LINEAR DIFFERENTIAL EQUATIONS WITH EXCEPTIONAL FUNDAMENTAL SETS

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Abstract

We will prove the following conjecture of Wittich: Let $L(w) = w^{(n)} + p_{n-2}(z)w^{(n-2)} + \cdots + p_0(z)w$ be a linear differential operator with polynomial coefficients and let $\{w_1, \ldots, w_n\}$ be a fundamental set of the equation $L(w) = 0$, each w_j having Borel exceptional value 0 (or being a polynomial). Then the coefficients of L are constants.

1 Introduction

Let

$$L(w) = w^{(n)} + p_{n-1}(z)w^{(n-1)} + \cdots + p_0(z)w \tag{1}$$

be a linear differential operator. If not the contrary is stated, it is always assumed that the coefficients of L are polynomials. Then any solution of

$L(w) = 0$ is either a polynomial or a transcendental entire function of positive finite order $\lambda(w)$.

A fundamental set

$$w_1, w_2, \ldots, w_n \tag{2}$$

of the equation $L(w) = 0$ is said to be exceptional (P), (B) or (N), if zero is an exceptional value of any transcendental w_j in the sense of Picard (w_j has only finitely many zeros), Borel (the exponent of convergence is less than the order) or Nevanlinna (zero has maximial deficiency $\delta(0, w_j) = 1$), respectively. Clearly, (P) $\Longrightarrow$ (B) $\Longrightarrow$ (N).

If L has constant coefficients, then there is a fundamental set exceptional (P) and, conversely, if L admits a fundamental set exceptional (P), then there is a polynomial Q (e.g., $Q' = -p_{n-1}/n$) such that the operator

$$L^*(v) = e^{-Q} L(e^Q v) \tag{3}$$

has constant coefficients (Frank [2]).

In [1] it is asked whether this result remains true if 'exceptional (P)' is replaced by 'exceptional (B)' and if p_{n-1} is a constant.

In this paper we will give a complete characterization of those operators L having a fundamental set exceptional (N): Either L has constant coefficients or the solution of $L^*(v) = 0$ have order less than $\deg Q$.

Before stating the result some partial results should be mentioned:

(a) The case $n = 1$ is trivial, while $n = 2$ is treated completely in [2].

(b) In [1] special operators L are considered: the hypotheses in [1] imply that there is a fundamental set exceptional (B) with

$$w_j = g_j e^{c_j z^\lambda / \lambda}, \quad 1 \le j \le n,$$

where $\lambda > \lambda(g_j)$ is fixed and the constants $c_j \ne 0$ are pairwise distinct.

(c) In [5] the problem is solved for third-order equations

$$w''' + p(z)w' + q(z)w = 0.$$

We remark that the corresponding problem for operators L having rational coefficients is completely solved in [5], but the solution is quite different from the polynomial case.

The main tool is the theory of asymptotic integration, dating back to the beginning of this century. Good references are Sternberg [6] or Wasow [7] for the matrix case. For notations of Nevanlinna theory the reader is referred to Hayman [3].

2 Results

As already mentioned, the main result is as follows.

Theorem 1 *Suppose that the differential equation $L(w) = 0$ has a fundamental set exceptional* (N)*. Then, either the coefficients of L are constants, or there is a nonlinear polynomial Q such that all solutions of*

$$L^*(v) = e^{-Q} L(e^Q v) = 0$$

have order less than $\deg Q$.

Remark. Actually one can choose $Q' = -p_{n-1}/n$ or any other polynomial with exactly the same leading term. It is clear that the second case may occur, one has only to start from an operator L^* and to choose a polynomial Q of sufficiently high degree. Clearly any fundamental set is exceptional (B).

We mention only two easy consequences of Theorem 1.

Corollary 1 *If the equation $L(w) = 0$ has a fundamental set exceptional* (N) *and if $1 + \deg p_{n-1}$ is less than the greatest order of any solution, then L has constant coefficients.*

Corollary 2 *If the equation $L(w) = 0$ has a fundamental set exceptional* (P), *then L^*, defined by* (3), *has constant coefficients if $Q' = -p_{n-1}/n$.*

Corollary 1 answers the question in [1], while Corollary 2 is identical with Satz 10 in [2].

