Sub-hyperbolic Rational Maps
And Algebraic Differential Equations

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Dedicated to the memory of Professor Chi-Tai Chuang

Abstract We give a new existence proof for a singular metric on a marked planar domain via first-order algebraic differential equations. This singular metric applies in complex dynamics to sub-hyperbolic rational functions.

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§ 1—Sub-hyperbolic rational maps—Let $R$ be a rational function on the Riemann sphere with $\deg R > 1$. Then $R$ is called hyperbolic if there exists some smooth metric $\varrho(w)\,|dw|$ on a neighbourhood of the Julia set $\mathcal{J} = \mathcal{J}_R$, such that $R$ is expanding on $\mathcal{J}$ with respect to that metric,

$$\frac{\varrho(R(z))\,|R'(z)|}{\varrho(z)} > 1$$

on $\mathcal{J}$. (1)

If $\varrho$ is allowed to have finitely many singularities $a_1, a_2, \ldots, a_r$ such that

$$\varrho(w) = O(\left| w - a_j \right|^{-\gamma_j}) \quad \text{as} \quad w \to a_j,$$

for some $\gamma_j < 1$, then $R$ is called sub-hyperbolic. It is well-known that sub-hyperbolicity is equivalent with the following: $R$ has no parabolic orbits, and each critical point on the Julia set $\mathcal{J}$ is eventually periodic, see [2], p. 92. The crucial part in the proof is to show that the latter condition implies (1) and
(2). This can be achieved by proving the following

**Theorem** Let \( D \subseteq \mathcal{U} \) be a domain and let \( \nu : D \to \mathbb{N} \) be a map such that \( \nu(a_j) = l_j > 1, 1 \leq j \leq p, \) and \( \nu(w) = 1 \) elsewhere. Then, with the exceptions \( D = \mathcal{U}, \ p = 2, \ l_1 = l_2 = 2 \) and \( D = \mathcal{U}, \ p = 1, \) there exists a branched covering map \( \Phi : \mathbb{D} \to D \) such that the local degree of \( \Phi \) satisfies

\[
\text{deg}_z \Phi = \nu(\Phi(z)) .
\]

Then the Poincaré metric of the unit disk will be carried over to \( D \) in the same manner as in the hyperbolic (non-branched) case:

\[
g(w)|dw| = \frac{2|dz|}{1 - |z|^2}, \quad w = \Phi(z) .
\]

By uniqueness of \( \Phi \) (if \( \Phi_1 \) is any other branched covering map, then \( \Phi_1 = \Phi \circ T \) with some Möbius transform \( T : \mathbb{D} \to \mathbb{D} \), a so-called deck transform) it is easy to see that \( g \) is well-defined, it satisfies (2) with \( \gamma_j = 1 - 1/l_j \).

Now suppose that \( \Phi : \mathbb{D} \to D \) is some branched covering map satisfying (3). Then, if we set

\[
m = \text{lcm}\{l_1, \ldots, l_p\} \quad \text{and} \quad m_j = m(1 - 1/l_j),
\]

it is easily seen that

\[
q(z) = \frac{(\Phi'(z))^m}{\prod_{j=1}^{p} (\Phi(z) - a_j)^{m_j}}
\]

is a zero-free holomorphic function in the unit disk \( \mathbb{D} \), and hence \( \Phi \) is a solution of the differential equation

\[
\Phi'^m = q(z) \prod_{j=1}^{p} (\Phi - a_j)^{m_j} .
\]

**§ 2—Algebraic differential equations**—The problem of determining \( \Phi \) is divided into two steps. First we consider the universal algebraic differential equation

\[
w'^m = P(w) := \prod_{j=1}^{p} (w - a_j)^{m_j}
\]

in \( \mathcal{U} \), and then take into account the domain \( D \) to determine the coefficient \( q(z) \).

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Local solutions For every \( a \neq a_1, \ldots, a_p \), the initial value problem (5), \( w(z_0) = a \), has exactly \( m \) distinct local solutions \( w, w_1, \ldots, w_{m-1} \) satisfying \( w_j(z) = w(z_0 + \varepsilon_j(z - z_0)) \) with \( \varepsilon_j = e^{2\pi j/m} \).

