ABSTRACTS

The Turán Problem for a Class of Polytopes

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Consider the class of positive–definite functions $f$ on $\mathbb{R}^d$ with $f(0) = 1$ and supported in a given convex centrally symmetric body $D$. The Turán problem is to find the least upper bound for integrals of such functions. We solve the Turán problem in the case where $D$ is a convex centrally symmetric polytope in $\mathbb{R}^d$ with the property that its translations over a lattice form a tiling of $\mathbb{R}^d$.

Joint work with V.V. Arestov (Yekaterinburg, Russia).

About Generalised Energy Functionals, Equidistributed Point Sets and Invariance Principles

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Heuristics expects that point sets with extremal energy are good point sets for numerical integration with Chebyscheff–type quadrature formula. A general class of energy functionals with some of their properties will be introduced. A generalisation of Stolarsky’s invariance principle is used to obtain invariance principles for this class of energy functionals.
Convergence of Multivariate Non–Stationary Vector Subdivision Schemes

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Any non–stationary subdivision scheme is associated with masks that may vary from one scale to the next finer one. We investigate the convergence of non–stationary vector subdivision schemes that arise when computing partial derivatives of vector–valued functions generated by subdivision schemes in $\mathbb{Z}^d$. In particular, we present a strategy for deriving non–stationary difference subdivision schemes whose zero convergence guarantee the convergence of the original schemes.

Multivariate Balanced Refinable Function Vectors and Multiwavelets*

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This is a joint work with Qingtang Jiang. Vanishing moments of sufficiently high order and compact supports of reasonable size contribute to the great success of wavelets in various areas of applications, particularly in signal and image processing. However, for multiwavelets, polynomial preservation by the refinable function vectors does not necessarily imply annihilation of polynomials by the corresponding orthogonal or bi–orthogonal multiwavelets. This led to the introduction of the notion of “balanced” multiwavelets by Lebrun and Vetterli, and later, generalization to higher–order balancing by Selesnick. Selesnick’s work is concerned only with orthonormal refinable function vectors and orthonormal multiwavelets. In this talk after giving a brief overview of the state–of–the–art, I will discuss our most recent contribution to this research area. Our goal is to derive a set of necessary and sufficient conditions that characterize the balancing property of any order for the general multivariate matrix–dilation setting. I will end my talk by demonstrating our theory with examples of bivariate splines on the four–directional mesh.

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Selecting Best Bases

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Suppose that we have a class of functions $K$ that we wish to approximate in some given norm $\| \cdot \|$. One way to do this is to choose a basis and then select a good or best $n$–term approximation from this basis. But which basis should we choose? This talk will explain recent results of Donoho and DeVore–Petrova–Temlyakov on characterizations of when a basis is best for a given function class in a given metric. This subject has much interplay between the Geometry of Banach Spaces, Information Theory and Approximation Theory.

Transfinite Interpolation by Blending Functions,
Best One–Sided $L^1$–Approximation and Cubature Formulae

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I. Blending functions of order $(m,n)$. The classical univariate algebraic polynomials have a natural multivariate extension by the so–called blending functions. The real vector space $B^{m,n}(I^2)$ of all blending functions of order $(m,n)$, on the unit square $I^2 := [-1,1]^2$, is defined as follows

$$B^{m,n}(I^2) := \left\{ h \in C^{m,n}(I^2) : D^{m,n} h := \frac{\partial^{m+n}}{\partial x^m \partial y^n} h = 0 \right\}.$$ 

The linear space $B^{m,n} := B^{m,n}(I^2)$ is infinite–dimensional in contrast to the fact that the linear space of all algebraic polynomials of degree at most $m – 1$ is of finite dimension $m$. It is easily seen that each blending function $h \in B^{m,n}$ can be represented in the form

$$h(x,y) = \sum_{k=0}^{m-1} a_k(y) x^k + \sum_{l=0}^{n-1} b_l(x) y^l.$$ 

However, note that the above representation is not unique.

II. Transfinite interpolation by blending functions. Interpolation schemes that yield interpolants that match a given function on a non denumerable set of points were said to be transfinite by W.J. Gordon and Ch.A. Hall. Such schemes provide the analyst with an ability to construct interpolants that pass through a specified network of curves, where more classical, pointwise, finite–dimensional interpolation schemes are restricted to a finite number of points of interpolation. Let us mention that for a given domain, there is no pointwise universal Chebyshev system in multivariate case. The Dirichlet problem is an example of a transfinite interpolation scheme.

