t \circ R_* : \mathbb{C} \rightarrow \mathbb{C}$$

is K -quasiregular, coincides with R_* outside $R_*^{-1}(\{w : |w - t_*| \leq 3\epsilon\})$, and is analytic on $\mathbb{C} \setminus R_*^{-1}(A)$, with $A = \{w : \epsilon \leq |w - t_*| \leq 3\epsilon\}$. Since the sets $R_*^{-k}(A)$ are mutually disjoint, each iterate g_t^k is also K -quasiregular (g_t is then called *uniformly K -quasiregular*), and by Shishikura’s qc-Lemma [15], g_t is quasiconformally conjugate to some rational function

$$G_t = h_t \circ g_t \circ h_t^{-1}.$$

(The fact that the sets $R_*^{-k}(A)$ are mutually disjoint is obvious for $n > 0$, since then $R_*^{-k}(A)$ belongs to the preimage of \mathcal{A}_t of generation $n + k$. In the case $n = 0$ define open sets $D_m = R_*^{-m}(D_0)$, starting with $D_0 = \{z : |z| > \rho\}$ such that $R_*(\overline{D_0}) \subset D_0$ and $A \subset D_\ell \setminus \overline{D_{\ell-1}}$ for some ℓ (by choosing ϵ sufficiently small). Then $R_*^{-k}(A) \subset D_{k+\ell} \setminus \overline{D_{k+\ell-1}}$.)

If we normalise the quasiconformal map h_t to fix the critical points $3/2, 0, \infty$, then h_t is uniquely determined and depends analytically on t . Although this does not, in general, imply that h_t^{-1} and G_t depend analytically on t , it turns out to be true in the present case, as follows easily from Lemma 3 below.

We note that h_t is regular in the open set $\mathbb{C} \setminus \overline{\bigcup_{k \in \mathbb{N}} R_*^{-k}(A)}$ which, in particular, contains the points $3/2, 0, t, t_*$, and ∞ (all of $\mathcal{A}_* = \mathcal{A}_{t_*}$ in the case where $n > 0$). By Lemma 3 below we have

$$G_t = R_{\tau(t)}$$

and

$$\tau(t)\Phi_{\tau(t)} = t_*\Phi_* \circ h_t^{-1},$$

where $t \mapsto \tau(t)$ is analytic and non-constant on $|t - t_*| < \epsilon$. Since the critical point 0 is fixed by h_t , it follows from

$$R_\tau(0) = R_\tau(h_t(0)) = G_t(h_t(0)) = h_t(g_t(0)) = h_t(t)$$

that $\tau(t) = h_t(t)$, and hence

$$\Xi_n(\tau(t)) = -\frac{4}{27}t_*\Phi_*(R_*^n(t)).$$

Thus Ξ_n is non-ramified at $t = t_*$ if we can show that the map $t \mapsto -(4/27)t_*\Phi_*(R_*^n(t))$ is univalent in some neighbourhood of t_* . Since Φ_* is locally univalent (and even univalent if $n > 0$) on \mathcal{A}_* , this map is univalent near t_* if and only if R_*^n is univalent in a neighbourhood of $t = t_*$, and this is true since the critical points $3/2$ and $0 = R_*(3/2)$ are predecessors of t^* , and thus do not belong to the orbit of t^* .

Since (every analytic continuation of) Φ_t satisfies $|\Phi_t(z)| > 27/4|t|$, the first part $|\Xi_n(t)| > 1$ of the last assertion follows immediately. Finally, $|\Xi_n(t)| \rightarrow 1$ as $t \rightarrow \partial E \cap \mathbb{C}^*$ follows in the case where $n = 0$ from Lemma 1 and the remark after it: for fixed $t_* \in \partial E \cap \mathbb{C}^*$ and $\delta > 0$ sufficiently small, the exhaustion (D_k) in Lemma 1 may be constructed uniformly with respect to $t \in E_\delta = E \cap \{t : |t - t_*| < \delta\}$. Then, setting

$$M = \sup \{|\Phi_t(z)| : t \in E_\delta, z \in D_0 \setminus R_t(D_0)\},$$

we see that Böttcher's functional equation $\Phi_t(R_t^k(z)) = -(4/27)t^{2^k-1}\Phi_t(z)^{2^k}$ yields

$$1 < \left| \frac{4}{27}t\Phi_t(t) \right| \leq \left(\frac{4}{27}|t|M \right)^{1/2^{k(t)}} \rightarrow 1 \quad \text{as } t \rightarrow t_* \text{ in } E_\delta.$$

In the case where $n > 0$ we take the same construction for the exhaustion (D_k) , but now we define $k(t)$ to be the first integer such that $Q_n(t) \in D_{k(t)}$ (there is no problem in defining $\Phi_t(Q_n(t))$). Then again $k(t) \rightarrow \infty$ as $t \rightarrow t_*$, and the proof runs along the same lines as in the case where $n = 0$. □

LEMMA 3. We have $G_t = R_{\tau(t)}$ for some analytic map $t \mapsto \tau(t)$, and

$$\tau(t)\Phi_{\tau(t)} = t_*\Phi_* \circ h_t^{-1} \quad \text{on } \mathcal{A}_{\tau(t)}.$$

Proof. By hypothesis, G_t has degree 3, a pole of order 2 at $z = \infty$, and a simple pole at some point $z_0 \neq 0, 3/2$, and hence

$$G_t(z) = P(z)(z - z_0)^{-1},$$

where $P(z) = a_0 + a_1z + a_2z^2 + a_3z^3$ is a polynomial of degree 3 with $P(z_0) \neq 0$. Also G_t has a critical point of order 2 at $z = 0$ and a critical zero at $z = 3/2$. Thus the term in brackets in the equation

$$G'_t(z) = [(z - z_0)P'(z) - P(z)](z - z_0)^{-2}$$

has to vanish once at $z = 3/2$, and twice at $z = 0$, yielding

$$P(z) = a_3 \left(z^3 - \frac{27}{4}z + \frac{27}{4} \right), \quad a_3 \neq 0, \quad \text{and} \quad z_0 = 1.$$

From $h_t(z) = \sigma_t z + O(1)$ and $G_t \circ h_t = h_t \circ R_*$ at $z = \infty$ it follows that $a_3\sigma_t = -(4/27)t_*$, and hence $G_t = R_{\tau(t)}$, with $\tau(t) = t_*/\sigma_t = -(27/4)a_3$ an analytic function of t . This also shows that h_t^{-1} and G_t depend analytically on the parameter t .

