# Sub-hyperbolic Rational Maps And Algebraic Differential Equations

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*Dedicated to the memory of Professor Chi-Tai Chuang*

**Abstract** We give a new existence proof for a singular metric on a marked planar domain via first-order algebraic differential equations. This singular metric applies in complex dynamics to sub-hyperbolic rational functions.

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**§ 1—Sub-hyperbolic rational maps**—Let $R$ be a rational function on the Riemann sphere with $\deg R > 1$. Then $R$ is called *hyperbolic* if there exists some smooth metric $\varrho(w)|dw|$ on a neighbourhood of the Julia set $\mathcal{J} = \mathcal{J}_R$, such that $R$ is expanding on $\mathcal{J}$ with respect to that metric,

$$\frac{\varrho(R(z))|R'(z)|}{\varrho(z)} > 1 \text{ on } \mathcal{J}. \tag{1}$$

If $\varrho$ is allowed to have finitely many singularities $a_1, a_2, \ldots, a_p$ such that

$$\varrho(w) = O(|w - a_j|^{-\gamma_j}) \quad \text{as} \quad w \to a_j \;, \tag{2}$$

for some $\gamma_j < 1$, then $R$ is called *sub-hyperbolic.* It is well-known that sub-hyperbolicity is equivalent with the following: $R$ has no parabolic orbits, and each critical point on the Julia set $\mathcal{J}$ is eventually periodic, see [2], p. 92. The crucial part in the proof is to show that the latter condition implies (1) and

(2). This can be achieved by proving the following

**Theorem** *Let $D \subset \mathbb{C}$ be a domain and let $\nu : D \longrightarrow \mathbb{N}$ be a map such that $\nu(a_j) = l_j > 1$, $1 \leq j \leq p$, and $\nu(w) = 1$ elsewhere. Then, with the exceptions $D = \mathbb{C}$, $p = 2$, $l_1 = l_2 = 2$ and $D = \mathbb{C}$, $p = 1$, there exists a branched covering map $\Phi : \mathbb{D} \longrightarrow D$ such that the local degree of $\Phi$ satisfies*

$$\deg_z \Phi = \nu(\Phi(z)) \ . \tag{3}$$

Then the Poincaré metric of the unit disk will be carried over to $D$ in the same manner as in the hyperbolic (non-branched) case:

$$\varrho(w)|dw| = \frac{2|dz|}{1 - |z|^2}, \quad w = \Phi(z) \, .$$

By uniqueness of $\Phi$ (if $\Phi_1$ is any other branched covering map, then $\Phi_1 = \Phi \circ T$ with some Möbiustransform $T : \mathbb{D} \longrightarrow \mathbb{D}$, a so-called decktransform) it is easy to see that $\varrho$ is well-defined, it satisfies (2) with $\gamma_j = 1 - 1/l_j$.
Now suppose that $\Phi : \mathbb{D} \to D$ is some branched covering map satisfying (3). Then, if we set

$$m = \mathrm{lcm}\,\{l_1, \ldots, l_p\} \quad \text{and} \quad m_j = m(1 - 1/l_j),$$

it is easily seen that

$$q(z) = \frac{(\Phi'(z))^m}{\prod\limits_{j=1}^{p} (\Phi(z) - a_j)^{m_j}}$$

is a zero-free holomorphic function in the unit disk $\mathbb{D}$, and hence $\Phi$ is a solution of the differential equation

$$\Phi'^m = q(z) \prod_{j=1}^{p} (\Phi - a_j)^{m_j} \ . \tag{4}$$

§ 2—**Algebraic differential equations**—The problem of determining $\Phi$ is divided into two steps. First we consider the universal algebraic differential equation

$$w'^m = P(w) := \prod_{j=1}^{p} (w - a_j)^{m_j} \tag{5}$$

in $\mathbb{C}$, and then take into account the domain $D$ to determine the coefficient $q(z)$.

**Local solutions** *For every* $a \neq a_1, \ldots, a_p$, *the initial value problem (5),* $w(z_0) = a$, *has exactly* $m$ *distinct local solutions* $w, w_1, \ldots, w_{m-1}$ *satisfying* $w_j(z) = w(z_0 + \varepsilon_j(z - z_0))$ *with* $\varepsilon_j = e^{2\pi j/m}$ .

