Airy Solutions of Painlevé's Second Equation

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Dedicated to the memory of Professor Dieter Gaier

Abstract. We prove that any transcendental solution of Painlevé's second equation $w'' = \alpha + zw + 2w^3$, which has the form $w = R(z, u)$, R rational in both variables and non-linear with respect to u, is obtained by repeated application of the Bäcklund transformation to some solution of the Riccati equation $U' = \pm(z/2 + U^2)$. In particular $\alpha = n + 1/2$, $n \in \mathbb{Z}$, and w has order of growth $\varrho = 3/2$. Moreover it is shown that u satisfies some Riccati differential equation $u' = a(z) + b(z)u + c(z)u^2$ with rational coefficients.

Keywords. Painlevé, Airy and Riccati differential equation, Bäcklund transformation

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1. Introduction

The solutions of *Painlevé's Second Equation*

$$\frac{d^2w}{dz^2} = \alpha + zw + 2w^3 \tag{II$_\alpha$}$$

are meromorphic functions in the plane, see Hinkkanen-Laine [5] and the author [9], and have order of growth $\varrho(w) \le 3$, see Shimomura [6, 7], the author [10, 11] and also the recent monograph [3] by Gromak, Laine and Shimomura. The *Second Painlevé Transcendents* have infinitely many poles p, this following from the observation, due to Wittich [12], that $m(r, w) = O(\log r)$, and have Laurent development

$$w(z) = \frac{\epsilon}{z-p} - \frac{\epsilon p}{6}(z-p) - \frac{\alpha+\epsilon}{4}(z-p)^2 + h(z-p)^3 + \cdots , \tag{1}$$

$\epsilon = \pm 1$, about every pole p. The coefficient h remains undetermined; for any fixed solution $h = O(|p|^2)$ as $p \to \infty$ is valid, see [10, 11].

2. Poles

It was shown by Gromak [3, Theorem 18.2] that every transcendental solution w has infinitely many poles with residue $+1$ and also with residue -1, except when $\alpha = \pm 1/2$ and w solves the *Riccati Equation*

$$\frac{dw}{dz} = \pm \frac{z}{2} \pm w^2. \tag{2}$$

This can be made more precise, see also [11], if we denote by $n_\epsilon(r, w)$ the number of poles of w in $|z| \leq r$ with residue ϵ, $\epsilon = \pm 1$, and by $N_\epsilon(r, w)$ the corresponding Nevanlinna counting function.

Theorem 1. *Let w be a transcendental solution of* (II_α)*, but not a solution of (2). Then $N_+(r, w) \leq 2N_-(r, w) + O(\log r)$, and, conversely, $N_-(r, w) \leq 2N_+(r, w) + O(\log r)$ hold.*

Proof. From (1) it follows that $\Phi = w' + w^2 + z/2$ is regular and even has a zero at any pole of w with residue $+1$, and has a double pole at any pole of w with residue -1, and no other poles. Hence, if $\Phi \not\equiv 0$ it follows that

$$N_+(r, w) \leq N(r, 1/\Phi) \leq T(r, \Phi) + O(1) = 2N_-(r, w) + O(\log r).$$

In the same manner $N_-(r, w) \leq 2N_-(r, w) + O(\log r)$ is proved. □

We remark that both inequalities are sharp. To this end we introduce the so-called

Bäcklund transformation $w \mapsto w_{\pm 1}$, being defined by

$$w_1 = \begin{cases} -w & \text{if } \alpha = -1/2 \\ -w - \dfrac{\alpha + 1/2}{w' + w^2 + z/2} & \text{if } \alpha \neq -1/2 \end{cases}$$

and

$$w_{-1} = \begin{cases} -w & \text{if } \alpha = 1/2 \\ -w + \dfrac{\alpha - 1/2}{w' - w^2 - z/2} & \text{if } \alpha \neq 1/2 \end{cases}$$

If w is any solution of (II_α), then $w_{\pm 1}$ is a solution of ($\mathrm{II}_{\alpha \pm 1}$).

For more details the reader is referred to [3].

