Zalcman Functions and Rational Dynamics

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Abstract

Application of Zalcman's Lemma to iteration sequences of rational functions R yields a rich class $\mathcal{Z}$ of new transcendental entire or meromorphic functions–called Zalcman functions. We prove the basic properties of these functions and determine completely their value distribution quantities. We also show that for rational functions with parabolic orbifold $\mathcal{O} = (\widehat{\mathbb{C}}, \nu)$, the class $\mathcal{Z}$ essentially consists of branched covering maps $\mathbb{C} \longrightarrow \mathcal{O}$.

1 Definition and Basic Properties of Zalcman Functions

We refer to **Zalcman's Lemma** [19] as follows: *Let $\mathcal{F}$ be a family of functions $F : D \longrightarrow \widehat{\mathbb{C}}$, meromorphic in some domain $D \subset \mathbb{C}$, which is not normal at $z = z_0 \in D$. Then there exist sequences $(F_n) \subset \mathcal{F}$, $(z_n) \subset D$, $z_n \to z_0$, and $(\rho_n) \subset \mathbb{C} \setminus \{0\}$, $\rho_n \to 0$, and some constant $C > 0$, such that, for every $r > 0$,*

$$|\rho_n| F_n^{\#}(z_n + \rho_n z) \le C$$

holds for $|z| \le r$, $n \ge n(r)$, where $F^{\#}$ denotes the spherical derivative of F.

Remark A similar result for normal functions can be found in Lohwater and Pommerenke [7]. Zalcman's Lemma is sometimes also called rescaling Lemma; for various generalizations and applications the reader is referred to Zalcman's overview [20] and Schiff's monograph [14].

Zalcman's Lemma applies to iteration sequences of rational functions as follows–for notation and various results on rational iteration used in this paper, the reader is referred to Milnor [8] or the author [15]:

Zalcman's Lemma for iteration sequences *Let R be rational of degree $d > 1$ and let $(\tilde{z}_n)$ be any complex sequence tending to the Julia set $\mathcal{J}_R$ of R. Then there exist sequences $(z_n) \subset \mathbb{C}$ with $z_n - \tilde{z}_n \to 0$, and $(\rho_n) \subset \mathbb{C} \setminus \{0\}$ with $\rho_n \to 0$,* *such that the sequence* $(R^n(z_n+\rho_n z))_{n\in\mathbb{N}}$ *is normal in* $\mathbb{C}$, *and the limit function of every convergent sub-sequence,*

$$f(z)=\lim_{k\to\infty} R^{n_k}(z_{n_k}+\rho_{n_k}z) \tag{1}$$

is a non-constant meromorphic function in the complex plane which has bounded spherical derivative,

$$f^{\#}(z)=|f'(z)|(1+|f(z)|^2)^{-1}\le \text{const.} \tag{2}$$

The conclusion about the whole sequence $(R^n(\tilde{z}_n+z))_{n\in\mathbb{N}}$ follows from two facts: (i) no sub-sequence is normal at $z=0$, and (ii) the Julia set $\mathcal{J}_R$ is perfect.

Remark Candidates for appropriate sequences $(\tilde{z}_n)$ are: $\tilde{z}_n\in\mathcal{J}_R$; any sequence such that $(R^n)^{\#}(\tilde{z}_n)\to\infty$.

Throughout this paper, R will always denote a rational function of degree $d>1$. Let $\mathcal{Z}(R)$ be the set of meromorphic functions $f:\mathbb{C}\longrightarrow\widehat{\mathbb{C}}$ which occur as limits of type (1), (2). For every point $z_0\in\mathcal{J}_R\setminus\{\infty\}$ let $\mathcal{Z}(z_0,R)$ denote the subset of $\mathcal{Z}(R)$ consisting of those functions (1) with $\lim_{k\to\infty} z_{n_k}=z_0$. We call the functions in $\mathcal{Z}(R)$ Zalcman functions associated with R. If R and S are conjugates of each other, $\phi\circ R=S\circ\phi$, ϕ a Möbiustransform, then it is easily seen that

$$\phi(\mathcal{Z}(z_0,R))=\mathcal{Z}(\phi(z_0),S).$$

We take the opportunity to define, for $\infty\in\mathcal{J}_R$,

$$\mathcal{Z}(\infty,R)=1/\mathcal{Z}(0,R_1),\quad R_1(z)=1/R(1/z)).$$

It is then obvious that $\mathcal{Z}(R)=\bigcup_{z_0\in\mathcal{J}}\mathcal{Z}(z_0,R)$.

There is one situation where there is no need for Zalcman's Lemma: suppose z_n^* is chosen as to maximize $(R^n)^{\#}(z)$. Then, with $1/\rho_n^*=(R^n)^{\#}(z_n^*)$ the sequence $\big(\rho_n^*(R^n)^{\#}(z_n^*+\rho_n^*z)\big)_{n\in\mathbb{N}}$ is bounded by $\rho_n^*(R^n)^{\#}(z_n^*)=1$, and for any admissible sequence (ρ_n) we have $\rho_n^*=O(|\rho_n|)$.

Remark There is, of course, no necessity to insist that Zalcman functions have bounded spherical derivative. Some of the proofs, of course, become easier, if this additional property has not to be proved. For the results in section 3 it would suffice to consider functions of finite order, in section 2 one could also deal with functions of infinite order. It is just a matter of taste to proceed as we did.