To prove the last assertion of Lemma 3 we set $\Psi = \sigma_t\Phi_* \circ h_t^{-1}$, and note that $\Psi(z) \sim z$ at $z = \infty$, and

$$\begin{aligned} \Psi \circ R_{\tau(t)}(z) &= \Psi \circ G_t(z) = \sigma_t\Phi_* \circ R_* \circ h_t^{-1}(z) \\ &= -\frac{4}{27}t_*\sigma_t(\Phi_* \circ h_t^{-1}(z))^2 = -\frac{4}{27}\tau(t)\Psi(z)^2 \end{aligned}$$

holds in $\mathcal{A}_{\tau(t)}$. Then the uniqueness of Böttcher's function yields the desired result $\Psi = \Phi_{\tau(t)}$. □

REMARK. From the representation

$$\Phi_{\mathbf{t}}(z) = \lim_{k \rightarrow \infty} \sqrt[2^k]{\frac{R_{\mathbf{t}}^k(z)}{(-4/27)^{d_{k-1}}}} = -\frac{27}{4\mathbf{t}} \lim_{k \rightarrow \infty} \sqrt[2^k]{\mathbf{t}R_{\mathbf{t}}^k(z)},$$

which holds for large $|z|$, it follows that

$$\Xi_n(\mathbf{t}) = \lim_{k \rightarrow \infty} \sqrt[2^k]{\mathbf{t}Q_{n+k}(\mathbf{t})}, \tag{6}$$

locally in \mathfrak{E}_n . (Note that $Q_k(\mathbf{t}) \rightarrow \infty$.)

5. The escape locus

Recall that the escape locus \mathfrak{E} is the set of parameters \mathbf{t} such that the sequence $(Q_n(\mathbf{t}))$, the orbit of \mathbf{t} , tends to infinity. It consists of the Cantor locus \mathfrak{E}_0 and the escape-connectedness locus $\mathfrak{E} \setminus \mathfrak{E}_0$.

In the first step we shall analyse \mathfrak{E}_0 and the dynamics of its members, following this with an analysis of $\mathfrak{E} \setminus \mathfrak{E}_0$.

THEOREM 1. *The Cantor locus \mathfrak{E}_0 is connected, and $\mathfrak{E}_0 \cup \{\infty\}$ is simply connected and is mapped conformally onto $\Delta = \{w : |w| > 1\}$ by either branch of*

$$\Xi_0^{1/2}(\mathbf{t}) = \lim_{k \rightarrow \infty} \sqrt[2^{k+1}]{\mathbf{t}Q_k(\mathbf{t})}.$$

REMARK. In what follows, we will assume that $\Xi_0^{1/2}$ is normalised by

$$\Xi_0^{1/2}(\mathbf{t}) \sim \frac{3\sqrt{3}}{2}i\mathbf{t} \quad \text{at } \mathbf{t} = \infty.$$

Proof of Theorem 1. Let E be any connected component of \mathfrak{E}_0 . Then Ξ_0 is holomorphic and satisfies $|\Xi_0(\mathbf{t})| > 1$ on E , while $|\Xi_0(\mathbf{t})| \rightarrow 1$ as $\mathbf{t} \rightarrow \partial E \cap \mathbb{C}^*$ by Lemma 2. By the minimum principle, ∞ is a boundary point of E . On the other hand it is easily shown that $|R_{\mathbf{t}}(z)| > 5$ for $|z| \geq 5$, $|\mathbf{t}| \geq 5$, such that the disc $|z| \geq 5$ belongs to $\mathcal{A}_{\mathbf{t}}$, and the punctured disc $\{\mathbf{t} : 5 \leq |\mathbf{t}| < \infty\}$ belongs to \mathfrak{E}_0 . Thus $\mathfrak{E}_0 = E$ is itself connected, and $\Xi_0 : \mathfrak{E}_0 \cup \{\infty\} \rightarrow \Delta$ is proper and non-ramified on \mathfrak{E}_0 , with a pole of order 2 at $\mathbf{t} = \infty$. Since Ξ_0 has degree 2 and only one (simple) critical point, the Riemann–Hurwitz formula shows that $\mathfrak{E}_0 \cup \{\infty\}$ is simply connected, and hence, by the monodromy theorem, either branch of $\Xi_0^{1/2}$ maps $\mathfrak{E}_0 \cup \{\infty\}$ conformally onto Δ . \square

THEOREM 2. *For $\mathbf{t} \in \mathfrak{E}_0$ the Fatou set of $R_{\mathbf{t}}$ coincides with the basin $\mathcal{A}_{\mathbf{t}}$. The Julia set $\mathcal{J}_{\mathbf{t}}$ is totally disconnected (a Cantor set).*

Proof. The super-attracting basin $\mathcal{A}_{\mathbf{t}}$ is completely invariant, since it contains $z = 0$ and $R_{\mathbf{t}}$ has local degree 3 at 0. It coincides with the Fatou set, since it contains all critical points. For the same reason, $\mathcal{A}_{\mathbf{t}}$ is infinitely connected, and the Julia set is a Cantor set, by the hyperbolicity of $R_{\mathbf{t}}$. \square