**Proof** Take any analytic $m$-th root of $w \mapsto P(w)$ in some neighbourhood of $w = a$. Then Picard's Existence and Uniqueness Theorem yields a unique analytic solution $z \mapsto w(z)$ in some neighbourhood of $z = z_0$ . The second statement is obvious, just note that $w_j'(z_0) = \varepsilon_j w'(z_0)$.

**Analytic continuation** *Each local solution admits unrestricted analytic continuation to the whole plane* $\mathbb{C}$ , *except for (algebraic) poles.*

**Proof** By Painlevé's theorem, see Bieberbach [1], p. 10, every local solution of (5) admits unrestricted analytic continuation except for algebraic singularities (and, of course, poles). If some solution $w$ is continued along an arc $t \mapsto z_t$, $0 \leq t < 1$ and $z_1$ is an algebraic singularity (not a pole), then $\lim\limits_{t \to 1-} w(z_t) = a_j$ for some $j$. Substituting $w - a_j = y^{l_j}$, we obtain the algebraic differential equation

$$y'^m = l_j^{-m} \prod_{k \neq j} (y^{l_j} + a_j - a_k)^{m_k}, \tag{6}$$

for which $y = 0$ is a regular point. It is obvious that, for some appropriate local solution $z \mapsto y(z)$ of the initial value problem (6), $y(z_1) = 0$, the map $z \mapsto a_j + y^{l_j}(z)$ provides the analytic continuation of $w$ into a neighbourhood of $z_1$ . Note that $z_1$ is a zero of $w(z) - a_j$ of order $l_j$, since $y'(z_1) \neq 0$.

**Inverse analytic continuation** *Let* $w$ *be any non-constant local solutions of (5),* $w(z_0) \neq a_j$. *Then* $w$ *has a local inverse*

$$Z = Z(w) = Z_0 + \int_{w(z_0)}^{w} P^{-1/m}(\xi) d\xi \ ,$$

*which obviously admits unrestricted analytic continuation to* $\mathbb{C} \setminus \{a_1, \ldots, a_p\}$ . *At* $w = a_j$, $w \mapsto Z$ *has an algebraic singularity.*

**Proof** Obviously all one has to do is to continue some branch of $P^{1/m}$ analytically along a given arc $\gamma$ starting at $w(z_0)$ and avoiding the set $\{a_1, \ldots, a_p\}$. At $w = a_j$, $w \mapsto Z$ has an algebraic singularity of type

$$Z(w) \sim z_1 + \ const \cdot (w - a_j)^{-1/l_j} \ .$$

**Poles** *Suppose* $\mu = \sum\limits_{j=1}^{p} m_j > m$. *Then every non-constant solution of (5) has infinitely many (possibly algebraic) poles. For* $\mu = m$ *every solution is a transcendental entire function, while, for* $\mu < m$, *every solution is algebraic.*

**Proof** Let $w$ be any non-constant solution of (5) with $w(0) = a \in \mathbb{C}$, and assume $\sum\limits_{j=1}^{p} m_j > m$. Then $w$ cannot be entire (see, e.g., Wittich [4]) and so has a pole $z_0 = z_0(a)$ of smallest modulus $R(a)$; note that $R(a)$ does not depend on the choice of $\arg w'(0)$ . It is obvious that $a \mapsto R(a)$ is continuous, this following from analytic dependence, and, for $|a|$ sufficiently large we have

$$R(a) \le |a|^{1-\mu/m} \int_1^\infty \prod_{j=1}^{p} \left(t - \left|\frac{a_j}{a}\right|\right)^{-1+\frac{1}{l_j}} dt,$$

which tends to 0 as $a \to \infty$ . Thus $R(a)$ has a maximum $\mathsf{P}$, and so, given $z_0$ and $w(z_0) = w_0$, every non-constant solution of (5) has a (algebraic) pole on $|z - z_0| \le \mathsf{P}$. This shows that there are infinitely many poles.
The inequality $\sum\limits_{j=1}^{p} m_j \le m$ is equivalent with $\sum\limits_{j=1}^{p} 1/l_j \ge p-1$ , and since each $l_j$ is at least two, this implies $p \le 2$ . In case $p = 2$ we have $w'^2 = (w-a_1)(w-a_2)$. This is essentially the cosine-equation, while in case $p = 1$ we have $w'^m = w^{m-q}$, $m/q = l_1$. The solutions of this equation are $w = \left(l_1^{-1}z + c\right)^{l_1}$ .