Starting with any solution $w = U_\pm$ of (2) we obtain by repeated application of the Bäcklund transformation solutions of ($\mathrm{II}_{\pm(n+1/2)}$) of type

$$w_{\pm n}(z) = \pm R_n(z, \pm U_\pm(z)),$$

where R_n is a rational function, of degree $2n + 1$ with respect to the second variable. We note in particular

$$R_1(z, U) = -\frac{1 + zU + 2U^3}{z + 2U^2}$$

and

$$R_2(z,U)=\frac{1-z^3+3zU-4z^2U^2+6U^3-4zU^4}{(z+2U^2)(1+2zU+4U^3)}.$$

Any function $w(z)=w_{\pm n}(z)$ satisfies some *first order differential equation* of *odd degree*

$$(w')^{2n+1}+P_1(z,w)(w')^{2n}+\cdots+P_{2n}(z,w)w'+P_{2n+1}(z,w)=0, \qquad (3)$$

where P_j is a polynomial in both variables and has degree $\le 2j$ with respect to w. These solutions of $(\mathrm{II}_{\pm(n+1/2)})$ are called *Airy Solutions*, the reason for this being that we can write $U=\mp v'/v$, where v is an appropriate solution of the modified *Airy Equation*

$$\frac{d^2v}{dz^2}+\frac{z}{2}v=0.$$

Airy solutions have order of growth $\varrho=3/2$; this follows for U from Wittich [12], and in the general case from *Valiron's Lemma*, see Bieberbach [1, p. 99/100], which says that

$$T(r,w_{\pm n})=(2n+1)T(r,U)+O(\log r).$$

It is not hard to see that

$$|N_+(r,w_{\pm n})-N_-(r,w_{\pm n})|=T(r,U)+O(\log r).$$

3. A new Characterization of Airy Solutions

The converse of the above statement on Airy solutions is also true:

Theorem (Gromak [3, Theorem 21.1]) *Any transcendental solution of* (II_α)*, which also satisfies some first order algebraic differential equation, is an Airy solution.*

We will yet prove another characterization of Airy solutions:

Theorem 2. *Let w be any transcendental solution of some equation* (II_α) *having the particular form*

$$w(z)=R(z,u(z))=\frac{P(z,u(z))}{Q(z,u(z))}, \qquad (4)$$

where u is a meromorphic function in $\mathbb{C}$, and R is rational with respect to both variables, and non-linear with respect to the variable u. Then w is an Airy solution, and u solves some Riccati differential equation $u'=a(z)+b(z)u+c(z)u^2$ with rational coefficients.

Proof. We start with the observation that (II_α) and (4) imply

$$u''=-\frac{R_{uu}(z,u)}{R_u(z,u)}(u')^2-2\frac{R_{zu}(z,u)}{R_u(z,u)}\,u'+N(z,u), \qquad (5)$$

where $N(z,u)$ is some rational function, and in the next steps determine the nature of w, by the way gathering information on the function u. Our considerations depend mainly on the Laurent expansion (1). We have to consider two major cases, which do not necessarily exclude each other.

4. Case A: Common poles of w and u

We assume that w and u have infinitely many poles in common, hence

$$R(z,u) = a_k(z)u^k + \cdots + a_1(z)u + a_0(z) + b_1(z)/u + \cdots \tag{6}$$

as $u \to \infty$, with $a_k(z) \not\equiv 0$ and $k \geq 1$. Conversely, if we assume (6), then there exist constants $r_0 > 0$ and $\beta > 0$ such that $|z| \geq r_0$ and $|u(z)| \geq |z|^\beta$ imply $k \log |u(z)| = \log |w(z)| + O(\log |z|)$. Thus we have $m(r,u) = O(\log r)$, and so u has infinitely many poles, all but finitely many being also poles of w. Since w has only simple poles this implies $k = 1$, and also that almost all poles of u are simple. We set $y = a_1(z)u + a_0(z)$ and obtain a new representation

$$w = R(z,u) = \tilde{R}(z,y) = y + c_1(z)/y + c_2(z)/y^2 + \cdots , \tag{7}$$

$c_1, c_2, \ldots$ rational. Then $m(r,y) = O(\log r)$ holds, and almost all poles of y are simple. Let p be any simple pole of y, which is not a pole of the remainder term in (7). Then from (1) and (7) it follows that

$$\begin{aligned} y(z) &= \frac{\epsilon}{z-p} - \frac{\epsilon}{6}(p + 6c_1(p))(z-p) \\ &\qquad - \frac{1}{4}(\epsilon + \alpha + 4\epsilon c_1'(p) + 4c_2(p))(z-p)^2 + \cdots \end{aligned} \tag{8}$$