Most of our results rely on the following:

Theorem 1 *Let* $\mathcal{Z}$ *denote any of the sets* $\mathcal{Z}(R)$ *or* $\mathcal{Z}(z_0,R)$, *respectively, and assume* $f\in\mathcal{Z}$. *Then also* $R\circ f\in\mathcal{Z}$, *and, more important, there exists* $f_1\in\mathcal{Z}$, *such that*

$$f=R\circ f_1. \tag{3}$$

Proof Suppose that f is given by (1). Then the sequence $(R^{n_k+1}(z_{n_k}+\rho_{n_k}z))_{k\in\mathbb{N}}$ tends to $R\circ f$ as $k\to\infty$. On the other hand, the sequence $(R^{n_k-1}(z_{n_k}+\rho_{n_k}z))_{k\in\mathbb{N}}$ is normal in $\mathbb{C}$, and hence a sub-sequence tends to some function f_1, locally uniformly in $\mathbb{C}$, with $f=R\circ f_1$. We have to show that $f^\#$ is bounded if and only if $f_1^\#$ is. Suppose first that $f^\#(z)\le M$ in $\mathbb{C}$ and let $(\hat{z}_n)$ be any sequence in $\mathbb{C}$. Since the sequence $(f(\hat{z}_n+z))_{n\in\mathbb{N}}$ is normal in $\mathbb{C}$ by Marty's Criterion (see, e.g., Ahlfors [1]), the sequence $(f_1(\hat{z}_n+z))_{n\in\mathbb{N}}$ is normal, too, and hence $f_1^\#(\hat{z}_n)$ is bounded, again by Marty's Criterion . This shows that $\sup f_1^\#(z)$ is finite. The proof of the converse is easier, since $f^\#(z)\le Hf_1^\#(z)$ with $H=\sup R^\#(w)(1+|w|^2)$. q.e.d.

Theorem 2 *The classes $\mathcal{Z}(R)$ and $\mathcal{Z}(z_0,R)$ are closed under post-composition with maps $z\mapsto az+b$, $a\neq 0$.*

Remark We call $f(az+b)$ a translate of f.

The **Proof** follows from replacing ρ_n by $a\rho_n$ and z_n by $z_n+\rho_n b/a$. q.e.d.

Theorem 3 *Suppose that $R'(z_0)\neq 0$. Then $\mathcal{Z}(z_0,R)=\mathcal{Z}(R(z_0),R)$.*

Proof Let $f\in\mathcal{Z}(z_0,R)$ be given by (1). Then, for $n=n_k$, we have

$$R^n(z_n+\rho_n z)=R^{n-1}\left(R(z_n)+R'(z_n)\rho_n z+O(|\rho_n|^2)\right),$$

locally uniformly, and hence $f\in\mathcal{Z}(R(z_0),R)$: replace, in (1), z_n with $\hat{z}_n=R(z_n)$ and ρ_n with $\hat{\rho}_n=R'(z_n)\rho_n$. The converse is also true since

$$R^{n-1}(\hat{z}_n+\hat{\rho}_n z)=R^{-1}\left(R^n(z_n+\rho_n z+O(|\rho_n|^2))\right)\sim R^{-1}(R^n(z_n+\rho_n z)),$$

where R^{-1} denotes the local inverse of R mapping $R(z_0)$ to z_0. q.e.d.

Remark The limit (1) depends sensitively on z_n, e.g., in the case $z_n\to z_0$, z_0 a parabolic fix-point (as will be seen later), but remains unaltered if ρ_n is replaced with $\rho_n(1+o(1))$.

Theorem 3 is not always true if $R'(z_0)=0$ (this will be shown in section 3). We call $f\in\mathcal{Z}(z_0,R)$ a Zalcman function of the first kind, if in (1) we have

$$z_{n_k}-z_0=O(|\rho_{n_k}|)\quad\text{as}\quad k\to\infty$$

(at least for some sub-sequence of (n_k)), and of the second kind otherwise. We may assume that, in the latter case,

$$\rho_{n_k}=o(|z_{n_k}-z_0|)\quad\text{as}\quad k\to\infty,$$

and $z_{n_k}=z_0+b\rho_{n_k}(1+o(1))$ for some $b\in\mathbb{C}$ in the former case, and thus

$$f(z)=\lim_{k\to\infty}R^{n_k}(z_{n_k}+\rho_{n_k}z)=\lim_{k\to\infty}R^{n_k}(z_0+\rho_{n_k}(z+b))=f_1(z+b),$$

with $f_1 \in \mathcal{Z}_1(z_0, R)$, this indicating the definition of $\mathcal{Z}_1(z_0, R)$.

The difference between first and second kind is essentially the following: in the first case there exists $r_0 > 0$ and $k_0 \in \mathbb{N}$, such that, for $r \geq r_0$, $\bigcap_{k \geq k_0} D_k(r) = \{z_0\}$, while $\bigcap_{k \geq k_0} D_k(r) = \emptyset$ for every $k_0 \in \mathbb{N}$ and every $r > 0$ in the second case.

Theorem 4 *Suppose that $f \in \mathcal{Z}_1(z_0, R)$ and $R^{(j)}(z_0) = 0$ for $1 \leq j < s$, but $R^{(s)}(z_0) \neq 0$. Then there exists some $f_1 \in \mathcal{Z}_1(R(z_0), R)$ such that $f(z) = f_1(z^s)$. On the other hand, any $f \in \mathcal{Z}(z_0, R)$ of the second kind belongs also to $\mathcal{Z}(R(z_0), R)$, and vice versa.*

Proof We first treat the case that $f \in \mathcal{Z}(z_0, R)$ is of the second kind. Then, for $n = n_k$, we have $R^n(z_n + \rho_n z) = R^{n-1}(R(z_n + \rho_n z))$. Since $\hat{\rho}_n = R'(z_n)\rho_n \sim sc(z_n - z_0)^{s-1}\rho_n$, $c \neq 0$, while $R(z_n + \rho_n z) - R(z_n) - \hat{\rho}_n z = o(|\hat{\rho}_n|)$, uniformly in $|z| < r$, we obtain

$$f(z) = \lim_{k \to \infty} R^{n_k - 1}(\hat{z}_{n_k} + \hat{\rho}_{n_k} z), \tag{4}$$

with $\hat{z}_{n_k} = R(z_{n_k})$, i.e., $f \in \mathcal{Z}(R(z_0), R)$. The converse is proved by reversing this argument: given $\hat{z}_n$ and $\hat{\rho}_n$ for $n = n_k$, such that (4) holds, choose $z_n \to z_0$ with $R(z_n) = \hat{z}_n$ and set $\rho_n = \hat{\rho}_n / R'(z_n)$.

Now assume $f \in \mathcal{Z}_1(z_0, R)$, and hence $f(z) = \lim_{k \to \infty} R^{n_k}(z_0 + \rho_{n_k} z)$. Then

$$f(z) = \lim_{k \to \infty} R^{n_k - 1}(R(z_0) + c\rho_{n_k}^s z^s) = f_1(z^s)$$

holds with $f_1 \in \mathcal{Z}_1(R(z_0), R)$. q.e.d.

