ON THE ORDER AND LOWER ORDER OF ENTIRE FUNCTIONS WITH RADially DISTRIBUTED ZEROS

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ABSTRACT. It is shown that the order and lower order of an entire function with zeros restricted to k distinct rays differ at most by k, if either k ≤ 2 or if the zeros or the rays are regularly distributed.

1. Introduction. Throughout this paper f will denote an entire function whose zeros are restricted to k distinct rays \( \arg z = \omega_j \) \((0 \leq \omega_1 < \omega_2 < \cdots < \omega_k < 2\pi)\). The order and lower order of \( f \) are defined to be

\[
\lambda(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \\
\mu(f) = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
\]

By \( N(r) \) and \( N_j(r) \) we denote the integrated counting functions of all zeros of \( f \) and of those on \( \arg z = \omega_j \), respectively.

It was proved by Edrei and Fuchs [2] (see also [1,3,4,6]) that the order and the lower order are cofinite. But, in general, no explicit upper bound for \( \lambda(f) \) is known. Edrei and Fuchs [2] constructed an entire function \( F \) with real negative zeros of prescribed order \( \lambda \) and lower order \( \mu \), subject only to \( 0 < \mu \leq \lambda < 1 \). Therefore, the least upper bound for \( \lambda(f) \) in terms of \( \mu(f) \) and \( k \), if there is any, is at least \( [\mu(f)] + k \), where \([\cdot]\) denotes the greatest integer function (consider \( f(z) = F(z^k) \), \( k \) a positive integer). The inequality

\[
\lambda(f) \leq [\mu(f)] + k
\]

is known to be true for \( k = 1 \) (see [1,3,6]). It will be shown that (1) is valid for \( k = 2 \), too, and also, for general \( k \), if the rays \( \arg z = \omega_j \) or the zeros of \( f \) are regularly distributed in some sense.

2. Statement of results. There is no loss of generality in assuming \( f(0) = 1 \). If \( f \) has finite lower order, let \( q \) be the smallest integer such that

\[
\liminf_{r \to \infty} \frac{T(r, f)}{r^q} < \infty.
\]

Clearly, \( q \leq \mu(f) < q + 1 \).
All our results will be derived from the following

**Theorem.** Under the hypotheses stated above there exists a sequence \( r_n \uparrow \infty \) such that, for any integer \( p \), \( p > q \),

\[
(2) \quad \sum_{j=1}^{k} e^{ip\omega} \int_{0}^{r_n} \frac{N_j(t)}{t^{p+1}} dt = O(1)
\]

as \( n \to \infty \).

**Corollary 1.** \( \mu(f) < \infty \) implies \( \lambda(f) < \infty \).

**Corollary 2.** \( k = 1 \) and \( \mu(f) < \infty \) imply \( \lambda(f) \leq q + 1 \leq [\mu(f)] + 1 \).

**Corollary 3.** \( k = 2 \) and \( \mu(f) < \infty \) imply either \( \lambda(f) \leq q + 1 \) or \( q + 1 < \lambda(f) \leq q + 2 \leq [\mu(f)] + 2 \). In this case, \( \omega_2 - \omega_1 = (2s + 1)\pi/(q + 1) \) (\( s, 0 \leq s \leq q \), an integer).

**Remark.** In [1] Abi-Khuzam proved for \( k = 2 \), \( \omega_1 = 0 \), \( \omega_2 = m\pi/\alpha \) (\( m \) and \( \alpha \) relative prime positive integers): \( \lambda(f) \leq [\mu(f)] + \alpha \) if \( m \) is even, and \( \lambda(f) \leq [\mu(f)] + 2\alpha \) if \( m \) is odd. As Corollary 3 shows, this is only sharp if \( \alpha = m = 1 \), i.e. if \( f \) has only real zeros (see also Volkmann [6]).