If w_1, w_2 are linearly independent solutions of

$$w'' + p(z)w = 0, \quad p \text{ a polynomial},$$

then the product $E = w_1 w_2$ satisfies

$$2EE'' - E'^2 + 4p(z)E^2 - c^2 = 0,$$

where $c = W(w_1, w_2)$ is a nonzero constant. From this it is easy to deduce that either $\lambda(w_1 w_2) = \max\ \lambda(w_j)$ or $w_1 w_2$ and p are constants.

The corresponding result in the n-th order case is

Theorem 2 *Suppose that the differential equation $L(w) = 0$ has a fundamental set* (2) *satisfying*

$$T(r, w_1 w_2 \cdots w_n) = o\left(\max_{\nu=1}^{n} T(r, w_\nu)\right).$$

Then $w_1 w_2 \cdots w_n$ is a polynomial and L has constant coefficients.

3 Auxiliary Results

We first state some facts from the theory of asymptotic integration (see [6], [7]) in a form which is sufficiently precise for our purpose.

To avoid unnecessarily complicated notations, we use the Landau symbol $o(1)$ in the following way: $\delta(z) = o(1)$ means that δ is analytic in some sector S : $|\arg z - \theta| < h$ and

$$\delta^{(\nu)}(z) = O\left(\left(\frac{1}{\log z}\right)^{(\nu)}\right) \quad \text{as } z \to \infty \text{ in } S \tag{4}$$

for $\nu = 0, 1, 2, \ldots$. Thus $(z \log z)\delta'(z) = o(1)$, and if R is a rational function of m variables with $R(0, 0, \ldots, 0) = 0$, then $R(o(1), \ldots, o(1)) = o(1)$ in this sense.

Main Theorem of Asymptotic Integration Theory. *Let*

$$M(u) = u^{(m)} + r_{m-1}(z)u^{(m-1)} + \cdots + r_0(z)u \tag{5}$$

be a linear differential operator with rational coefficients. Then there exist polynomials P_j ($P_j(0) = 0$), complex numbers ρ_j and nonnegative integers μ_j such that

$$u_j(z) = e^{P_j(z^{1/p})} z^{\rho_j} (\log z)^{\mu_j} (1 + o(1)), \quad 1 \le j \le m, \tag{6}$$

represents a fundamental set of $M(u) = 0$ in a sufficiently small sector S around a given ray $\arg z = \theta$. *Here $p \ge 1$ is some integer, and the triples (P_j, ρ_j, μ_j) are pairwise distinct.*

Remark. Actually $(\log z)^{\mu_j + k_j}(1+o(1))$ (for some integer $k_j \ge 0$) is a polynomial in $\log z$ whose coefficients have asymptotic representations (divergent power series in $z^{-1/p}$) in S. For our purpose it is enough to know that (6) may be differentiated and that u_j' has a similar representation. This fact is expressed by (4).

Lemma 1 *Let $u_1, u_2, \ldots, u_m$ be given by (6) in some sector S and set $y_j = u_j(1 + o(1))$, $1 \le j \le m$. Then*

$$\frac{W(y_1, y_2, \ldots, y_m)}{W(u_1, u_2, \ldots, u_m)} \to 1 \quad as \quad z \to \infty \quad in S.$$

Here, as usual,

$$W(u_1, u_2, \ldots, u_m) = \begin{vmatrix} u_1 & \ldots & u_m \\ u_1' & \ldots & u_m' \\ \vdots & & \vdots \\ u_1^{(m-1)} & \ldots & u_m^{(m-1)} \end{vmatrix}$$

denotes the Wronskian of the system $u_1, u_2, \ldots, u_m$. It is easily seen that the functions $u_1, u_2, \ldots, u_m$ are linearly independent if, as it is assumed, the triples (P_j, ρ_j, μ_j) are pairwise distinct.

Proof (by induction). Since only local considerations are made, we may and will assume $p = 1$, otherwise the variable z will be replaced by z^p.