THEOREM 3. *Every component E of \mathfrak{E}_n , $n > 0$, is simply connected and contains a unique solution \mathbf{t}_* of the equation $Q_{n-1}(\mathbf{t}) = 1$ (its centre), and is mapped conformally onto $\Delta = \mathbb{C} \setminus \overline{\mathbb{D}}$ by Ξ_n . In particular, \mathfrak{E}_n has exactly $(3^n - 1)/2 = \deg Q_{n-1}$ escape components.*

Proof. Let E be any connected component of \mathfrak{E}_n . Then, by Lemma 2, the map Ξ_n , defined by (5) (or, equivalently, (6)) has the following properties: it is non-ramified on E and satisfies $|\Xi_n(t)| \rightarrow 1$ as $t \rightarrow \partial E$ and $|\Xi_n(t)| > 1$ on E . By the minimum principle, Ξ_n has at least one (simple) pole, this being a pole of Q_n and a zero of $Q_{n-1} - 1$. Altogether, $\Xi_n : E \rightarrow \Delta$ is non-ramified and proper, and hence a conformal map. Obviously every zero of $Q_{n-1} - 1$ belongs to \mathfrak{E}_n , and different zeros belong to different components. Suppose that t_* is a ν -fold zero of $Q_{n-1} - 1$. Then it is also a pole of $Q_{n+k}(t) = R_t^{k+1}(Q_{n-1}(t))$ of order $2^k\nu$, and thus a pole of $\Xi_n(t) = \lim_{k \rightarrow \infty} \sqrt[2^k]{tQ_{n+k}(t)}$ of order ν ; this proves $\nu = 1$, and thus the statement about the number of components of \mathfrak{E}_n . \square

6. Kernel convergence

The concept of kernel convergence in the sense of Carathéodory may be described as follows. Let (D_n) be any sequence of domains, each containing some base point z_0 . The *kernel* $\ker(D_n) = \mathfrak{K}$ of (D_n) with respect to z_0 is then the union of all simply connected domains D , such that $z_0 \in D$ and $\overline{D} \subset D_n$ for $n \geq n_0(D)$. If no such D exists we set $\mathfrak{K} = \{z_0\}$. The sequence (D_n) is said to *converge* to \mathfrak{K} in the sense of Carathéodory, if \mathfrak{K} is also the kernel of every subsequence (D_{n_k}) . If all domains D_n are simply connected and if f_n denotes the conformal map of the unit disc \mathbb{D} onto D_n , normalised by $f_n(0) = z_0$ and $f'_n(0) > 0$, then $D_n \rightarrow \mathfrak{K} \neq \{z_0\}$ in the sense of Carathéodory is equivalent to $f_n \rightarrow f$ locally uniformly, where f is the normalised conformal map $\mathbb{D} \rightarrow \mathfrak{K}$; for $\mathfrak{K} = \{z_0\}$ the sequence (f_n) tends to the constant z_0 . For more details the reader is referred to Carathéodory [3]; the original definition for simply connected domains carries over to arbitrary domains D_n .

Every map Q_n has a super-attracting fixed point at ∞ :

$$Q_n(t) = \mu_n t^{d_n} + \dots \quad (d_n = 2^{n+1} - 1, \mu_n = -(4/27)^{d_n-1}),$$

and hence a super-attracting basin \mathfrak{A}_n about ∞ , and a corresponding Böttcher function Θ_n , defined in some neighbourhood of infinity and satisfying

$$\Theta_n(Q_n(t)) = \mu_n \Theta_n(t)^{d_n}, \quad \text{with } \Theta_n \sim t \text{ as } t \rightarrow \infty. \tag{7}$$

In the preceding section we have implicitly shown that the conditions $|Q_{n-1}(t)| > 5$ and $|t| > 5$ imply that $|Q_n(t)| > 5$. Hence every basin \mathfrak{A}_n contains $\{t : |t| > 5\}$, and the kernel of the sequence (\mathfrak{A}_n) is a domain about ∞ . If the basin \mathfrak{A}_n is simply connected, then Θ_n maps \mathfrak{A}_n conformally onto the disc $|w| > |\mu_n|^{1/(d_n-1)} \sim \frac{2}{3}\sqrt{3}$ as $n \rightarrow \infty$. In this case the statements $\Theta_n \rightarrow \Theta_0$ and $\mathfrak{A}_n \rightarrow \mathfrak{K} = \ker(\mathfrak{A}_n)$ are equivalent.

THEOREM 4. *The sequence (Θ_n) tends to $(-(27/4)\Xi_0)^{1/2}$, locally uniformly in $\mathfrak{E}_0 \cup \{\infty\}$, while the sequence (\mathfrak{A}_n) tends to its kernel $\mathfrak{K} = \mathfrak{E}_0 \cup \{\infty\}$ with respect to ∞ .*