**Remark** In the same way it can be shown that every non-constant solution assumes every value $\neq a_j$ infinitely often, provided $\mu > m$ , thus $w(z_1) = w(z_2) = \ldots = a$ . It is obvious that there exist indices $j, k$ with $w'(z_j) = w'(z_k)$, and so $w(z) = w(z + z_j - z_k)$ by uniqueness: (analytic continuation of) any non-constant solution is periodic.

**§ 3—Branched uniformization**—We assume in the sequel $\sum\limits_{j=1}^{p} 1/l_j < p - 1$. Let $D \subseteq \mathbb{C}$ be any domain with marked points $a_1, \ldots, a_p$ . Suppose that $a$ is any point in $D \setminus \{a_1, \ldots, a_p\}$ and that $z \mapsto w(z)$ is any local solution of (5), $w(0) = a$ . Then we obtain a domain $H$, which consists of all values obtained by analytic continuation of the local inverse $w \mapsto Z$, $a \mapsto 0$, in $D \setminus \{a_1, \ldots, a_p\}$ , together with all limits of $Z(w)$ as $w$ tends to some $a_j$, $1 \le j \le p$. The domain $H$ is hyperbolic, since it does not contain any of the infinitely many poles of $w$. Hence there exists a unique universal covering map $\psi : \mathbb{D} \longrightarrow H$, normalized by $\psi(0) = 0$ and $\arg \psi'(0) = -\arg w'(0)$ . Then $\Phi = w \circ \psi$ is locally defined at $z = 0$ and admits unrestricted analytic continuation in $\mathbb{D}$. Hence $\Phi$ is analytic in $\mathbb{D}$ by the Monodromy Theorem, and is indeed a branched covering map $\mathbb{D} \longrightarrow D$ satisfying (3). Finally, the Schwarz Lemma shows that $\Phi$ is uniquely determined by the condition $\Phi(0) = a$, $\Phi'(0) > 0$ .

§ 4—**Summary**—By our method the problem of constructing a branched covering map $\Phi : \mathbb{D} \to D$, and so of constructing a singular metric $\varrho(w)|dw|$ is broken into two parts: $\Phi$ is a composition of a (non-branched) universal covering map $\psi : \mathbb{D} \longrightarrow H$ and a branched covering map $H \to D$, defined by (continuation of) a solution of (5). This branched map is universal insofar it does not take into account the special shape of the domain $D$, but only the map $\nu : D \longrightarrow \mathbb{N}$. On the other hand, $\psi$ takes into account the domain $H$ only. And so is the construction of $\varrho$: Since the Poincaré density of $H$ at $\zeta = \psi(z)$ is given by $\varrho_H(\zeta)|\psi'(z)| = 2(1-|z|^2)^{-1}$, and since $|\Phi'(z)| = |\psi'(z)| \prod_{j=1}^{p} |w-a_j|^{1-1/l_j}$ , we obtain

$$\varrho(w) = \frac{\delta(w)}{\prod_{j=1}^{p} |w-a_j|^{1-1/l_j}} ,$$

where $\delta(w) = \varrho_H(\psi(z))$ is a well-defined and smooth function of $w$.

**Example**—$R(w) = w^2 + i$ . The postcritical orbit is $\{ i, -1+i, -i\}$ with $l_1 = l_2 = l_3 = 2$ . The differential equation (5) is given by

$$w'^2 = (w^2+1)(w+1-i)$$

and has solutions $w(z) = \wp(z/2+c) - (1+i)/3$. The domain $D$ is a large disk $|z| < R$, $H$ is a component of $w^{-1}(D)$, and with $\psi$ a universal covering map $\mathbb{D} \longrightarrow H$ and $\delta(w) = \varrho_H(\psi(z))$ we have $\varrho(w) = \delta(w)|(w^2+1)(w+1-i)|^{-1/2}$. In [2] the equivalent density $\varrho^*(w) = |(w^2+1)(w+1-i)|^{-1/2}$ —a good guess—is constructed 'by hand'.

## References

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[4] H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen,* Springer-Verlag 1965.