To descend from m to $m - 1$ we proceed as follows (obviously $m = 1$ is trivial):

The transformation formula (see [4], p. 113)

$$\begin{aligned} W(u_1, \ldots, u_m) &= u_m^m W\left(\frac{u_1}{u_m}, \ldots, \frac{u_{m-1}}{u_m}, 1\right) \\ &= (-1)^{m-1} u_m^m W\left(\left(\frac{u_1}{u_m}\right)', \ldots, \left(\frac{u_{m-1}}{u_m}\right)'\right) \end{aligned}$$

yields
$$\frac{W(y_1,\ldots,y_m)}{W(u_1,\ldots,u_m)} = \frac{W(\eta_1',\ldots,\eta_{m-1}')}{W(\omega_1',\ldots,\omega_{m-1}')}\,(1+o(1)), \tag{7}$$
where $\eta_j = (y_j/y_m)$ and $\omega_j = (u_j/u_m)$, and we have only to show that the functions ω_j' and η_j' satisfy the hypotheses of Lemma 1.

From
$$\omega_j' = \omega_j\left[(P_j - P_m)' + \frac{\rho_j-\rho_m}{z} + \frac{\mu_j-\mu_m}{z\log z} + \frac{o(1)}{z\log z}\right] \tag{8}$$
follows the representation
$$\omega_j' = c_j e^{Q_j} z^{\sigma_j}(\log z)^{\nu_j}(1+o(1)), \tag{9}$$
$Q_j(0)=0$, $c_j\neq 0$ a constant. We have to distinguish three cases:

(i) $Q_j = P_j - P_m \not\equiv 0$: $\sigma_j = \rho_j + \deg Q_j'$, $\nu_j = \mu_j$.

(ii) $Q_j = P_j - P_m \equiv 0$, $\rho_j \neq \rho_m$: $\sigma_j = \rho_j - 1$, $\nu_j = \mu_j$.

(iii) $Q_j = P_j - P_m \equiv 0$, $\rho_j = \rho_m$: $\sigma_j = -1$, $\nu_j = \mu_j - 1$.

It is proved by inspection that the triples (Q_j,σ_j,ν_j) are pairwise distinct and that $\eta_j' = \omega_j'(1+o(1))$. Thus the assertion follows from (7) by induction.

Remark. If $y_j = z^{\alpha_j}u_j(1+o(1))$, where α_j is an integer such that $\alpha_j = \alpha_k$ if $P_j = P_k$, then $W(y_1,\ldots,y_m) = z^{\alpha_1+\cdots+\alpha_m}W(u_1,\ldots,u_m)(1+o(1))$ as $z\to\infty$ in S. This is proved in the same manner.

Lemma 2 *Let M be given by* (5) *and assume that all solutions of $M(u)=0$ are of the form $u = ve^{az^\lambda/\lambda}$ with fixed az^λ and $\lambda(v)<\lambda$. Then*
$$r_j(z) = \binom{m}{j}(-az^{\lambda-1})^{m-j}\left(1+O\left(\frac{1}{z}\right)\right) \quad \textit{as} \quad z\to\infty. \tag{10}$$

Proof. The leading term $a_j z^{\lambda_j}/\lambda_j$ of $P_j(z^{1/p})$ in (6) is obtained as follows. Let $y_1,\ldots,y_m$ be the solutions of the algebraic equation
$$H(z,y) = y^m + r_{m-1}(z)y^{m-1} + \cdots + r_0(z) = 0.$$
Then $y_j(z) = a_j z^{\lambda_j - 1} + \cdots$ near $z=\infty$. Since under the assumptions of Lemma 2 every nontrivial solution has the dominating factor $e^{az^\lambda/\lambda+\cdots}$, this must also be true for the formal solutions and so $a_j z^{\lambda_j} = az^\lambda$ for $j=1,\ldots,m$. This gives
$$H(z,y) = (y - az^{\lambda-1}+\cdots)\cdots(y-az^{\lambda-1}+\cdots)$$
and so (10).