**Corollary 4.** If \( \omega_j - \omega_{j-1} = 2\pi/k \) (\( 1 \leq j \leq k; \omega_0 = \omega_k - 2\pi \)), then \( \mu(f) < \infty \) implies \( \lambda(f) \leq q + l \leq [\mu(f)] + k \), where \( l, 1 \leq l \leq k \), is the smallest integer satisfying \( q + l = ks \) (\( s \) an integer).

**Corollary 5.** If the limits \( \alpha_j := \lim_{r \to \infty} N_j(r)/N(r) \) exist, then \( \lambda(f) \leq q + k \leq [\mu(f)] + k \).

**Remark.** Given \( 0 < \alpha \leq \beta < 1 \), Edrei and Fuchs [2] constructed an entire function \( F \) of lower order \( \alpha \) and order \( \beta \), having only negative zeros. Thus, \( f(z) = F(-z^k) \) has lower order \( \mu = k\alpha \) and order \( \lambda = k\beta \), which may be chosen arbitrarily near to \( [\mu] + k \). The zeros of \( f \) are regularly distributed on \( k \) distinct rays \( \arg z = 2\pi j/k \) (\( 0 \leq j < k \)), and so this example does not only prove that inequality (1) is sharp for every \( k \) (if it is true in general), but also the sharpness of the statements of Corollaries 2-5.

**3. Proof of the Theorem.** Let \( a_1, a_2, \ldots \) be the zeros of \( f \) repeated according to multiplicity and assume \( \log f(z) = c_1z + c_2z^2 + \cdots \) near \( z = 0 \). Then for any integer \( p = 0, 1, 2, \ldots \)

\[
(3) \quad \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| e^{-ip\theta} d\theta = \frac{c_p}{2} r^p + \frac{1}{2p} \sum_{|a_r| < r} \left( \left( \frac{r}{a_r} \right)^p - \left( \frac{\bar{a_r}}{r} \right)^p \right)
\]

(see F. Nevanlinna [5]). Clearly,

\[
(4) \quad \left| \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| e^{-ip\theta} d\theta \right| \leq 2T(r, f)
\]
and

\[(5) \quad \sum_{|\omega_j| \leq \rho} \left( \frac{\alpha_j}{\rho} \right)^p \leq n(\rho) \leq T(\rho) .\]

With obvious notation,

\[(6) \quad \sum_{|\omega_j| \leq \rho} a_j \cdot e^{-ip\omega_j} = \frac{e^{-ip\omega_j}}{\rho} \int_0^\rho \frac{dn_j(t)}{t^p}.
\]

Integrating by parts twice we get

\[ \int_0^\rho \frac{dn_j(t)}{t^p} = \frac{n_j(r)}{r^p} + p \frac{N_j(r)}{r^p} + p^2 \int_0^r \frac{N_j(t)}{t^{p+1}} dt, \]

which yields together with (3)–(6), after passing to conjugate values,

\[ \sum_{j=1}^k c_j e^{ip\omega_j} \int_0^\rho \frac{N_j(t)}{t^{p+1}} dt = O \left( \frac{T(\rho)}{\rho^p} + 1 \right) \quad \text{as } r \to \infty. \]

To complete the proof, we have only to choose \( r_n \to \infty \) such that \( T(\rho_n, f) \) remains bounded as \( n \to \infty \).

4. Proof of the corollaries. In all cases we may replace \( \lambda(f) \) by \( \lambda(0, f) := \limsup_{r \to \infty} \frac{\log N(r)}{\log r} \), since \( \lambda(0, f) < \lambda(f) \) implies \( \lambda(f) = \mu(f) \). This is easily seen from \( f = Pe^g \), where \( P \) is a canonical product of order \( \lambda(0, f) < \lambda(f) \) (or a polynomial). Since \( e^g \) is of regular growth, the same is true for \( f \). The method gives slightly more. Instead of inequalities of type \( \lambda(0, f) \leq b = q + 1, q + 2, q + l \) and \( q + k \), respectively) we will prove that \( \int_0^\infty \frac{N(t)}{t^{p+1}} dt \) converges (implying \( N(r)/r^h \to 0 \) as \( r \to \infty \)).