Remark For $f(z) = f_1(z^s)$ to have bounded spherical derivative it is necessary and sufficient that $f_1^{\#}(z) = O(|z|^{-1+1/s})$ as $|z| \to \infty$. Under that additional hypothesis the converse is also true, namely that $f \in \mathcal{Z}_1(z_0, R)$ if $f_1 \in \mathcal{Z}_1(R(z_0), R)$.

The next theorem only states what everyone would expect.

Theorem 5 *For hyperbolic R and arbitrary $z_0 \in \mathcal{J}_R$ we have $\mathcal{Z}(z_0, R) = \mathcal{Z}(R)$.*

Proof Let $z_0, \hat{z}_0 \in \mathcal{J}_R$, and let $f \in \mathcal{Z}(z_0, R)$ be given by (1). Since R is hyperbolic, there is some disk D around z_0, such that every inverse map of R^n, $n \in \mathbb{N}$, exists in D. Now $\bigcup_p R^{-p}(\{z_n\})$ is dense in $\mathcal{J}_R$, and hence there exists $p_k \in \mathbb{N}$ and $\hat{z}_{n_k} \to \hat{z}_0$ with $R^{p_k}(\hat{z}_{n_k}) = z_{n_k}$. Set $1/\tau_k = (R^{p_k})'(\hat{z}_{n_k})$ and note that $\tau_k \to 0$ as $k \to \infty$. Then the Koebe Distortion Theorem, see, e.g., [10], applies to the inverse map ϕ_k of R^{p_k}, which maps z_{n_k} to $\hat{z}_{n_k}$ and exists in D. Set $D_k(r) = D(z_{n_k}, r) = \{z : |z - z_{n_k}| < r|\rho_{n_k}|\}$ and $\widehat{D}_k(r) = D(\hat{z}_{n_k}, r|\tau_k|)$. Then, for every $r > 0$ we have

$$\widehat{D}_k(r/2) \subset \phi_k(D_k(r)) \subset \widehat{D}_k(2r)$$

as $k \to \infty$. With $\hat{\rho}_{n_k} = \tau_k \rho_{n_k}$ we thus obtain $f(z) = \lim_{k\to\infty} R^{n_k+p_k}(\hat{z}_{n_k} + \hat{\rho}_{n_k} z)$, i.e., $f \in \mathcal{Z}(\hat{z}_0, R)$. q.e.d.

Remark We note that the methods are not restricted to iteration sequences of rational functions. As an example we consider the polynomial sequence (Q_n), recursively defined by $Q_0(w) = w$ and $Q_{n+1}(w) = Q_n(w)^2 + w$ for $n \geq 0$. The sequence (Q_n) generates the Mandelbrot set for the degree-two polynomial family $\mathcal{P} = \{P_w : P_w(z) = z^2 + w, \quad w \in \mathbb{C}\}$,

$$\mathcal{M} = \{w \in \mathbb{C} : \sup_{n \in \mathbb{N}_0} |P_w^n(w)| \leq 2\}.$$

The family $\mathcal{P}$ is non-normal exactly at points of $\partial\mathcal{M}$. Let w_0 be any point in $\partial\mathcal{M}$. Then Zalcman's Lemma applies to the sequence $(Q_n(w_0 + z))_{n\in\mathbb{N}}$, and also to any sequence $(Q_n(\tilde{w}_n + z))_{n\in\mathbb{N}}$ with $\tilde{w}_n \to w_0$. If

$$f(z) = \lim_{k\to\infty} Q_{n_k}(w_{n_k} + \rho_{n_k} z),$$

then, obviously,

$$f(z) = f_1(z)^2 + w_0 = P_{w_0}(f_1(z))$$

holds with $f_1(z) = \lim_{k\to\infty} Q_{n_k-1}(w_{n_k} + \rho_{n_k} z)$, i.e., the main result in Theorem 1 remains valid, and so do the results of section 2.

2 Value Distribution

We denote by $\mathcal{E}_R$, $\mathcal{C}_R$ and $\mathcal{C}_R^+$, respectively, the set of exceptional values of R, the set of its critical points and its critical orbit $\{R^n(c) : c \in \mathcal{C}_R,\ n \in \mathbb{N}\}$. The exceptional set consists of at most two points. For $\mathcal{E}_R = \{a\}$, R is conjugate to a polynomial P with $\mathcal{E}_P = \{\infty\}$, while for $\mathcal{E}_R = \{a, b\}$, R is conjugate to $z \mapsto z^{\pm d}$ with exceptional set $\{0, \infty\}$.

We use the standard notation of Nevanlinna theory, see, e.g., Hayman [5] or Nevanlinna [10]: for f meromorphic and non-constant in the plane, $m(r, f) = m(r, \infty, f)$, $N(r, f) = N(r, \infty, f)$ and $T(r, f) = m(r, f) + N(r, f)$ denote the proximity function, the counting function of poles and the Nevanlinna characteristic of f, respectively; for $a \in \mathbb{C}$ we set $m(r, a, f) = m(r, \infty, (f - a)^{-1})$ and $N(r, a, f) = N(r, \infty, (f - a)^{-1})$. Then Nevanlinna's First Main Theorem states that

$$m(r, a, f) + N(r, a, f) = T(r, f) + O(1) \quad \text{as} \quad r \to \infty.$$

For functions of finite order, i.e., $\varrho(f) = \limsup_{r\to\infty} \log T(r, f)/\log r < +\infty$, Nevanlinna's Second Main Theorem can be stated as follows:
Given mutually distinct values $a_1, \ldots, a_p \in \mathbb{C}$, then

$$\sum_{j=1}^{p} m(r, a_j, f) \leq m(r, 0, f') + O(\log r)$$

and

$$T(r,f') \le T(r,f) + \overline{N}(r,\infty,f) + O(\log r)$$

hold, where $\overline{N}(r,a,f)$ "counts" the a-points of f, each point being counted simply. Finally we mention Valiron's Lemma, which says that

$$T(r, R\circ f) = (\deg R)\, T(r,f) + O(1) \quad \text{as} \quad r\to\infty.$$

Each Zalcman function f has order of growth at most two, this following from the Ahlfors-Shimizu Theorem (see, e.g., Hayman [5]):

$$T(r,f) = \frac{1}{\pi}\int_0^r A(t)\frac{dt}{t} + O(1),$$

with $A(t) = \int_{|z|<t} f^{\#}(z)dxdy$ being the spherical measure of the image under f of the disk $|z| < t$. It is less obvious that f has order of growth at most one if $\mathcal{E}_R \ne \emptyset$. A proof for f entire, say, can be found in [2]. Moreover, from [11, Thm 4] one can deduce that $f^{\#}(z) \le C$ implies $|f(z)| \le \max(|f(0)|,1)e^{2C|z|}$, and hence $T(r,f) \le 2Cr + O(1)$ as $r\to\infty$.