Lemma 3 *Let the operator M have coefficients r_j, given by* (10) *and let*

$$w = e^{bz^\mu/\mu+\cdots} z^\rho (\log z)^\nu (1+o(1))$$

in some sector S, where $bz^\mu \neq az^\lambda$. Then

$$M(w) = c^m z^{m\alpha} w(1+o(1)) \quad \text{as } z\to\infty \text{ in } S,$$

where $cz^\alpha = bz^{\mu-1}$ if $\mu > \lambda$, $cz^\alpha = -az^{\lambda-1}$ if $\lambda > \mu$ and $cz^\alpha = (b-a)z^{\lambda-1}$ if $\mu = \lambda$.

Proof. It is easily seen that ($\mu \geq 1$)

$$w^{(j)} = (bz^{\mu-1})^j w(1+o(1)) \quad \text{as } z\to\infty \text{ in } S,$$

and so

$$\begin{aligned} M(w) &= \sum_{j=0}^{m} \binom{m}{j} (-az^{\lambda-1})^{m-j} (bz^{\mu-1})^j (1+o(1))w \\ &= (bz^{\mu-1} - az^{\lambda-1})^m (1+o(1))w. \end{aligned}$$

This proves Lemma 3, since for $\mu = 0$ the proof is even easier. One observes that $w^{(j)} = O(\frac{w}{z})$ as $z\to\infty$ in S for $j \geq 1$.

4 Proof of Theorem 1 and its Corollaries

Let(2) be a fundamental set exceptional (N) for the equation $L(w) = 0$. Then (2) is also exceptional (B), as follows from [5, Corollary 2]. Also, Theorem 3 in [5] yields the decomposition

$$V(L) = V(M_1) \oplus \cdots \oplus V(M_k)$$

of the solution space $V(L)$ of $L(w) = 0$ into solution spaces $V(M_\kappa)$ of certain linear differential equations $M_\kappa(w) = 0$, where M_κ has rational coefficients. Moreover, any $w \in V(M_\kappa)$ has the representation

$$w = v \exp{(a_\kappa z^{\lambda_\kappa}/\lambda_\kappa)}, \tag{11}$$

where v is an entire function of order less than λ_κ (this makes only sense if $\lambda_\kappa \geq 1$; if $\lambda_\kappa = 0$ then (11) means $w = v$ is a polynomial). The monomials $a_\kappa z^{\lambda_\kappa}$ are pairwise distinct.

The case $k = 1$ is very simple: Every solution has the form $w = ve^{az^\lambda/\lambda}$ and any polynomial $Q = az^\lambda/\lambda + \cdots$ may be taken ($\lambda \geq 2$; if $\lambda = 1$ then L has constant coefficients).

We may now assume $k \geq 2$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Then $M = M_1$ has rational coefficients $r_0, r_1, \ldots, r_{m-1}$ with property (10) ($\lambda = \lambda_1$, $a = a_1$). We will derive a contradiction if $\lambda > 1$ as follows: In

$$W(w_1,\ldots,w_n) = W(w_1,\ldots,w_m) W(M(w_{m+1}),\ldots,M(w_n)) \tag{12}$$

the left hand side is zero-free, while the first Wronskian on the right hand side possibly has finitely many zeros. These zeros are regular-singular points of M and thus may be cancelled out by poles of the second Wronskian.

However, we will show that the second Wronskian has at least $m(n-m)(\lambda-1)$ zeros, and this gives a contradiction if $\lambda \geq 2$. Thus all solutions of $L(w)=0$ have order at most one, which implies that L has constant coefficients (see [8]).

To compute the second Wronskian on the right hand side of (12), we first remark that its logarithmic derivative is invariant under a change of the fundamental set. Thus we may replace the functions w_{m+j} $(1 \leq j \leq n-m)$ in any sufficiently narrow sector by a distinguished fundamental set

$$u_j = \exp\,(a_\kappa z^{\lambda_\kappa}/\lambda_\kappa + \cdots) z^{\rho_j} (\log z)^{\nu_j} (1+o(1))$$

$(j=1,2,\ldots,n-m$; the index κ indicates that u_j belongs to $V(M_\kappa)$, $2 \leq \kappa \leq k)$. Now Lemma 3 gives

$$M(u_j) = (c_\kappa z^{\lambda-1})^m u_j (1+o(1)) = (c_\kappa z^{\lambda-1})^m y_j \tag{13}$$

and so

$$W(M(w_{m+1}),\ldots,M(w_n)) = cz^{m(n-m)(\lambda-1)} W(y_1,\ldots,y_{n-m}), \tag{14}$$

c a nonzero constant.