Proof. The idea of proof is due to Busse [2], who proved the corresponding theorem for the Mandelbrot set; see also [17]. Böttcher's function Θ_n , the solution of Böttcher's functional equation (7), is given by

$$\Theta_n(t) = \lim_{k \rightarrow \infty} \sqrt[2^k]{Q_n^k(t) / \mu_n^{1+d_n+\dots+d_n^{k-1}}} \sim t \quad \text{as } t \rightarrow \infty,$$

at least in $|t| > 5$. Since Q_n has no zeros and poles in \mathfrak{E}_0 , and $Q_n(t) \rightarrow \infty$, locally uniformly in $\mathfrak{E}_0 \cup \{\infty\}$, Θ_n is eventually defined in every simply connected domain D with $\overline{D} \subset \mathfrak{E}_0 \cup \{\infty\}$ and $\infty \in D$. Thus $\overline{D} \subset \mathfrak{A}_n$ for $n \geq n_0$ (and $D \subset \mathfrak{K} = \ker(\mathfrak{A}_n)$), and the sequence

$$\Psi_n(t) = \sqrt[2^n]{tQ_n(t)}$$

tends to $\Xi_0^{1/2}$, uniformly in D . From $Q_n(t) \rightarrow \infty$,

$$\mu_n \Theta_n(t)^{d_n} = \Theta_n(Q_n(t)) = Q_n(t) + O(1),$$

and $\Psi_n(t)^{d_n} = Q_n(t)\Psi_n(t)/t$, it follows that

$$\mu_n \Theta_n(t)^{d_n} - \Psi_n(t)^{d_n} \frac{t}{\Psi_n(t)} = O(1);$$

thus

$$\mu_n^{1/d_n} \Theta_n(t) - \Psi_n(t) \rightarrow 0$$

as $n \rightarrow \infty$. Noting that $\mu_n^{1/d_n} = (-4/27)^{d_n-1/d_n} \rightarrow 2i/(3\sqrt{3})$, we have *a fortiori*

$$\frac{2i}{3\sqrt{3}} \Theta_n \rightarrow \Xi_0^{1/2},$$

locally uniformly in $\mathfrak{E}_0 \cup \{\infty\}$ and, in addition, $\mathfrak{E}_0 \cup \{\infty\} \subset \ker(\mathfrak{A}_n)$.

Conversely, let D be any simply connected domain containing ∞ and such that $\overline{D} \subset \tilde{\mathfrak{K}}$, where $\tilde{\mathfrak{K}}$ is the kernel of any subsequence $(\mathfrak{A}_{n(\nu)})$. Then $\overline{D} \subset \mathfrak{A}_{n(\nu)}$ for $\nu \geq \nu_0$, and the sequence $(Q_{n(\nu)}^k)_{k \in \mathbb{N}}$ tends to infinity as $k \rightarrow \infty$, uniformly in D , so that $\Theta_{n(\nu)}$ is defined in D for ν large. Also, the sequence $(\Theta_{n(\nu)})_{\nu \geq \nu_0}$ is normal in D , since

$$|\Theta_{n(\nu)}(t)| \geq |\mu_{n(\nu)}|^{1/(d_{n(\nu)}-1)} \sim \frac{2}{3}\sqrt{3} \quad \text{in } D.$$

By Vitali's theorem on the convergence of normal sequences, the sequence $(\Theta_{n(\nu)})$ converges to Θ , locally uniformly in D , with $|\Theta(t)| > 2/3\sqrt{3}$ in D and $\Theta(t) = -(27/4)\Xi_0(t)^{1/2}$ in $D \cap \mathfrak{E}_0$. Since $|\Xi_0(t)| \rightarrow 1$ as $t \rightarrow \partial\mathfrak{E}_0$, this shows that $D \cap \partial\mathfrak{E}_0 = \{\infty\}$, and hence $D \subset \mathfrak{E}_0 \cup \{\infty\}$ and $\tilde{\mathfrak{K}} \subset \mathfrak{E}_0 \cup \{\infty\}$. Putting things together, we have shown that the sequence (\mathfrak{A}_n) tends to its kernel $\mathfrak{E}_0 \cup \{\infty\}$. \square

THEOREM 5. *Let $t_n \in \mathbb{C}^*$, $n \geq 1$, be a root of the equation $Q_{n-1}(t) = 1$, and let \mathfrak{A}'_k , $k \geq n$, denote the Fatou component of Q_k containing t_n . (\mathfrak{A}'_k may coincide with \mathfrak{A}_k .) Then the sequence $(\mathfrak{A}'_k)_{k \geq n}$ tends to its kernel \mathfrak{K}' with respect to the point t_n , and \mathfrak{K}' is the connected component of \mathfrak{E}_n containing t_n .*

The proof of Theorem 5 is almost the same as the proof of Theorem 4 and will therefore be omitted here.

7. Sierpiński or non-Sierpiński

In the case of the McMullen family $F_\lambda(z) = z^m + \lambda/z^\ell$, the escape locus \mathfrak{E} consists of the Cantor locus \mathfrak{E}_0 and the *Sierpiński locus*; that is, for every parameter $\lambda \in \mathfrak{E} \setminus \mathfrak{E}_0$ the Julia set is a Sierpiński curve; see, for example, [5].

In the present family the situation is more complicated. We assume that R_t is hyperbolic and $t \notin \mathfrak{E}_0$ (or just assume that the Julia set \mathcal{J}_t is locally connected and contains no critical point). Then $\partial\mathcal{A}_t$ is a closed curve, and the following holds.

LEMMA 4. *Under that hypothesis, $\partial\mathcal{A}_t$ is a Jordan curve if and only if $\overline{\mathcal{A}_t} \cap \overline{\mathcal{A}_t^*} = \emptyset$, where \mathcal{A}_t^* denotes the unique preimage of \mathcal{A}_t under R_t . Moreover, in that case any two preimages of \mathcal{A}_t of any order have disjoint closures. In particular, for $t \in \mathfrak{E} \setminus \mathfrak{E}_0$ the Julia set is a Sierpiński curve if and only if $\partial\mathcal{A}_t$ is a Jordan curve.*

Proof. First of all, if \mathcal{A}_t is a Jordan domain, then so are its preimages of every order. If U_1 and U_2 are preimages, then Lemma 5 below applies to $R = R_t^n$, where n is the smallest integer such that $R_t^n(U_1) = R_t^n(U_2) = \mathcal{A}_t$, thus showing that U_1 and U_2 have disjoint closures. In particular, this applies to $U_1 = \mathcal{A}_t$ and $U_2 = \mathcal{A}_t^*$, and hence $\overline{\mathcal{A}_t} \cap \overline{\mathcal{A}_t^*} = \emptyset$.