Theorem 6 *Let f be any Zalcman function associated with R. Then $\mathcal{E}_R$ coincides with the set of Picard exceptional values of f (in the strict sense). For every $a \notin \mathcal{E}_R$ we have*

$$m(r,a,f) = o(T(r,f)) \qquad \text{as } r\to\infty. \tag{5}$$

Proof We first note that $\mathcal{E}_R$ is a subset of the Fatou set $\mathcal{F}_R$. Let $r > 0$ be arbitrary. Then, for $k \ge k(r)$, the disk $|z - z_{n_k}| < r|\rho_{n_k}|$ does not meet $\mathcal{E}_R$, so that $R^{n_k}(z_{n_k} + \rho_{n_k} z)$ never assumes values from $\mathcal{E}_R$ in $|z| < r$. Thus, by Hurwitz' Theorem, f never assumes values from $\mathcal{E}_R$, since $r > 0$ was arbitrary.

Now suppose $a \notin \mathcal{E}_R$. Iterating (3) we obtain, for every $n \in \mathbb{N}$, some Zalcman function f_n such that

$$f = R^n \circ f_n \tag{6}$$

holds. Then, under R^n, a has distinct pre-images $a_1, a_2, \ldots, a_s$, $s = s(n)$, and each a_j has multiplicity at most κ^n, where $\kappa < d$, for $n \ge n_0$. Thus, Nevanlinna's Second Main Theorem yields

$$m(r,a,f) \le \kappa^n \sum_{j=1}^{s(n)} m(r,a_j,f_n) + O(1) \le 2\kappa^n T(r,f_n) + O(\log r) \tag{7}$$

as $r\to\infty$. Since, by Valiron's Lemma, $T(r,f_n) = d^{-n}T(r,f) + O(1)$ as $r\to\infty$, (7) gives

$$m(r,a,f) \le \left(\frac{\kappa}{d}\right)^n T(r,f) + O(\log r) \quad \text{as} \quad r\to\infty,$$

and so (5) follows. q.e.d.

Theorem 7 *For every $a \notin \mathcal{C}_R^+$ we have*

$$\overline{N}(r,a,f) \sim T(r,f) \quad \text{as} \quad r \to \infty,$$

and so $\Theta(a) = 1 - \lim_{r\to\infty} \overline{N}(r,a,f)/T(r,f) = 0$. For $a \in \mathcal{C}^+$, $\Theta(a)$ is a positive rational number, only depending on R, but not on f, with

$$\sum_{a\in\widehat{\mathbb{C}}} \Theta(a) = 2. \tag{8}$$

Remark $\overline{N}(r,a,f) \sim T(r,f)$ means that $\overline{N}(r,a,f) = T(r,f)(1+o(1))$ as $r \to \infty$. We will give a recursive procedure in the proof to compute the ramification index $\Theta(a)$ explicitly. Note that $\mathcal{E}_R \subset \mathcal{C}^+$, with $\Theta(a) = 1$ for $a \in \mathcal{E}_R$.

Proof For $a \notin \mathcal{C}_R^+$ the equation $R^n(\zeta) = a$ has d^n distinct solutions $a_1, \ldots, a_{d^n}$, so that

$$\begin{aligned}\overline{N}(r,a,f) = \sum_{j=1}^{d^n} \overline{N}(r,a_j,f_n) &\geq (d^n-2)T(r,f_n) + O(\log r)\\ &= (1-2d^{-n})T(r,f) + O(\log r),\end{aligned}$$

again by Nevanlinna's Second Main Theorem and Valiron's Lemma. This gives the desired result about $\overline{N}(r,a,f)$, since $n \in \mathbb{N}$ was arbitrary.

Now suppose $a \in \mathcal{C}_R^+$ and set $n(a) = \sup\{n \in \mathbb{N} : f^n(c) = a \text{ for some } c \in \mathcal{C}_R\}$. Note that $n(a) = +\infty$ iff $a \in \mathcal{C}_R^+$ is periodic. We also set $n(a) = 0$ for $a \notin \mathcal{C}_R^+$ and remark that, for $1 \leq n(f(a)) < \infty$, $n(f(a)) \geq n(a) + 1$ holds.

For $n(a) = 0$ we have shown that $\Theta(a) = 0$. Now suppose that $n(a) \in \mathbb{N}$. Then, under R, a has distinct pre-images $a_1, \ldots, a_s$, $s \leq d$, and so

$$\overline{N}(r,a,f) = \sum_{i=1}^{s} \overline{N}(r,a_i,f_1)$$

holds. By assumption we have $\overline{N}(r,a_i,f_1) \sim \frac{1}{d}\overline{N}(r,a_i,f)$ (the proof is by induction on $n = n(a)$; note that $n(a_i) \leq n(a) - 1$), and so

$$1 - \Theta(a) = \frac{1}{d} \sum_{i=1}^{s} (1 - \Theta(a_i)), \tag{9}$$

i.e., $\Theta(a)$ is a rational number of type $k(a)d^{-n(a)}$, $k(a) \in \mathbb{N}$.