From (12) and Lemma 1 then follows

$$W(w_1,\ldots,w_n) = r(z) W(w_1,\ldots,w_m) W(w_{m+1},\ldots,w_n),$$

where r is a rational function with a pole of order $m(n-m)(\lambda-1)$ at $z=\infty$. (Note that, although all considerations were made locally, all functions occurring are meromorphic in the plane.) This proves Theorem 1, since $r(z)$ has at least $m(n-m)(\lambda-1)$ zeros.

To prove the corollaries, we have to rule out the second alternative of the theorem. If the fundamental set exceptional (B) or (P) has the form $w_j = e^Q v_j$, $\lambda(v_j) < \deg Q$ with a fixed polynomial Q, then

$$W(w_1,\ldots,w_n) = e^{nQ} W(v_1,\ldots,v_n) = e^{nQ+P}$$

with $\deg P < \deg Q$, and so the degree of

$$p_{n-1} = -nQ' - P'$$

is $\deg Q'$. This contradicts the hypothesis in Corollary 1.

The same is true for Corollary 2. If we choose $Q' = -p_{n-1}/n$, then

$$L^*(v) \equiv v^{(n)} + q_{n-2}(z) v^{(n-2)} + \cdots + q_0(z) v = 0$$

has a fundamental set exceptional (P) with constant Wronskian, and Corollary 1 says that L^* has constant coefficients.

5 Proof of Theorem 2

Let W denote the Wronskian of $w_1, w_2, \ldots, w_n$. Then Nevanlinna's lemma on the proximity function of the logarithmic derivative gives

$$\begin{aligned} m(r,W) &\leq m\left(r, \frac{W}{w_1 \cdots w_n}\right) + m(r, w_1 \cdots w_n) \\ &= O(\log r) + o\left(\max_{\nu=1}^{n} T(r, w_\nu)\right), \end{aligned}$$

and thus the degree of $p_{n-1} = -W'/W$ is less than $\lambda = \max_{\nu=1}^{n} \lambda(w_\nu)$.

Note that zero is a Borel exceptional value of any w_j of order λ.

Let c be any constant such that $y_j = e^{cz^\lambda} w_j$ has order λ for $1 \leq j \leq n$. Then $y_1, y_2, \ldots, y_n$ is a fundamental set exceptional (B) for the equation

$$K(y) := e^{cz^\lambda} L(e^{-cz^\lambda} y) = 0.$$

If $\lambda > 1$, then by Theorem 1 there must exist a polynomial Q of degree λ such that any solution of $K(y) = 0$ has the form ve^Q, where $\lambda(v) < \deg Q$ and so $w_j = v_j e^{Q - cz^\lambda}$, $\lambda(v_j) < \lambda$ for $1 \leq j \leq n$. Since

$$w_1 w_2 \cdots w_n = e^{n(Q - cz^\lambda)} v_1 \cdots v_n,$$

this is only possible if the degree of $Q - cz^\lambda$ is less than λ, and this implies $\lambda(w_j) < \lambda$ contradicting $\lambda = \max \lambda(w_j)$. However, if $\lambda \leq 1$, then the coefficients of L are constants, the w_j are exponential sums, and the same is true for $w_1 \cdots w_n$. Since $T(r, w_1 \cdots w_n) = o(r)$, $w_1 \cdots w_n$ must be a polynomial.

Remark. After having written down this paper, I learned at the "Tag der Funktionentheorie" at Karlsruhe (May 4–6, 1989) that F. Brüggemann independently has given a proof of Theorem 1 as a part of his thesis (RWTH Aachen). I also learned from G. Frank that the problem dealt with in this paper goes back to H. Wittich, at least in an informal way.

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