Conversely, if $\overline{\mathcal{A}_t} \cap \overline{\mathcal{A}_t^*} = \emptyset$ is assumed, let D_1 denote the connected component of $\mathbb{C} \setminus \overline{\mathcal{A}_t}$ containing the pole $z = 1$, and thus containing $\overline{\mathcal{A}_t^*}$. Then $R_t(\partial D_1) \subset R_t(\partial \mathcal{A}_t) = \partial \mathcal{A}_t = R_t(\partial \mathcal{A}_t^*)$, thus $\partial R_t(D_1) \subset R_t(\partial D_1) \setminus R_t(\overline{\mathcal{A}_t^*}) = \emptyset$. Hence $R_t(D_1) = \widehat{\mathbb{C}}$, so D_1 contains $\{0\} = R_t^{-1}(t)$. By our assumption, $D_1' = D_1 \setminus \overline{\mathcal{A}_t^*}$ is a conformal annulus containing the origin and is mapped by R_t onto some complementary component of $\overline{\mathcal{A}_t}$, say D_2 (note that $R_t(\partial D_1') \subset \partial \mathcal{A}_t$ and $R_t(D_1') \cap \mathcal{A}_t = \emptyset$). Since D_2 is simply connected, the Riemann–Hurwitz formula applied to $R_t : D_1' \rightarrow D_2$ yields $\deg R_t|_{D_1'}$ equal to the number of critical points in D_1' . Hence $\deg R_t|_{D_1'} = 3$, D_1' contains three critical points $0, 0$, and $3/2$, and D_2 also contains $0 = R_t(3/2)$. This yields $D_1 = D_2$ and $R_t^{-1}(D_1) = D_1' \subset D_1$. By Morosawa’s result [12] mentioned in the first section, $\partial \mathcal{A}_t$ is a Jordan curve. \square

It remains to prove the following lemma.

LEMMA 5. *Let R be rational, and let D be any Jordan domain. If ∂D contains no critical value of R , then any two distinct preimages of D have disjoint closures.*

Proof. Every boundary point of D is accessible; that is, for any $w_0 \in \partial D$ there exists an arc $\gamma : [0, 1) \rightarrow D$, such that $\lim_{t \rightarrow 1} \gamma(t) = w_0$. Assume that D_1 and D_2 are preimages of D under R with common boundary point z_0 , and let γ be any such arc in D ending at $w_0 = R(z_0)$. Then γ has two lifts under R , $\gamma_1 \subset D_1$ and $\gamma_2 \subset D_2$, both ending at z_0 ; this is true since D , and hence D_1 and D_2 , are Jordan domains. Then $R(\gamma_1) = R(\gamma_2) = \gamma$ which, of course, contradicts $R'(z_0) \neq 0$. \square

Computer experiments for $t \in (\mathfrak{E} \setminus \mathfrak{E}_0) \cup \mathfrak{H}$ show three different topologies: $\mathbb{C} \setminus \overline{\mathcal{A}_t}$ consists of one, two, or infinitely many connected components; that is, $\partial \mathcal{A}_t$ is a Jordan curve, or a figure-eight curve, or else has infinitely many cut-points. This turns out to be true in general.

THEOREM 6. *For $t \in \mathfrak{E} \setminus \mathfrak{E}_0$ the complement of $\overline{\mathcal{A}_t}$ consists of one, two, or infinitely many connected components. In the first case the Julia set is a Sierpiński curve, while in the second case $\partial \mathcal{A}_t$ is a figure-eight curve, with all R_t^n -lifts being Jordan curves.*

REMARK. Part of the above assertion also holds for $t \in \mathfrak{H} \setminus \mathfrak{H}_1$, namely: $\mathbb{C} \setminus \overline{\mathcal{A}_t}$ consists of one, two, or infinitely many connected components. Equivalently: $\partial \mathcal{A}_t$ is either a Jordan curve, or a figure-eight curve, or else has infinitely many cut-points. For $t \in \mathfrak{H}_1$, $\partial \mathcal{A}_t$ is always a Jordan curve; see Theorem 8.

Proof of Theorem 6. For $t \in \mathfrak{E} \setminus \mathfrak{E}_0$ the Julia set is connected and locally connected, and hence \mathcal{A}_t is simply connected and $\partial \mathcal{A}_t$ is locally connected. If $\mathbb{C} \setminus \overline{\mathcal{A}_t}$ is connected, then $\partial \mathcal{A}_t$ is a Jordan curve. Hence we have to discuss the case where $\mathbb{C} \setminus \overline{\mathcal{A}_t}$ consists of $n \geq 2$ components D_1, \dots, D_n , and we have to show that $n = 2$.