We now assume that $a \in \mathcal{C}_R^+$ is periodic, say $a = a_p \in \alpha$, where $\alpha = \{a_1, \dots, a_p\}$ is a p-cycle. Denote the distinct pre-images of a_j under R, other than a_{j-1}, by $a_i^{(j)}$. Then $n(a_i^{(j)}) < \infty$ and

$$\begin{aligned} \overline{N}(r,a,f) &= \overline{N}(r,a_{p-1},f_1) + \sum_i \overline{N}(r,a_i^{(p)},f_1) \\ &\sim \overline{N}(r,a_{p-1},f_1) + d^{-1}\sum_i \overline{N}(r,a_i^{(p)},f) \end{aligned} \tag{10}$$

holds. Iterating this relation yields

$$\overline{N}(r,a,f) \sim \overline{N}(r,a,f_p) + \sum_{j=1}^{p} d^{j-p-1}\sum_i \overline{N}(r,a_i^{(j)},f). \tag{11}$$

If it were already known that $\overline{N}(r,a,f_p) \sim d^{-p}N(r,a,f)$ this would imply

$$(1-d^{-p})\overline{N}(r,a,f) \sim \sum_{j=1}^{p} d^{j-p-1} \sum_i \overline{N}(r,a_j^{(j)},f),$$

and so

$$1-\Theta(a) = \sum_{j=1}^{p} \frac{d^{j-1}}{d^p-1}\sum_i \left(1-\Theta(a_i^{(j)})\right).$$

This is indeed true, following, in the limit $n\to\infty$, from (11) by iteration:

$$\overline{N}(r,a,f) \sim \overline{N}(r,a,f_{np}) + \sum_{j=1}^{p} d^{j-p-1}\frac{1-d^{-np}}{1-d^{-p}}\sum_i \overline{N}(r,a_i^{(j)},f)$$

for $n\in\mathbb{N}$; note that $\overline{N}(r,a,f_{np}) \le d^{-np}T(r,f)$.

To prove (8) we first assume $\mathcal{E}_R = \emptyset$, so that f has no deficient values. For the sake of simplicity we also assume that $\infty \notin \mathcal{C}_R \cup \mathcal{C}_R^+$. Let $a_1, \dots, a_{2d^n-2}$ denote the zeros of $(R^n)'$, written down according to their multiplicities, and let $v_1, \dots, v_m$ be the (mutually distinct) critical values of R^n. Then, by our previous result, for $n\in\mathbb{N}$,

$$\begin{aligned} T(r,f)\sum_{i=1}^{m}\Theta(v_i) &\sim \sum_{i=1}^{m}\big(N(r,v_i,f)-\overline{N}(r,v_i,f)\big) \\ &\ge \sum_{i=1}^{2d^n-2} N(r,a_i,f_n) = 2(1-d^{-n}+o(1))T(r,f), \end{aligned}$$

and

$$\sum_{i=1}^{m}\Theta(v_i) \ge 2-2d^{-n} \tag{12}$$

hold. This gives (8), since the total ramification is always at most 2. The proof is the same if $\mathcal{E}_R \neq \emptyset$. For $\mathcal{E}_R = \{\infty\}$, say, we have $\Theta(\infty) = 1$ and (12) has to be replaced with $\sum_{i=1}^{m} \Theta(v_i) \geq 1 - d^{-n}$, while for $\mathcal{E}_R = \{0, \infty\}$ we have $\Theta(0) = \Theta(\infty) = 1$, and $\Theta(a) = 0$ else. q.e.d.

Examples

- $R(z) = z^2 + i: \ \Theta(i) = 1/2, \ \Theta(-1+i) = 1/3, \ \Theta(-i) = 1/6, \ \Theta(\infty) = 1.$
- $R(z) = z^2 + z: \ c_n = R^n(-1/2): \ \Theta(c_n) = 2^{-n}, \ \Theta(\infty) = 1.$
- $R(z) = (1 - 2/z)^2: \ \Theta(0) = 1/2, \ \Theta(\infty) = \Theta(1) = 3/4.$
- $R(z) = i(z - 1/z)/2: \ \Theta(-1) = \Theta(1) = \Theta(0) = \Theta(\infty) = 1/2.$
- $R(z) = 2z^2 - 1: \ \Theta(-1) = \Theta(1) = 1/2, \ \Theta(\infty) = 1.$
- $R(z) = (z^3 + \omega)/(\omega z^3 + 1), \ \omega^3 = 1, \omega \neq 1: \ \Theta(1) = \Theta(\omega) = \Theta(\overline{\omega}) = 2/3.$

3 Parabolic Orbifolds

The usually "exceptional"cases in rational dynamics may be described algebraically as follows: there exists a map

$$\nu : \widehat{\mathbb{C}} \to \mathbb{N} \cup \{\infty\}$$

such that, for $a \in \widehat{\mathbb{C}}$,

$$\nu(a)\deg_{(a)} R = \nu(R(a)),$$

where $\deg_{(a)} R$ is the local degree of R at $z = a$, and

$$\sum_{a \in \widehat{\mathbb{C}}} (1 - 1/\nu(a)) = 2. \tag{13}$$

Then $\mathcal{O} = (\widehat{\mathbb{C}}, \nu)$ is called a parabolic orbifold, see Thurston [17] or Eremenko [3]. It admits a ramified covering map $F : \mathbb{C} \longrightarrow \mathcal{O}$, which is branched exactly over those a with $1 < \nu(a) < \infty$, in such a way that F omits every value a with $\nu(a) = \infty$, and assumes every other value a infinitely often, always with multiplicity $\nu(a)$: $\deg_{(z)} f = \nu(f(z))$. Note that $1 - 1/\nu(a) = \Theta(a) = \Theta(a, F)$ is the ramification index of a for F.

Set $m = \mathrm{lcm}\{\nu(a) : \nu(a) < \infty\}$ and $m(a) = m(1 - 1/\nu(a))$. Then

$$F'^m \prod_{m(a)>0} (F - a)^{-m(a)}$$

is an entire function without zeros, and so is an exponential $\exp g(z)$, g entire (we have tacitly assumed that $\nu(\infty) = 1$, hence $m(\infty) = 0$, for simplicity; if $m(\infty) > 0$, the product has to be taken over all finite a with $m(a) > 0$). Introducing new coordinates $\zeta = \int \exp(g(z)/m)dz$ and $f(\zeta) = F(z)$ gives $f'^m = \prod_{m(a)>0} (f-a)^{m(a)}$ with $\sum_{m(a)>0} m(a) = 2m$. Hence, branched covering maps always satisfy so-called binomial differential equations, and vice versa! They have several interesting characterizations: *every meromorphic solution of some Poincaré equation which is either periodic or else satisfies some algebraic differential equation is a branched covering of some parabolic orbifold,* see Ritt [12, 13].