A standard calculation then shows that

$$\Phi(z) = y''(z) - 2y^3(z) - (z + 6c_1(z))y(z)$$

is regular at $z = p$ and, moreover, satisfies

$$\Phi(z) = [\alpha + 4c_2(p) - 2\epsilon c_1'(p)] - 3\epsilon c_1''(p)(z-p) + \cdots \tag{9}$$

near $z = p$. Since Φ has at most finitely many poles and satisfies $m(r,\Phi) = O(\log r)$, it is a rational function.

We have to distinguish two different cases:

Sub-case (a) $\epsilon = \epsilon(p)$ *varies.*

Thus we suppose that there exist infinitely many poles p of y with $\epsilon(p) = 1$, and also infinitely many poles with $\epsilon(p) = -1$. Then from (9) follows

$$\Phi(p) = \alpha + 4c_2(p) - 2\epsilon(p)c_1'(p) \quad \text{and} \quad \Phi'(p) = -3\epsilon(p)c_1''(p),$$

and hence $c_1(z) \equiv c_1$ and $c_2(z) \equiv c_2$ are constants. Thus y satisfies the differential equation

$$y'' = (\alpha + 4c_2) + (z + 6c_1)y + 2y^3. \tag{10}$$

Comparing this with (5) and having $y = a_1(z)u + a_0(z)$ in mind we are led to some first order equation

$$(u')^2 + A(z,u)u' + B(z,u) = 0, \tag{11}$$

where A and B are rational; note that (5) contains the term $(u')^2$, but (10), when transformed to the variable u, does not.

It is now easily seen that $v(z) = y(z - 6c_1)$ satisfies $v'' = (\alpha + 4c_2) + zv + 2v^3$, and also some first order equation

$$(v')^2 + \tilde{A}(z,v)v' + \tilde{B}(z,v) = 0,$$

and hence is itself an Airy solution. Since Airy solutions, however, always solve first order equations of *odd* degree, this equation must be reducible, which means that v satisfies some Riccati equation. In this case, however, the residues of v never vary, in other words, sub-case (a) never occurs, but nevertheless had to be treated.

Sub-case (b) *$\epsilon = \epsilon(p)$ does not vary.*

This means that $\epsilon(p) = \epsilon \in \{-1, 1\}$ is constant for almost all poles p of y. If we take $\epsilon = 1$ for simplicity, it follows easily from the considerations in sub-case (a) that y satisfies

$$y'' = (\alpha + 4c_2(z) - 2c_1'(z)) + (z + 6c_1(z))y + 2y^3$$

where $4c_2' = -c_1''$. Also $\Psi(z) = y'(z) + y^2(z)$ is regular at $z = p$ with series expansion

$$\Psi(z) = -\frac{1}{2}(p + 6c_1(p)) - (1 + \alpha + 4c_1'(p) + 4c_2(p))(z - p) + \cdots$$

about almost every pole of y. Moreover, Ψ is rational, this following from $m(r, \Psi) = O(\log r)$ and the fact that Ψ has only finitely many poles. Hence we have

$$y' + y^2 = -z/2 - 3c_1(z)$$

and $-4c_2(z) = c_1'(z) + 1/2 + \alpha$. In this case u itself satisfies some Riccati equation

$$u' = a(z) + b(z)u + c(z)u^2 \tag{12}$$

with rational coefficients. For the sake of completeness we mention that in case $\epsilon = -1$ the Riccati differential for y is $y' = z/2 + 3c_1(z) + y^2$.

Finally it follows from

$$w = R(z,u) \quad\text{and}\quad w' = R_z(z,u) + R_u(z,u)\left[a(z) + b(z)u + c(z)u^2\right]$$

that w and w' are algebraically dependent over the field $\mathbb{C}(z)$ of rational functions, and hence w satisfies some first order equation (3). In other words, w is an Airy solution.