Proof Take any analytic \( m \)-th root of \( w \mapsto P(w) \) in some neighbourhood of \( w = a \). Then Picard’s Existence and Uniqueness Theorem yields a unique analytic solution \( z \mapsto w(z) \) in some neighbourhood of \( z = z_0 \). The second statement is obvious, just note that \( w_j'(z_0) = \varepsilon_j w'(z_0) \).

Analytic continuation Each local solution admits unrestricted analytic continuation to the whole plane \( \mathbb{C} \), except for (algebraic) poles.

Proof By Painlevé’s theorem, see Bieberbach [1], p. 10, every local solution of (5) admits unrestricted analytic continuation except for algebraic singularities (and, of course, poles). If some solution \( w \) is continued along an arc \( t \mapsto z_t, 0 \leq t < 1 \) and \( z_1 \) is an algebraic singularity (not a pole), then \( \lim_{t \to 1^-} w(z_t) = a_j \) for some \( j \). Substituting \( w - a_j = y^j \), we obtain the algebraic differential equation

\[
y'^m = l_j^{-m} \prod_{k \neq j} (y^j + a_j - a_k)^{m_k},
\]

for which \( y = 0 \) is a regular point. It is obvious that, for some appropriate local solution \( z \mapsto y(z) \) of the initial value problem (6), \( y(z_1) = 0 \), the map \( z \mapsto a_j + y^j(z) \) provides the analytic continuation of \( w \) into a neighbourhood of \( z_1 \). Note that \( z_1 \) is a zero of \( w(z) - a_j \) of order \( l_j \), since \( y(z_1) \neq 0 \).

Inverse analytic continuation Let \( w \) be any non-constant local solutions of (5), \( w(z_0) \neq a_j \). Then \( w \) has a local inverse

\[
Z = Z(w) = Z_0 + \int_{w(z_0)}^w P^{-1/m}(\xi)d\xi,
\]

which obviously admits unrestricted analytic continuation to \( \mathbb{C} \setminus \{a_1, \ldots, a_p\} \).

At \( w = a_j \), \( w \mapsto Z \) has an algebraic singularity.

Proof Obviously all one has to do is to continue some branch of \( P^{1/m} \) analytically along a given arc \( \gamma \) starting at \( w(z_0) \) and avoiding the set \( \{a_1, \ldots, a_p\} \). At \( w = a_j \), \( w \mapsto Z \) has an algebraic singularity of type

\[
Z(w) \sim z_1 + \text{const} \cdot (w - a_j)^{-1/l_j}.
\]
Poles Suppose \( \mu = \sum_{j=1}^{p} m_j > m \). Then every non-constant solution of (5) has infinitely many (possibly algebraic) poles. For \( \mu = m \) every solution is a transcendental entire function, while, for \( \mu < m \), every solution is algebraic.

Proof Let \( w \) be any non-constant solution of (5) with \( w(0) = a \in \Phi \), and assume \( \sum_{j=1}^{p} m_j > m \). Then \( w \) cannot be entire (see, e.g., Wittich [4]) and so has a pole \( z_0 = z_0(a) \) of smallest modulus \( R(a) \); note that \( R(a) \) does not depend on the choice of \( \arg w'(0) \). It is obvious that \( a \mapsto R(a) \) is continuous, this following from analytic dependence, and, for \( |a| \) sufficiently large we have
\[
R(a) \leq |a|^{-\mu/m} \int_{1}^{\infty} \prod_{j=1}^{p} \left( t - \left| \frac{a_j}{a} \right| \right)^{-1+\frac{1}{q}} dt,
\]
which tends to 0 as \( a \to \infty \). Thus \( R(a) \) has a maximum \( P \), and so, given \( z_0 \) and \( w(z_0) = w_0 \), every non-constant solution of (5) has a (algebraic) pole on \( |z - z_0| \leq P \). This shows that there are infinitely many poles.

The inequality \( \sum_{j=1}^{p} m_j \leq m \) is equivalent with \( \sum_{j=1}^{p} 1/l_j \geq p - 1 \), and since each \( l_j \) is at least two, this implies \( p \leq 2 \). In case \( p = 2 \) we have \( w'^2 = (w-a_1)(w-a_2) \), this is essentially the cosine-equation, while in case \( p = 1 \) we have \( w'^m = w^{m-q} \), \( m/q = l_1 \). The solutions of this equation are \( w = \left( l_1^{-1} z + c \right)^{l_1} \).