The domains D_j are bounded by Jordan curves, and $\partial \mathcal{A}_t$ contains one or more cut-points z_j , such that $\partial \mathcal{A}_t \setminus \{z_j\}$ consists of $\nu_j \geq 2$ components, with $n - 1 = \sum_j (\nu_j - 1)$. From Lemma 4 we know that $\overline{\mathcal{A}_t} \cap \overline{\mathcal{A}_t^*}$ is non-empty, and from our assumption we know that this intersection

is finite. To prove $n = 2$ we may assume that $1 \in D_1$, and hence $\mathcal{A}_t^* \subset D_1$. Then

$$\partial R_t(D_j) \subset R_t(\partial D_j) \subset \partial \mathcal{A}_t \quad \text{and} \quad R_t(D_j) \cap \mathcal{A}_t = \emptyset \quad \text{for } 1 < j \leq n,$$

while $R_t(D_1) \supset \mathbb{C} \setminus R_t(\overline{\mathcal{A}_t} \cap \overline{\mathcal{A}_t^*})$. In the first case we have $R_t(D_j) = D_{k(j)}$ for some $k(j)$, while $R_t(D_1)$ covers a finitely punctured sphere. From $R_t^{-1}(t) = \{0\}$ and $t \notin \mathcal{J}_t$ it follows that $0 \in D_1$. Also $R_t^{-1}(0) = \{-3, 3/2\}$ implies that $-3 \in D_1$ or else $3/2 \in D_1$, but not both $-3 \in D_1$ and $3/2 \in D_1$ simultaneously: in combination with Morosawa's theorem [12] the latter would imply that $\partial \mathcal{A}_t$ is a Jordan curve. Thus if D_2 denotes the component that contains $3/2$, then D_1 has two preimages: the domain D_2 , and the unique component D'_1 of $D_1 \setminus \overline{\mathcal{A}_t^*}$ containing -3 ; alternatively, if D_2 denotes the component that contains -3 , then the two preimages of D_1 are the domain D_2 and the unique component D'_1 of $D_1 \setminus \overline{\mathcal{A}_t^*}$ containing $3/2$. Each of the components D_j contains points of the Julia set, and also poles of certain iterates; thus $\{D_1, \dots, D_n\}$ contains no R_t -invariant sub-cycles, and so any domain D_j , $j > 1$, is mapped eventually onto D_1 :

$$D_j \xrightarrow{(1)} D_{j_1} \xrightarrow{(1)} \dots \xrightarrow{(1)} D_2 \xrightarrow{(k)} D_1, \quad k = 1 \text{ or } k = 2. \tag{8}$$

Every boundary curve ∂D_j contains at least one cut-point, and cut-points are mapped onto cut-points. For obvious reasons there exist at least two domains with only one cut-point; this, by (8), is true for D_1 , and thus for all domains D_j (note that all domains D_j are Jordan domains). This proves that \mathcal{A}_t is the complement of a flower with petals $\overline{D_j}$. The petals have a common boundary point, a cut-point z_0 of order n (that is also a fixed point of R_t). Thus $\overline{\mathcal{A}_t} \cap \overline{\mathcal{A}_t^*}$ consists of two points z'_0 and z''_0 , namely the preimages of z_0 , and $D_1 \setminus \overline{\mathcal{A}_t^*}$ consists of two components D'_1 and D''_1 . Then on the one hand R_t maps the domains D'_1 and D''_1 onto components $D_{j'}$ and $D_{j''}$, while on the other hand, $R_t(D_1)$ covers a finitely punctured sphere and hence covers all components D_1, \dots, D_n . This implies that $n = 2$, \mathcal{A}_t is a figure-eight curve, and any preimage of \mathcal{A}_t is a Jordan domain, since the cut-point z_0 has order 1. \square

REMARK. The above argument works for the following reason: for $t \in \mathfrak{E} \setminus \mathfrak{E}_0$ any component D_j is eventually mapped onto D_1 . This is also true, however, for $t \in \mathfrak{H} \setminus \mathfrak{H}_1$, since then the critical point $z = 0$ belongs to some (super)-attracting cycle of Fatou components, and every component D_j contains also Fatou components that are eventually mapped into that cycle. Thus the first statement of Theorem 6 holds for $t \in \mathfrak{H} \setminus \mathfrak{H}_1$.

8. Hyperbolic components

A finite (super)-attracting cycle $\{U_0, \dots, U_{n-1}\}$ of R_t is of either the *first* or the *second kind*; that is, the degree of $R_t : U_j \rightarrow U_j$ is either 3 or 6. We mention that in any hyperbolic component H_n , with the exception of $n = 1$, there exists a unique analytic *multiplier map* $t \mapsto \lambda_t$, mapping H_n properly onto the unit disc: for $|t| > 0$ small, R_t has a fixed point $z_t = t + O(t^4)$ with multiplier $\lambda_t = R'_t(z_t) = (4/9)t^3 + O(t^4)$, and hence there is some hyperbolic component of period 1 about the origin $t = 0$ (but $t = 0$ does not belong to the parameter plane). In the classical Mandelbrot case the multiplier maps are even conformal. We will see that this is never the case here.

THEOREM 7. *The set \mathfrak{H}_1 is connected, and $\mathfrak{H}_1 \cup \{0\}$ is a simply connected domain that is mapped onto the unit disc \mathbb{D} by the multiplier map, with mapping degree 3.*

Proof. Since none of the maps R_t has a finite super-attracting fixed point, \mathfrak{H}_1 consists of a single component, and $t \mapsto \lambda_t$ is a proper map $\mathfrak{H}_1 \rightarrow \mathbb{D} \setminus \{0\}$ of degree 3; this follows from

$\lambda_t = (4/9)t^3 + O(t^4)$ as $t \rightarrow 0$ (and $\lambda_t \neq 0$ in \mathfrak{H}_1). We have to prove that λ_t has no critical points in \mathfrak{H}_1 . To do this we start from the equation $z_t = R_t(z_t) = tR_1(z_t)$. On the one hand we obtain

$$t = z_t/R_1(z_t) \quad \text{and} \quad \dot{z}_t = \frac{R_1(z_t)}{1 - tR_1'(z_t)}$$

(where the dot means differentiation with respect to t) and

$$\lambda_t = R_t'(z_t) = tR_1'(z_t).$$

On the other hand,

$$\dot{\lambda}_t = R_1'(z_t) + tR_1''(z_t)\dot{z}_t = R_1'(z_t) + \frac{z_t R_1(z_t) R_1''(z_t)}{R_1(z_t) - z_t R_1'(z_t)}.$$