Before proceeding further we consider an **Example**. Suppose

$$w = R(z,u) = \frac{1 + zu + 2u^3}{z + 2u^2}$$

is a solution of $(*)\ w'' = -3/2 + zw + 2w^3$. In this case we have

$$R(z,u) = u + \frac{1/2}{u^2} - \frac{z/4}{u^4} + \cdots$$

and hence $c_1 = c_3 = 0$, $c_2 = 1/2$, $c_4(z) = -z/4$. Thus u is necessarily a solution of $u' = z/2 + u^2$, and obviously this is also sufficient for w to be a solution of equation $(*)$.

5. Case B: Common poles of w and $1/Q(z, u(z))$.

We now come to the second case, in which the methods of Case A, with simple modifications, also apply, the main difference being that we have to deal with *algebraic* and *algebroid* rather than *rational* and *meromorphic* functions.

We first note that Q need not be irreducible, and denote its factorization into irreducible polynomials by $Q = Q_1^{k_1} \cdots Q_n^{k_n}$. Let $s = \sigma(z)$ be any of the branches defined by the equation $Q_1(z,s) = 0$, say, and set $u - \sigma(z) = 1/v$. Then v is (some branch of) an *algebroid* function with finitely many algebraic singularities, and in a neighbourhood of $v = \infty$ we have

$$w = R(z,u) = a_{k_1}(z)v^{k_1} + \cdots + a_1(z)v + a_0(z) + b_1(z)/v + \cdots ,$$

$a_{k_1}(z) \not\equiv 0$. The coefficients a_j, b_j are now *algebraic.* We need some information about w, which can be found in [10], and, in more detail, in [11]:

Proposition *Let (p_ν) denote the sequence of non-zero poles of w and set $\Delta_\nu = \{z : |z - p_\nu| < \delta|p_\nu|^{-1/2}\}$. Then for $\delta > 0$ sufficiently small the disks Δ_ν are mutually disjoint, and $w(z) = O(|z|^{1/2})$, $w'(z) = O(|z|)$ and $w''(z) = O(|z|^{3/2})$ hold as $z \to \infty$ outside these disks.*

In particular, it is easy to construct closed curves Γ_r (approximately Γ_r coincides with the circle $|z| = r$), such that $w(z) = O(|z|^{1/2})$ holds on Γ_r. From this we may conclude that there exists some $\beta > 0$, such that $v(z) = O(|z|^\beta)$ holds on Γ_r. This and the fact that u is transcendental then leads to the conclusion that (every branch of) v has infinitely many regular poles p, which are not poles of the remainder term in (7), and hence are also poles of w.

As in Case A we conclude that $k_1 = 1$ and set $y = a_1(z)v + a_0(z)$; then y is also an algebroid function with finitely many algebraic singularities. As in Case A, (1) and (7) yield the development (8), and again we consider

$$\Phi(z) = y''(z) - 2y^3(z) - (z + 6c_1(z))y(z).$$

Then Φ is algebroid with at most finitely many algebraic singularities and also at most finitely many poles, since a standard calculation shows that Φ is regular at $z = p$; moreover (9) holds in this case, too.

Again from $w(z) = O(|z|^{1/2})$ outside the disks Δ_ν, and the corresponding estimates $w'(z) = O(|z|)$ and $w''(z) = O(|z|^{3/2})$, see [11], which yield analogous estimates for y, y' and y'', it follows that $\Phi(z) = O(|z|^\beta)$ as $z \to \infty$, for some $\beta > 0$, and hence Φ is an algebraic function, while y satisfies

$$y'' = \Phi(z) + (z + 6c_1(z))y + 2y^3.$$

As in Case A we distinguish two sub-cases:

Sub-case (a) $\epsilon = \epsilon(p)$ *varies.*

Then from

$$\Phi(p) = \alpha + 4c_2(p) - 2\epsilon(p)c_1'(p) \quad \text{and} \quad \Phi'(p) = -3\epsilon(p)c_1''(p)$$

it again follows that $4c_2'(p) = -\epsilon(p)c_1''(p)$, and hence $c_1(z) \equiv c_1$ and $c_2(z) \equiv c_2$ are constants. Otherwise, analytic continuation of c_1 and c_2 along any path starting at any pole p of y and terminating at any pole $\tilde{p}$ with $\epsilon(\tilde{p}) = -\epsilon(p)$ then yields branches $\tilde{c}_1$ and $\tilde{c}_2$ with $4\tilde{c}_2'(\tilde{p}) = -\epsilon(p)\tilde{c}_1''(\tilde{p})$. On the other hand, analytic continuation of Φ along the same path leads to $4\tilde{c}_2'(\tilde{p}) = -\epsilon(\tilde{p})\tilde{c}_1''(\tilde{p})$, which yields a contradiction unless c_2 and c_1' are constants. This and $\Phi(z) = \alpha + 4c_2 - 2\epsilon(p)c_1'$ at almost every pole $z = p$ of y also implies that Φ, and hence c_1, is a constant.

Thus, y satisfies the differential equation (10), and hence $y = a_1 + a_0/(u - \sigma)$ is a transcendental *meromorphic* function in the plane. By analytic continuation this implies that σ is a rational function, hence the irreducible factor $Q_1(z, u)$ is linear, and all coefficients are rational rather than algebraic.

Then, on one hand, u satisfies equation (5), while, on the other hand, it follows from $y = a_0(z) + \dfrac{a_1(z)}{u - \sigma(z)} = T(z, u)$ and (10) that u also satisfies

$$u'' = -\frac{T_{uu}(z, u)}{T_u(z, u)}(u')^2 - 2\,\frac{T_{zu}(z, u)}{T_u(z, u)}\,u' + S(z, u),$$

where S is rational. This equation is different from (5), since otherwise R would be linear with respect to u. Thus u solves some first order algebraic differential equation (11) of the second degree and with rational coefficients; this equation again reduces to some Riccati equation with rational coefficients.

Sub-case (b) $\epsilon = \epsilon(p)$ *does not vary.*

We take $\epsilon = 1$ for simplicity and consider $\Psi(z) = y'(z) + y^2(z)$. Then Ψ is regular at almost every pole $z = p$, and using the same arguments as in sub-case (a) for Φ, we may conclude that Ψ is algebraic and equals $-z/2 + 3c_1(z)$, so that y satisfies

$$y' + y^2 = -z/2 - 3c_1(z) \; ; \tag{13}$$

moreover we have $-4c_2(z) = c_1'(z) + 1/2 + \alpha$.

Thus

$$u = \sigma(z) + \frac{a_1(z)}{y - a_0(z)}$$

also satifies some Riccati equation

$$u' = a(z) + b(z)u + c(z)u^2$$

with algebraic coefficients. However, since u is a transcendental *meromorphic* function in the plane, the coefficients a, b and c, $c(z) \not\equiv 0$, have to be rational functions, since otherwise analytic continuation would lead to some non-trivial algebraic equation for u. We mention that in case $\epsilon = -1$ the differential equation for (the branches of) y is $y' - y^2 = z/2 + 3c_1(z)$.

We thus may conclude that again w is an Airy solution of (II_α). The remark at the end of A(a) remains valid, this sub-case B(a) never occurs. $\square$

6. The nature of u remains open

We continue discussion of our **Example**[1], begun at the end of Case A: Suppose that $w = \dfrac{1 + zu + 2u^3}{z + 2u^2}$ satisfies $w'' = -3/2 + zw + 2w^3$. We set $u = (-z/2)^{1/2} + v^{-1}$ to obtain $R(z,u) = \frac{1}{4}(-z/2)^{-1/2}v + (-z/2)^{1/2} + \frac{1}{4}z^{-1} + O(|v|^{-1})$ as $v \to \infty$, and hence, for $y = \frac{1}{4}(-z/2)^{-1/2}v + (-z/2)^{1/2} + \frac{1}{4}z^{-1}$, the Riccati equation $y' = -\frac{1}{2}z - \frac{3}{4}(-z/2)^{-1/2} - \frac{3}{16}z^{-2} - y^2$ with algebraic coefficients. In accordance with (13) we have $R(z,u) = y + c_1(z)/y + c_2(z)/y^2 + O(1/|y|^3)$, $c_1(z) = \frac{1}{4}(-z/2)^{-1/2} + \frac{1}{16}z^{-2}$, $c_2(z) = \frac{1}{4} + \frac{1}{64}(-z/2)^{-3/2} + \frac{1}{32}z^{-3}$ and $c_1'(z) + \frac{1}{2} - \frac{3}{2} + 4c_2(z) = 0$.