Remark In the same way it can be shown that every non-constant solution assumes every value \( \neq a_j \) infinitely often, provided \( \mu > m \), thus \( w(z_1) = w(z_2) = \ldots = a \). It is obvious that there exist indices \( j, k \) with \( w'(z_j) = w'(z_k) \), and so \( w(z) = w(z + z_j - z_k) \) by uniqueness: (analytic continuation of) any non-constant solution is periodic.

§ 3—Branched uniformization—We assume in the sequel \( \sum_{j=1}^{p} 1/l_j < p - 1 \). Let \( D \subseteq \Phi \) be any domain with marked points \( a_1, \ldots, a_p \). Suppose that \( a \) is any point in \( D \setminus \{ a_1, \ldots, a_p \} \) and that \( z \mapsto w(z) \) is any local solution of (5), \( w(0) = a \). Then we obtain a domain \( H \), which consists of all values obtained by analytic continuation of the local inverse \( w \mapsto Z \), \( a \mapsto 0 \), in \( D \setminus \{ a_1, \ldots, a_p \} \), together with all limits of \( Z(w) \) as \( w \) tends to some \( a_j \), \( 1 \leq j \leq p \). The domain \( H \) is hyperbolic, since it does not contain any of the infinitely many poles of \( w \).

Hence there exists a unique universal covering map \( \psi : \mathbb{D} \to H \), normalized by \( \psi(0) = 0 \) and \( \arg \psi'(0) = -\arg w'(0) \). Then \( \Phi = w \circ \psi \) is locally defined at \( z = 0 \) and admits unrestricted analytic continuation in \( \mathbb{D} \). Hence \( \Phi \) is analytic.
in $\mathbb{ID}$ by the Monodromy Theorem, and is indeed a branched covering map $\mathbb{ID} \longrightarrow D$ satisfying (3). Finally, the Schwarz Lemma shows that $\Phi$ is uniquely determined by the condition $\Phi(0) = a$, $\Phi'(0) > 0$.

§ 4—Summary—By our method the problem of constructing a branched covering map $\Phi : \mathbb{ID} \rightarrow D$, and so of constructing a singular metric $\varrho(w) |dw|$ is broken into two parts: $\Phi$ is a composition of a (non-branched) universal covering map $\psi : \mathbb{ID} \longrightarrow H$ and a branched covering map $H \rightarrow D$, defined by (continuation of) a solution of (5). This branched map is universal insofar it does not take into account the special shape of the domain $D$, but only the map $\nu : D \longrightarrow \mathbb{N}$. On the other hand, $\psi$ takes into account the domain $H$ only. And so is the construction of $\varrho$: Since the Poincaré density of $H$ at $\zeta = \psi(z)$ is given by $\varrho_H(\zeta) |\psi'(z)| = 2(1 - |z|^p)^{-1}$, and since $|\Phi'(z)| = |\psi'(z)| \prod_{j=1}^{p} |w - a_j|^{-\frac{1}{\beta_j}}$, we obtain

$$\varrho(w) = \frac{\delta(w)}{\prod_{j=1}^{p} |w - a_j|^{-\frac{1}{\beta_j}}} ,$$

where $\delta(w) = \varrho_H(\psi(z))$ is a well-defined and smooth function of $w$.

Example—$R(w) = w^2 + i$. The postcritical orbit is $\{ i, -1 + i, -i \}$ with $l_1 = l_2 = l_3 = 2$. The differential equation (5) is given by

$$w^2 = (w^2 + 1)(w + 1 - i)$$

and has solutions $w(z) = \varphi(z/2 + e) - (1 + i)/3$. The domain $D$ is a large disk $|z| < R$, $H$ is a component of $w^{-1}(D)$, and with $\psi$ a universal covering map $\mathbb{ID} \longrightarrow H$ and $\delta(w) = \varrho_H(\psi(z))$ we have $\varrho(w) = \delta(w)\left(|w^2 + 1|(w + 1 - i)\right)^{-\frac{1}{2}}$. In [2] the equivalent density $\varrho^*(w) = \left(|w^2 + 1|(w + 1 - i)\right)^{-\frac{1}{2}}$—a good guess—is constructed 'by hand'.

References