ON THE FUNCTIONAL EQUATION $\psi(x) = \psi(px) + \psi(qx+p)$

Norbert Steinmetz

Presented by F.V. Atkinson, F.R.S.C.

Abstract. Let $p, q > 0$ be fixed real constants with $p+q = 1$. It is shown that any absolutely continuous solution of the functional equation $\psi(x) = \psi(px) + \psi(qx+p)$, $0 \leq x \leq 1$, is linear.

On the other hand, there exist continuous nonlinear solutions.

1. The functional equation

$$\psi(x) = \frac{1}{2} \psi\left(\frac{x}{2}\right) + \frac{1}{2} \psi\left(\frac{x+1}{2}\right)$$

plays some role in deducing the duplication formula of the Gamma function (Artin [1], p.33) and the resolution of the cotangent into partial fractions (Mohr [3], Walter [4]) by real variable methods.

Equation (1) is a special case of

$$\psi(x) = p\psi(px) + q\psi(qx+p), \quad 0 \leq x \leq 1,$$

(with fixed $p, q > 0$, $p+q = 1$) which may be derived from Cauchy's equation

$$f(x+y) = f(x) + f(y), \quad x \text{ and } y \text{ real},$$
as follows: Setting $x = pt-1$, $y = qt+1$, equation (3) changes into the one variable functional equation

$$f(t) = f(pt-1) + f(qt+1).$$

Defining $\psi(x) = f(x-1)$ for $x$ real, this gives

$$\psi(x) = \psi(px) + \psi(qx+p), \quad p, q > 0, \quad p+q = 1,$$
and thus equation (2) by differentiation.

2. One might expect that all continuous solutions of (5) are linear. This is not the case as is shown by the following example. Let \( \phi_0 \) be an arbitrary continuous solution of

\[
\phi_0(px) + \phi_0(qx+p) = 0, \quad 0 \leq x \leq 1,
\]

such that \( \phi_0(0) = \phi_0(p) = 0 \) (\( \phi_0 \) may be prescribed in \( 0 < x < p \)), and let \( \phi_n \) be defined inductively by

\[
\phi_n(x) = \begin{cases} 
\frac{1}{2} \phi_{n-1}(\frac{x}{p}), & 0 \leq x \leq p, \\
\frac{1}{2} \phi_{n-1}(\frac{x-p}{q}), & p < x \leq 1.
\end{cases}
\]

Then

\[
\phi(x) = \sum_{n=0}^{m} \phi_n(x), \quad 0 \leq x \leq 1,
\]

is a continuous solution of (5), which is nonlinear, if, e.g., \( \phi_0(p^2) \neq 0 \). In the case \( p = q = \frac{1}{2} \), a much simpler example is

\[
\phi(x) = \sum_{n=1}^{m} 2^n \sin(2^n x)
\]

(Artin [1], p. 35).

3. We shall prove:

**Theorem.** Any absolutely continuous solution of (5), \( 0 \leq x \leq 1 \), is linear: \( \phi(x) = c(x-p) \).

**Remark.** In the case \( p = q = \frac{1}{2} \), Mohr [3] proved the Theorem for solutions with Riemann-integrable first derivative.
The proof of the Theorem is based on a lemma dealing with equation (2).

Lemma. Let \( \psi_0 \) be Lebesgue-integrable over \( [0,1] \) and let \( (\psi_n) \) be defined inductively by

\[
\psi_n(x) = p\psi_{n-1}(px) + q\psi_{n-1}(qx) + c.
\]

Then,

\[
\int_0^1 |\psi_n(x) - c| \, dx \to 0 \quad \text{as} \quad n \to \infty,
\]

where

\[
c = \int_0^1 \psi_0(x) \, dx,
\]

and even

\[
\psi_n(x) \to c \quad \text{as} \quad n \to \infty,
\]

uniformly in \( 0 \leq x \leq 1 \), if \( \psi_0 \) is continuous.

Remark. In the case \( p = q = \frac{1}{2} \), a somewhat stronger result of Jessen [2] could be used, which states that \( \psi_n(x) = 2^{-n} \sum_{k=0}^{2^n-1} \psi_0\left(\frac{x+k}{2^n}\right) \) tends to \( \int_0^1 \psi_0(t) \, dt \) almost everywhere in \( 0 \leq x \leq 1 \) as \( n \to \infty \).

4. To prove the lemma we will first assume that \( \psi_0 \) is continuous in \( 0 \leq x \leq 1 \). If

\[
\omega_n(h) = \max \{|\psi_n(x) - \psi_n(y)|; \ 0 \leq x, y \leq 1, \ |x-y| \leq h\}
\]
denotes the smallest modulus of continuity of \( \psi_n \), then from (10) easily follows

\[
\omega_n(h) \leq p\omega_{n-1}(ph) + q\omega_{n-1}(qh) + \omega_{n-1}(rh),
\]

where \( r = \max (p, q) < 1 \). Thus, by mathematical induction,
(16) \[ w_n(h) \leq w_0(r^n h) \leq w_0(r^n) , \]
which tends to zero as \( n \to \infty \). Especially,

(17) \[ \psi_n(x) - \psi_n(0) \to 0, \quad \text{uniformly as } n \to \infty, \]
and so, since, by (10), \( c = \int_0^1 \psi_n(x) \, dx \) is independent of \( n \),

(18) \[ \psi_n(x) \to c , \quad \text{uniformly as } n \to \infty. \]

To prove the main part, given \( \varepsilon > 0 \), we choose a continuous function
\( \theta_0 \) in \( 0 \leq x \leq 1 \), such that

(19) \[ \int_0^1 |\psi_n(x) - \theta_0(x)| \, dx < \varepsilon \]
and

(20) \[ \int_0^1 \theta_0(x) \, dx = c , \]
and define the sequence \( (\theta_n) \) in the same way as \( (\psi_n) \). Since

\[ \int_0^1 |\psi_n(x) - \theta_n(x)| \, dx \]
is nonincreasing as \( n \) increases, (19) gives togeth-er with \( \theta_n(x) + c \), uniformly in \( 0 \leq x \leq 1 \),

(21) \[ \int_0^1 |\psi_n(x) - c| \, dx < \varepsilon \]
for sufficiently large \( n \), and the lemma is proved.

5. To prove our Theorem, we assume that \( \psi \) is an absolutely con-
tinuous solution of (5) in \( 0 \leq x \leq 1 \). Then \( \psi = \psi' \) exists and is a
Lebesgue-integrable solution of (2) almost everywhere. The preceding
lemma shows that \( \psi \) is necessarily constant almost everywhere (set
\( \psi_0 = \psi \)). Thus, \( \phi \) is linear and, since \( \phi(x) = 0 \) (set \( x = 0 \) in (5)),
\( \phi(x) = c(x-p) \). This proves the theorem.

Finally, I would like to thank Professor P. Volkmann for
stimulating my interest in the topics of this note.
References


Mathematisches Institut I
Universität Karlsruhe
Englerstraße 2
D-7500 Karlsruhe

Received September 16, 1982