We will prove:

Theorem 8 *Let R be a rational function with parabolic orbifold $\mathcal{O} = (\widehat{\mathbb{C}}, \nu)$. Then every Zalcman function $f \in \mathcal{Z}(z_0, R)$ with $\nu(z_0) = 1$ satisfies*

$$f'^m = c \prod_{m(a)>0} (f-a)^{m(a)}, \quad m(a) = m(1 - 1/\nu(a)) = m\Theta(a), \tag{14}$$

for some constant $c \neq 0$, and hence is a ramified cover $\mathbb{C} \longrightarrow \mathcal{O}$.

Remark Of course also $\nu(\infty) > 1$ is possible; in (14), however, only the finite values a with $\nu(a) > 1$ appear. It will turn out that Zalcman functions $f \in \mathcal{Z}(a, R)$ with $\nu(a) > 1$ are definitely not branched covers $\mathbb{C} \longrightarrow (\widehat{\mathbb{C}}, \nu)$.

There are six (more or less) canonical forms of differential equations (14):

$$\begin{array}{llcll}
(A) & f' & = & f, & \nu(0) = \nu(\infty) = \infty \\
(B) & f'^2 & = & 1 - f^2, & \nu(-1) = \nu(1) = 2, \ \nu(\infty) = \infty \\
(C) & f'^2 & = & (f^2-1)(f-a), \quad a^2 \neq 1, & \nu(\infty) = \nu(1) = \nu(-1) = \nu(a) = 2 \\
(D) & f'^3 & = & (1-f^2)^2, & \nu(1) = \nu(-1) = \nu(\infty) = 3 \\
(E) & f'^4 & = & (1-f^2)^3, & \nu(1) = \nu(-1) = 4, \ \nu(\infty) = 2 \\
(F) & f'^6 & = & (f-1)^3(f+1)^4 & \nu(1) = 2, \ \nu(-1) = 3, \ \nu(\infty) = 6
\end{array}$$

Proof of Theorem 8. Let a be any value in $\mathcal{C}_R^+$ and assume $f(\hat{z}_0) = a$ for some Zalcman function $f \in \mathcal{Z}(z_0, R)$, given by (1), say. Then there exists some sequence $\hat{z}_k \to \hat{z}_0$, such that $R^{n_k}(z_{n_k} + \rho_{n_k}\hat{z}_k) = a$ for $k \geq k_0$, by Hurwitz' Theorem. Set $a_k = z_{n_k} + \varrho_{n_k}\hat{z}_k$, then

$$\nu(a) = \nu(a_k) \deg_{(a_k)} R^{n_k}. \tag{15}$$

Since a_k tends to z_0 as $k \to \infty$, with $\nu(z_0) = 1$, this implies $\nu(a_k) = 1$ for almost all k. Thus, (15) shows that $z = \hat{z}_0$ is a zero of $f - a$ of order (at least) $\nu(a)$, again by Hurwitz' Theorem, and hence

$$\varphi = f'^m \prod_{m(a)>0} (f-a)^{-m(a)} \tag{16}$$

is an entire function. The Lemma on the logarithmic derivative, see [5] or [10], then gives (here we need $\varrho(f) < \infty$)

$$m(r,\varphi) = O(\log r),$$

i.e., φ is a polynomial. Let b be any value with $\nu(b) = 1$ and (ζ_n) any sequence with $f(\zeta_n) = b$. Then from (16) it follows that $\varphi(\zeta_n)$ is bounded since $|f'(\zeta_n)| = f^{\#}(\zeta_n)(1+|b|^2)$ is, and so φ is a constant c, which implies (14). q.e.d.

Remark Differential equations similar to (14) were used by the author [16] to construct, in an elementary way, singular metrics for so-called sub-hyperbolic rational functions.

To illustrate Theorem 8 we give several **examples** to the cases (B)-(E); case (F) requires a degree-six rational function with critical orbit relation

$$\begin{array}{ccccc} & \diamondsuit & & \diamondsuit \\ & \downdownarrows & & \downarrow\downarrow\downarrow \\ \diamondsuit \rightrightarrows & \clubsuit & \Rrightarrow \heartsuit \rightleftarrows \heartsuit \Lleftarrow & \clubsuit \\ & \upuparrows & & \uparrow\uparrow\uparrow \\ & \diamondsuit & & \diamondsuit \end{array}$$

$\diamondsuit$ means critical point, $\clubsuit$ means both: critical point and value, and $\heartsuit$ is a repelling fix-point and also a critical value.

First consider $R(z) = 2z^2 - 1$, the second Chebychev polynomial, with $\nu(\infty) = \infty$, $\nu(\pm 1) = 2$, corresponding to equation (B). A ramified cover $\mathbb{C} \longrightarrow (\widehat{\mathbb{C}}, \nu)$ is $f(z) = \cos z$, which belongs to all classes $\mathcal{Z}(x,R)$, $-1 < x < 1$; indeed, $\mathcal{Z}(x,R) = \{z \mapsto \cos(az+b) : a, b \in \mathbb{C},\ a \neq 0\}$ consists of all translates of the Cosine. The Poincaré function for the repelling fix-point at $z = 1$ with multiplier $\lambda = 4$, i.e., the normalized solution of $R \circ \Phi(z) = \Phi(4z)$, is $\Phi(z) = \cosh(\sqrt{2z})$ (and not $\Phi(z) = \cos z$, as one could believe), which has order of growth $\varrho = 1/2$. It is obvious that $\Phi(z) = \lim_{n\to\infty} R^n(1 + 4^{-n}z)$ belongs to $\mathcal{Z}_1(1,R)$.

On the other hand, the iterate R^2 has local degree 2 at $z = 0$, and $\mathcal{Z}(0,R)$ consists of the translates $f(z) = \cos(az+b)$, which can be written as $f(z) = \Phi(-(az+b)^2/2)$, see Theorem 4 in section 1.