Solving, with or without the support of MAPLE, we see that

$$R_1'(z) + \frac{zR_1(z)R_1''(z)}{R_1(z) - zR_1'(z)} = 0 \quad \text{and} \quad t = z/R_1(z),$$

however, yields $t = (-17 \pm 4\sqrt{13})/9$ and $\frac{d}{dt}R_t'(\zeta(t)) = 0$ for some repelling fixed point $z = \zeta(t)$ of R_t . This proves Theorem 7. \square

THEOREM 8. *For $t \in \mathfrak{H}_1$ the Fatou set of R_t consists of the simply connected domain \mathcal{A}_t , its simply connected preimages of any order, and the completely invariant attracting basin \mathcal{S}_t (Schröder domain) about the finite attracting fixed point z_t of R_t . The basin \mathcal{S}_t is infinitely connected, while the unbounded component \mathcal{A}_t of $\mathbb{C} \setminus \overline{\mathcal{S}_t}$ is bounded by a Jordan curve.*

Proof. For $t \in \mathfrak{H}_1$, R_t has an attracting fixed domain \mathcal{S}_t that contains the critical value $z = t$, and hence contains its unique preimage $z = 0$, and $R_t : \mathcal{S}_t \rightarrow \mathcal{S}_t$ has full degree 3. Thus \mathcal{S}_t is completely invariant and contains three critical points $3/2, 0, 0$; by the Riemann–Hurwitz formula, \mathcal{S}_t is infinitely connected. Finally, the hyperbolicity of R_t , the simple connectivity of \mathcal{A}_t , and the complete invariance of \mathcal{S}_t , together with Morosawa’s result, show that \mathcal{A}_t is a Jordan domain; it is also the unbounded component of $\mathbb{C} \setminus \overline{\mathcal{S}_t}$. \square

THEOREM 9. *The sets \mathfrak{H}_n , $n \geq 2$, are non-empty and have simply connected components H_n , each containing exactly one zero t_0 of $Q_{n-2}(t) + 3$ (first kind), or of $Q_{n-2}(t) - 3/2$ (second kind), respectively. The multiplier map has degree 2 or 5, respectively, and is ramified exactly at the centre t_0 . The components of the (super)-attracting finite cycles of either kind are simply connected.*

Proof. We will adapt the procedure due to Douady, Hubbard, and Sullivan (see [6]) which, in the case of the quadratic family $P_c(z) = c + z^2$, shows that the multiplier map $c \mapsto \lambda_c$ provides a conformal map of any hyperbolic component H of the Mandelbrot set onto the unit disc \mathbb{D} . This will, however, cause some difficulties since we expect $t \mapsto \lambda_t$ not to be univalent. We assume that (R_t) is any rational family, parametrised over some domain $T \subset \mathbb{C}$, such that every R_t has a (super)-attracting n -cycle $\{z_0, \dots, z_{n-1}\}$ with associated Fatou cycle $\{U_0, \dots, U_{n-1}\}$ and multiplier λ_t satisfying $|\lambda_t| \rightarrow 1$ as $t \rightarrow \partial T$. We also assume that $R_t^n : U_0 \rightarrow U_0$ has degree m and a single $(m - 1)$ -fold critical point c_t . This implies that

$$R_t : U_0 \xrightarrow{(m_1)} U_1 \xrightarrow{(m_2)} \dots \xrightarrow{(m_{n-1})} U_{n-1} \xrightarrow{(m_n)} U_n = U_0,$$

$m = m_1 m_2 \dots m_n$, and R_t^n has a single $(m - 1)$ -fold critical point $R_t^j(c_t)$ in U_j . (The ordering of the domains U_j is, of course, not uniquely determined. In the case of the family

$P_c(z) = z^d + c$, and also in our particular case, there is a natural ordering: U_0 is the unique component containing the critical point $z = 0$. If, however, there is no distinguished critical point, and no *a priori* knowledge that T is simply connected, then one has to work with a local ordering. Fortunately this suffices, as may be seen in what follows.)

Then U_0 is simply connected, and if $\Psi_t : U_0 \rightarrow \mathbb{D}$ denotes any Riemann map such that $\Psi_t(z_0) = 0$, then $B = \Psi_t \circ R_t^n \circ \Psi_t^{-1}$ is a Blaschke product that fixes the origin and has a single, $(m - 1)$ -fold, critical point in \mathbb{D} . By normalising $\Psi_t'(z_0)$ suitably we may assume that B is given by

$$B(\zeta) = B_a(\zeta) = \frac{M(\zeta)^m - a^m}{1 - \bar{a}^m M(\zeta)^m}$$

with $M(\zeta) = (\zeta + a)/(1 + \bar{a}\zeta)$, critical point $\zeta = -a$, and

$$\lambda_t = B'(0) = \frac{1 - |a|^2}{1 - |a|^{2m}} ma^{m-1}. \tag{9}$$

We note that a is not uniquely determined, since the map $\omega\Psi_t$ with $\omega^{m-1} = 1$ conjugates R_t to $B_{a\bar{\omega}}$ with critical point $-a\bar{\omega}$. Locally, we may choose Ψ_t in such a way that the *critical point map* $t \mapsto \Psi_t(c_t) = -a$ is continuous. It is, however, impossible to define *a priori* a continuous map $T \rightarrow \mathbb{D}$, $t \mapsto \Psi_t(c_t)$, except in the case $m = 2$, where

$$B_a(\zeta) = \zeta \frac{\zeta + \lambda}{1 + \bar{\lambda}\zeta}, \quad \lambda = \frac{2a}{1 + |a|^2}, \quad a = \frac{1 - \sqrt{1 - |\lambda|^2}}{|\lambda|^2} \lambda.$$

We fix $t = t_0$ such that $\lambda_{t_0} \neq 0$, set $R = R_{t_0}$, $\Psi = \Psi_{t_0}$, and $-a_0 = c(t_0)$, and consider the family $(B_a)_{\{|a| < 1 - \epsilon\}}$ for any $0 < \epsilon < 1 - |a_0|$. We also choose $r < 1$ such that for $|a| < 1 - \epsilon$ the closed disc $|\zeta| \leq r$ is mapped by B_a into the open disc $\Delta = \{\zeta : |\zeta| < r\}$. Then $B_a : B_a^{-1}(\Delta) \rightarrow \Delta$ is a proper map of degree m with $\bar{\Delta} \subset B_a^{-1}(\Delta)$.