We return to the general case and assume that $w(z) = R(z, u(z))$ satisfies

$$w'' = n + 1/2 + zw + 2w^3$$

with $n \in \mathbb{N}_0$, say. Applying the Bäcklund transformation $w \mapsto w_{-1}$, thereby making use of $u' = a(z) + b(z)u + c(z)u^2$, we obtain $w_{-1}(z) = R_1(z, u(z))$, where

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[1] Computation done with *Maple V Release 5.1.*

R_1 is rational. Hence after n steps we arrive at $U(z) = S(z, u(z))$, where S is a rational function and U is some solution of $U' = z/2 + U^2$.

Any Riccati equation $u' = a(z) + b(z)u + c(z)u^2$ may be transformed into some Riccati equation of the form $v' = a(z) + v^2$, a rational, by a suitably chosen Möbius transformation, having rational coefficients, of the dependent variable. We assume that this has already be done and write again u instead of v, and hence have

$$u' = a(z) + u^2. \tag{14}$$

The considerations in sub-cases A(b) and B(b) then show that the following is true:

A) $S(z, \infty) \equiv \infty$ and (14) imply $S(z, u) = u + O(|u|^{-1})$ as $u \to \infty$.

B) On the other hand, if $S = P/Q$ and $Q(z, \sigma(z)) = 0$, we set $u = \sigma(z) + 1/v$ to obtain $U = R(z, \sigma(z) + 1/v) = a_1(z)v + a_0(z) + O(1/v)$ as $v \to \infty$. If we finally set $y = a_1(z)v + a_0(z)$, we obtain in the same manner

$$y' = b(z) + y^2, \tag{15}$$

where now y is an algebroid function with finitely many algebraic singularities and infinitely many poles, and b is an algebraic function. This differential equation is preserved under analytic continuation.

The relation between u and (any branch) y may be written in symmetric form

$$(u - \sigma(z))(y - \tau(z)) = \gamma(z), \tag{16}$$

from which we derive that $b(z) + \tau(z)^2 - \tau'(z) = -\gamma(z)$ holds at almost every pole $z = p$ of u, and hence for all z. By symmetry we also obtain $a(z) + \sigma(z)^2 - \sigma'(z) = -\gamma(z)$, and hence

$$u' = \sigma'(z) - \sigma(z)^2 - \gamma(z) + u^2 \quad \text{and} \quad y' = \tau'(z) - \tau(z)^2 - \gamma(z) + y^2.$$

In order that equations (14) and (15) are transformed by (16) into each other, it is necessary and sufficient that

$$\gamma'/\gamma = 2(\sigma + \tau) \tag{17}$$

holds.

This can be done for every branch $s = \sigma_k(z)$ defined by $Q(z, s) = 0$, and hence we have either

$$U = S(z, u) = u + \sum_{k=1}^{n} \frac{\gamma_k}{u - \sigma_k} = u + \sum_{k=1}^{n} (y_k - \tau_k),$$

or else

$$U = S(z, u) = \rho + \sum_{k=1}^{n} \frac{\gamma_k}{u - \sigma_k} = \rho + \sum_{k=1}^{n} (y_k - \tau_k),$$

where ρ is rational. Moreover, $a(z) = \sigma_k'(z) - \sigma_k^2(z) - \gamma_k(z)$ is rational and independent of k, and $\gamma_k'/\gamma_k = 2(\sigma_k + \tau_k)$ holds for every k. It remains open whether or not these conditions imply $a(z) = z/2$, and hence $u = U$. This seems to be a non-trivial question. Although it seems impossible that the great number of conditions imposed on the various coefficients a, σ_k, τ_k and γ_k, the meromorphic function u and the algebroid functions y_k, can be fulfilled, we note that they *are* indeed fulfilled for Airy solutions $w \neq U$.

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