Our second example is $R(z) = (1 - 2/z)^2$ with critical point $z = 2$, critical orbit $2 \rightrightarrows 0 \rightrightarrows \infty \to 1 \rightleftarrows 1$, and repelling fix-point $z = 1$ with multiplier $\lambda = -4$. Here we have $\nu(0) = 2$, $\nu(\infty) = \nu(1) = 4$, i.e., we are in case (E). For $z_0 \neq 0, 1, \infty$, every $f \in \mathcal{Z}(z_0, R)$ satisfies

$$f'^4 = c^4 f^2 (f-1)^3, \tag{17}$$

and hence is a branched cover $\mathbb{C} \longrightarrow (\widehat{\mathbb{C}}, \nu)$. It is not hard to show that $f(z) = 1 - \wp^2(cz/4 + b)$, $b \in \mathbb{C}$ suitably chosen; $\wp$ is the Weierstrass P-function for a rectangular lattice.

The Poincaré function $\Phi(z) = \lim_{n\to\infty} R^n(1 + (-4)^{-n}z)$ for the fix-point $z = 1$, i.e., the normalized solution of $R \circ \Phi(z) = \Phi(-4z)$, has order of growth $\varrho(\Phi) = \log 2/\log 4 = 1/2$ (see section 4), and hence does not belong to any class $\mathcal{Z}(z_0, R)$, $z_0 \neq 0, \infty, 1$. Since R^3 has degree 4 at the critical point $z = 2$, it follows from Theorem 4 in section 1 that $\Phi(z) = f_1(\sqrt[4]{z})$ for some suitably chosen $f_1 \in \mathcal{Z}_1(2, R)$.

Besides $z = 1$, R has two additional fix-points $z = \pm 2i$ with multipliers $\lambda = 1 \pm i$. Here the corresponding Poincaré function belongs to $\mathcal{Z}_1(\pm 2i, R)$ and satisfies the differential equation (17).

We consider one more example, namely $R(z) = i(z - 1/z)/2$, with critical orbit

$$i \rightrightarrows -1 \to 0 \to \infty \rightleftarrows \infty \leftarrow 0 \leftarrow 1 \leftleftarrows -i,$$

and $\nu(-1) = \nu(1) = \nu(0) = \nu(\infty) = 2$. In that case the corresponding differential equation for the branched cover $f : \mathbb{C} \longrightarrow (\widehat{\mathbb{C}}, \nu)$ is

$$f'^2 = 4c^2 f(f^2 - 1), \tag{18}$$

corresponding to (C), with non-constant solutions $\wp(cz + b)$. The Poincaré function for the fix-point $z_0 = \infty$ is $\wp(\sqrt{z})$.

Our last example is $R(z) = (z^3 + \omega)/(\omega z^3 + 1)$, with $\omega \neq 1$ a third root of unity (R is a bi-critical map in the terminology of Milnor's paper [9]). Then R has critical points 0 and ∞, which are mapped to ω and $\overline{\omega}$, respectively. Both critical values are mapped to the repelling fix-point $z = 1$ with multiplier $\lambda = -3i\sqrt{3}$; here, in case (D), the ramified cover satisfies the differential equation

$$f'^3 = c^3(1 - f^3)^2. \tag{19}$$

The Poincaré function for the fix-point $z = 1$ is $\Phi(z) = 1 + 1/\wp'(\sqrt[3]{-2z})$, which has order of growth $\varrho(\Phi) = 2/3$.

4 Miscellanea

Repelling fix-points. Let z_0 be a repelling fix-point of R,

$$R(z) = z_0 + \lambda(z - z_0) + a_2(z - z_0)^2 + \cdots, \quad |\lambda| > 1.$$

Then Schröder's functional equation

$$R \circ \Phi = \Phi \circ \lambda, \quad \lambda(z) = \lambda z, \tag{20}$$

has a unique solution

$$\Phi(z) = \lim_{n\to\infty} R^n(z_0 + \lambda^{-n} z),$$

normalized by $\Phi(0) = z_0$ and $\Phi'(0) = 1$; it is meromorphic in the plane, with order of growth $\varrho(\Phi) = \log d/\log|\lambda|$, and is also called Poincaré function for R at the repelling fix-point z_0.

Remark To compute the order one applies Valiron's Lemma to (20), this giving $T(|\lambda|r, f) = dT(r, f) + O(1)$. Iterating this relation yields $h(|\lambda|^n r) = h(r) + O(r^{-\varrho})$ as $r \to \infty$, uniformly with respect to $n \in \mathbb{N}$, with $\varrho = \log d / \log|\lambda|$ and $h(r) = T(r, f) r^{-\varrho}$, and thus $0 < c < h(r) < C$ for $r \geq r_0$.

For $\Phi \in \mathcal{Z}(z_0, R)$ it is necessary that $d \leq |\lambda|^2$ ($d \leq |\lambda|$ in the polynomial case). It is not known whether this condition is also sufficient. It is, however, easy to see that $\Phi \in \mathcal{Z}(z_0, R)$ implies $\Phi \in \mathcal{Z}_1(z_0, R)$, and this is equivalent to $\mathcal{Z}_1(z_0, R) \neq \emptyset$, in fact, $\mathcal{Z}_1(z_0, R) = \{z \mapsto \Phi(az) : a \neq 0\}$.

We remark that the results on the value distribution in section 2 remain valid for Poincaré functions; all what was needed there was a relation like $\Phi(z) = R(\Phi_1(z))$, here fulfilled with $\Phi_1(z) = \Phi(z/\lambda)$, and $\varrho(\Phi) < +\infty$, a minor requirement; for Theorem 6 in the corresponding case see also Eremenko and Sodin [4].

Parabolic fix-points. We assume that R has a parabolic fix-point at z_0 with multiplier 1,

$$R(z) = z_0 + (z - z_0)(1 + c(z - z_0)^s + \cdots),$$

$c \neq 0$, $s \in \mathbb{N}$, and consider $f \in \mathcal{Z}(z_0, R)$, given by (1). We claim that

$$\rho_{n_k} = o(|z_{n_k} - z_0|^{s+1}),$$

and, in particular, that $\mathcal{Z}_1(z_0, R) = \emptyset$.