We also set $D = \Psi^{-1}(\Delta)$ and $D^* = (R^n)^{-1}(D) \cap U_0$, and define a diffeomorphism $\phi_a : D^* \rightarrow B_a^{-1}(\Delta)$ to satisfy

$$B_a \circ \phi_a = \Psi \circ R^n \quad \text{on } \partial D^* \quad \text{and} \quad \phi_a = \Psi \quad \text{on } D,$$

and such that ϕ_a depends smoothly on a , $|a| < 1 - \epsilon$, with $\phi_{a_0} = \Psi$.

Set $D_* = R^{-(n-1)}(D) \cap U_1$ such that $R^{n-1} : D_* \rightarrow D$ is a proper map of degree m/m_1 , ramified only at $R_t(c_t)$, while $\Psi^{-1} \circ B_a \circ \phi_a : D^* \rightarrow D$ is a non-analytic proper map of degree m , which is ramified only at $z = c_t$ and coincides with R^n on ∂D^* . Thus we may define a non-analytic proper map $g_a : D^* \rightarrow D_*$ of degree m_1 that satisfies

$$R^{n-1} \circ g_a = \Psi^{-1} \circ B_a \circ \phi_a \quad \text{on } D^*.$$

It is obvious that we obtain a continuously differentiable map $g_a : \mathbb{C} \rightarrow \mathbb{C}$ by setting $g_a = R$ outside D^* . Moreover, this map is non-analytic only in $\bar{D}^* \setminus D$, and *uniformly quasiregular*, which means that each iterate g_a^n is K -quasiregular with fixed K . Again by Shishikura's qc-lemma [15] there exists a quasiconformal map $\psi_a : \mathbb{C} \rightarrow \mathbb{C}$, depending continuously on a and conjugating g_a to some rational map

$$\tilde{R}_a = \psi_a \circ g_a \circ \psi_a^{-1};$$

\tilde{R}_a has an attracting n -cycle with multiplier $\lambda = B'_a(0)$.

It now depends on the family (R_t) , and the normalisation of ψ_a , as to whether or not one can prove that

$$\tilde{R}_a = R_{\tau(a)}, \tag{10}$$

with $a \mapsto \tau(a)$ continuous ($a \in \mathbb{D}$, since $\epsilon > 0$ was arbitrary). If so, this implies that $\Psi_{\tau(a)}(c_{\tau(a)}) = -\tilde{a}$ with

$$\frac{m(1 - |\tilde{a}|^2)}{1 - |\tilde{a}|^{2m}} \tilde{a}^{m-1} = \lambda,$$

and thus $\tilde{a} = \omega a$ for some root of unity $\omega = \omega(a)$. From $\phi_{a_0} = \Psi$, and hence $\tilde{R}_{a_0} = R_{t_0}$, it follows that $\tilde{a}_0 = a_0$ and $\omega(a_0) = 1$, which implies that $\omega(a) = 1$, at least in some neighbourhood of a_0 . This, however, suffices, since we have shown that the map $\mathfrak{t} \mapsto \lambda_{\mathfrak{t}}$ is ramified at most over $\lambda = 0$. The Riemann–Hurwitz formula then yields that T is simply connected, and λ has only one zero ($\mathfrak{t} = t_0$, say), and is unramified on $T \setminus \{t_0\}$. We may now conclude that $\mathfrak{t} \mapsto \Psi_{\mathfrak{t}}(c_{\mathfrak{t}})$ is a homeomorphism $T \rightarrow \mathbb{D}$ with inverse $a \mapsto \tau(-a)$, and that the multiplier map has degree $m - 1$; this follows from

$$\lambda_{\tau(a)} = \frac{m(1 - |a|^2)}{1 - |a|^{2m}} a^{m-1} \sim a^{m-1} \quad \text{as } a \rightarrow 0.$$

It is easily seen that condition (10) is fulfilled in case of the Morosawa–Pilgrim family (2) for any $d \geq 3$, and also for the family (3), by normalising ψ_a to fix z equal to 0, 1, or ∞ (this is also true for the family $P_c(z) = c + z^d$: the multiplier map $c \mapsto \lambda_c$ has degree $d - 1$ on the hyperbolic components of the connectedness locus \mathcal{M}_d , and is ramified only over $\lambda = 0$). The last assertion follows from the fact that $R_t^n : U_j \rightarrow U_j$ has only one critical point (of order 2 or 5); this follows from $R_t(3/2) = 0$. \square

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The referee encouraged me to make available computer graphics depicting various phenomena revealed in this paper. Since, however, in our case grey-scaled computer graphics carry comparatively little information, the interested reader is referred to the web site <http://www.mathematik.tu-dortmund.de/steinmetz/MP.html>

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Norbert Steinmetz
Technische Universität (TU) Dortmund
Fakultät für Mathematik
D-44221 Dortmund
Germany

stein@math.tu-dortmund.de