**Proof of Corollary 1.** Assume that \( \int_0^\infty \frac{N(t)}{t^{p+1}} dt \) diverges for some \( p > q \).

We divide equation (2) by \( \int_0^\infty \frac{N(t)}{t^{p+1}} dt \) and choose a subsequence of \( r_n \) (still denoted by \( r_n \)) such that the limits

\[ \lim_{n \to \infty} \int_0^{r_n} \frac{N_i(t)}{t^{p+1}} dt / \int_0^{r_n} \frac{N(t)}{t^{p+1}} dt =: \alpha_j \]

exist. Then we get \( \Sigma_{j=1}^k \alpha_j e^{ip\omega_j} = 0 \) \( (\alpha_j \geq 0, \Sigma_{j=1}^k \alpha_j = 1) \), which shows that the origin belongs to the convex hull of \( \{ e^{ip\omega_j}; 1 \leq j \leq k \} \) for any such \( p \). But this is not possible for every \( p \), since by Weyl's equidistribution theorem [7] there exist arbitrarily large \( p \) such that all \( e^{ip\omega_j}, 1 \leq j \leq k \), belong to an open halfplane. This proves Corollary 1.

**Proof of Corollary 2.** We mention only that (2) implies the boundedness of \( \int_0^\infty \frac{N(t)}{t^{q+2}} dt \) \( (N_i(t) = N(t)) \) for \( r = r_n \) and so, by monotonicity, for any \( r \geq 0 \).

**Proof of Corollary 3.** If \( \int_0^\infty \frac{N(t)}{t^{q+2}} dt \), we are done. If not, both integrals \( \int_0^\infty \frac{N_j(t)}{t^{q+2}} dt, j = 1, 2 \), must diverge. Thus, dividing by \( \int_0^\infty \frac{N_j(t)}{t^{q+2}} dt \) and letting \( n \) tend to infinity, (2) gives \( -e^{(q+1)(\omega_2 - \omega_1)} = 1 \) and so \( \omega_2 - \omega_1 = (2s + 1)\pi/(q + 1) \), \( s \) an integer. In the same way, \( \int_0^\infty \frac{N(t)}{t^{q+3}} dt = + \infty \) would imply \( \omega_2 - \omega_1 = (2s + 1)\pi/(q + 2) \), which is impossible.
Proof of Corollary 4. We may assume $\omega_j = 2(j - 1)\pi/k$ $(1 \leq j \leq k)$. If $l, 1 \leq l \leq k$, is chosen in such a way that $q + l = ks$, then for $p = q + l$ we have $e^{i\omega_j} = 1$ and (2) gives $\int_0^T N(t)/t^{q+l+1} \, dt = O(1)$, which proves Corollary 4.

Proof of Corollary 5. We assume that $\int_0^T N(t)/t^{p+1} \, dt$ diverges for any integer $p, q < p \leq q + k$, and will derive a contradiction. By l'Hospital's rule we get

$$\int_0^T N(t)/t^{p+1} \, dt / \int_0^T N(t) \, dt \to \alpha_j \quad \text{as } n \to \infty$$

(even if $\alpha_j$ is zero). Thus, (2) gives

$$\sum_{j=1}^k \alpha_j e^{i\omega_j} = 0 \quad (p = q + 1, \ldots, q + k)$$

which is impossible, since (8) has no nontrivial solution (the determinant $| e^{i\omega_j} | = 1, \ldots, p = q + 1, \ldots, q + k$ does not vanish). This proves Corollary 5.

Remark. Obviously, Corollary 5 holds true under the following weaker hypothesis: For every $p, q < p \leq q + k$, there exists a subsequence of $r_n$ (still denoted by $r_n$) such that

$$\int_0^{r_n} N_j(t) \, dt / \int_0^{r_n} N(t) \, dt$$

tends to $\alpha_j$ (independent of $p$) as $n \to \infty$.

References


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