Otherwise we may assume that, for some $b \in \mathbb{C}$, $(z_{n_k} - z_0)^{s+1} \sim b\rho_{n_k}$ holds. Thus, from (1) it follows that $R(f(z)) \sim R^{n_k}(z_{n_k} + \rho_{n_k}(z + b))$, hence $R(f(z)) = f(z + b)$. Obviously, $b = 0$ is impossible, hence we may assume $b = 1$, i.e., $f \in \mathcal{Z}(z_0, R)$ is a solution of Abel's equation

$$R(\Phi(z)) = \Phi(z + 1). \tag{21}$$

This, however is impossible, since any non-constant solution of (21) has infinite order of growth (see Yanagihara [18]). This example shows how sensitive the limit in (1) may depend on (z_n): *replacing z_n by $R(z_n) = z_n + O(|z_n - z_0|^{s+1})$, while ρ_n remains unaltered, leads to the limit function $R \circ f$.*

Remark It is easily seen from $T(r, R \circ \Phi) = dT(r, \Phi) + O(1)$ (Valiron's Lemma), and $T(r + 1, \Phi) \geq (1 + o(1))T(r, \Phi(z + 1))$ (see, e.g., Jank-Volkmann [6]) that, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ with

$$T(r, \Phi) \geq \delta(d - \epsilon)^r, \quad r \geq r_0(\epsilon).$$

As was pointed out in [18], no upper estimate for $T(r, \Phi)$ is possible. To see this let p be any non-constant 1-periodic entire function. Then, obviously, $\Psi(z) = \Phi(z + p(z))$ also is a solution of (21), and thus there are solutions having arbitrarily large Nevanlinna characteristic.

Blaschke products. Let R be a (finite) Blaschke product; its Julia set is contained in (generically: coincides with) the unit circle $\mathbb{T}$. Suppose $f \in \mathcal{Z}(R)$ is given by (1). Then every disk $|z - z_{n_k}| < r|\rho_{n_k}|$, r and n_k large, intersects the unit circle $\mathbb{T}$, and hence there exists a straight line L dividing the plane into two half-planes, which are mapped by f into the interior and exterior of $\mathbb{T}$, respectively. Replacing f by an appropriate translate we thus may assume that $f(\overline{z})\overline{f(z)} = 1$ and $|f(z)| < 1$ in $\text{Im } z > 0$. Then from $f^{\#}(\overline{z}) = f^{\#}(z) \leq |f'(z)| \leq 1/\text{Im } z$ in $\text{Im } z > 0$ we easily obtain $T(r, f) = O(r)$, and hence $\sum_n |c_n|^{-1-\epsilon} < +\infty$, $\epsilon > 0$ arbitrary, where (c_n) denotes the sequence of zeros of f. By analogy with infinite Blaschke products in the unit disk we may deduce that

$$f(z) = e^{icz} \prod_n \frac{|c_n|^2 - \overline{c}_n z}{|c_n|^2 - c_n z} \tag{22}$$

with $c \geq 0$, $\text{Im } c_n > 0$, $\sum_n \text{Im } (1/\overline{c}_n) < +\infty$ and $\sum_n |c_n|^{-1-\epsilon} < +\infty$. Now suppose in addition that $R(z) \not\equiv z^d$ is real and has non-negative zeros a_ν. It is then obvious that

$$|R'(z)| < R'(1) = \lambda = \sum_\nu (1 + a_\nu)/(1 - a_\nu,), \quad \lambda > \deg R,$$

in the unit disk, and from $(R^n)^{\#}(1/\overline{z}) \leq 2|z|^2 (R^n)^{\#}(z)$, $z \in \mathbb{D}$, it follows that the Poincaré function

$$\Phi(z) = \lim_{n \to \infty} R^n(1 + \lambda^{-n} z)$$

for the repelling fix-point $z = 1$ belongs to $\mathcal{Z}_1(R, 1)$; it has order of growth $\log \deg R / \log \lambda < 1$. In particular we have $\sum_n |c_n|^{-1} < +\infty$.

Conversely, suppose that f is given by (22). Then $f^{\#}$ is bounded in $\mathbb{C}$ if and only if f' is bounded in $\text{Im } z > 0$. From

$$\frac{f'(z)}{f(z)} = ic + \sum_n \frac{2i \text{ Im } c_n}{(c_n - z)(\overline{c}_n - z)}$$

and $|f(z)| = |(c_n - z)/(\overline{c}_n - z)| Q_n(z)$, $Q_n(z) \leq 1$, now follows that

$$|f'(x + iy)| \leq c + \sum_n \frac{2 \text{ Im } c_n}{|x + iy - \overline{c}_n|^2} \leq c + \sum_n \frac{2 \text{ Im } c_n}{|x - c_n|^2} = |f'(x)|$$

for $x \in \mathbb{R}$ and $y > 0$. Thus f has bounded spherical derivative if and only if f' is bounded on $\mathbb{R}$.

5 Open questions and concluding remarks

Several problems have been left open. Here are some of them:

- When are Poincaré functions of R also Zalcman functions? Is this characteristic for rational functions with parabolic orbifolds?

- In which cases are Zalcman functions periodic, or solutions of algebraic differential equations (with the same additional question)?

- When is $\mathcal{Z}_1(z_0, R)$ non-empty?

- If R has a Cremer-point, can one gain information about the dynamics of (R^n) near that point from the corresponding Zalcman class?

- More general, what can be said about the dynamics if one has information about the Zalcman class(es)?

- Conversely, how strong is the influence of the dynamics of R^n on the classes $\mathcal{Z}(z_0, R)$? Of course, R determines $\mathcal{Z}$, but this could happen in a non-dynamical way.

- In which non-hyperbolic case is it true that $\mathcal{Z}(R) = \mathcal{Z}(z_0, R)$ for all $z_0 \in \mathcal{J}_R$. Or, to be less rigid, can one determine the set $\{z_0 : \mathcal{Z}(R) = \mathcal{Z}(z_0, R)\}$ in certain non-hyperbolic cases? The answer is "yes" if R has a parabolic orbifold, and possibly also "yes" in the more general sub-hyperbolic case.

- Suppose $f \in \mathcal{Z}(R)$, so that $f = R^n \circ f_n$, $f_n \in \mathcal{Z}(R)$. Does this imply $f(z) = \lim_{n\to\infty} R^n(z_n + \rho_n z)$ (the whole sequence is involved) for suitably chosen sequences (z_n), (ρ_n)?

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