III. Best one–sided $L^1$–approximation by blending functions. The following result is based on appropriate transfinite interpolation by blending functions.
Theorem. Let a real valued function \( f(x,y) \in C^{2,2}(I^2) \) satisfy
\[
\frac{\partial^4}{\partial x^2 \partial y^2} f(x,y) \geq 0 \text{ on } [-1,1]^2.
\]
Then, the best one-sided \( L^1 \)-approximant to \( f \) from \( B^{2,2}(I^2) \) of the so-called blending functions of order \((2,2)\) is, in fact, the unique blending Hermite interpolant \( h^f \in B^{2,2}(I^2) \), which satisfies the transfinite interpolation conditions
\[
h^f|_x \equiv f|_x \text{ and } \text{grad } h^f|_x \equiv \text{grad } f|_x
\]
on the canonical point set \( x := \{ (x,y) \in I^2 : |x| = |y| \} \).

**IV. Approximate integration of bivariate integrals over \([-1,1]^2\) by finite linear combinations of univariate ones.**

We discuss blending type approximate integration based on functional traces (interpolation information) on manifolds of lower dimension (not only points). Here is a blending analog of the well-known univariate, Newton–Côtes trapezoidal compound cubature. Let us consider the points \( x_\mu = y_\mu = -1 + 2\mu/m, \mu = 0, 1, \ldots, m \) and let
\[
G_{x_{m+1},y_{m+1}} := \{ (x,y) \in I^2 : \prod_{s=0}^m (x - x_s) \prod_{k=0}^m (y - y_k) = 0 \}
\]
be the blending grid associated with the nodes
\[
x_m := \{ x_s, 0 \leq s \leq m \} \text{ and } y_m := \{ y_s, 0 \leq s \leq m \}.
\]
Then, for each \( f \in C^{2,2}(I^2) \), the following cubature for approximate integration holds
\[
\int_{I^2} f = \left[ \frac{2}{m} \int_{G_{x_{m+1},y_{m+1}}} f - \frac{1}{m} \int_{\partial I^2} f \right]
- 2 \frac{m}{m^2} \sum_{\mu=1}^{m-1} (f(-1,y_\mu) + f(1,y_\mu) + f(x_\mu,-1) + f(x_\mu,1))
+ 4 \frac{m^2}{m^2} \sum_{\mu=1}^{m-1} \sum_{\nu=1}^{m-1} f(x_\mu,y_\nu)
+ \frac{1}{m^2} (f(-1,-1) + f(-1,1) + f(1,-1) + f(1,1))
+ \frac{4}{9m^2} D^{2,2} f(\xi,\eta), \quad (\xi,\eta) \in I^2.
\]

References:

Spectral Interpolation

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An interpolation scheme for refining the samples of a narrowband oscillatory signal is presented. The scheme is exact for signals in a space spanned by a certain finite set of exponentials, determined by spectral properties of the samples. This interpolation scheme is significantly better than local polynomial interpolation of the same complexity. In case of a narrowband oscillatory signal with added white noise, this interpolation scheme yields errors with variance tending to zero as $2^{-k}(\Delta w)^2$, with $k$ the refinement level and with $\Delta w$ the maximal width of the disjoint narrowband supports of the signal.

Shift Invariant Operators

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We deal with operators of the form

$$Qf(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy,$$

where the kernel

$$K(x, y) = \sum_{\alpha \in \mathbb{Z}^d} G(x - \alpha)F(y - \alpha),$$

is determined by the two measurable functions $G, F$ decreasing exponentially: there are $C > 0$ and $0 < q < 1$ such that for all $x \in \mathbb{R}^d$

$$|F(x)| < Cq^{|x|}, \quad |G(x)| < Cq^{|x|}.$$

For operators $Q_h = \sigma_h \circ Q \circ \sigma_{1/h}$, $h > 0$, where $\sigma_h f(x) = f(x/h)$, we show the following:

(i) $r$ is the rate of convergence iff $Q$ has the polynomial order $r$;
(ii) the asymptotic formula;
(iii) in the box–spline case I show the special role of Bernoulli-Stöckler splines;
(iv) weak saturation theorem with smoothing phenomena obtained by bootstrap method;
(v) non–homogeneous scaling;
(vi) application in statistics.

References:

Polynomial Approximation of Fundamental Solutions

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The famous theorem of Carl Runge (1885) asserts that each solution of the homogeneous Cauchy–Riemann equation $\partial u = 0$ in the neighbourhood of a compact set $K \subset \mathbb{C}$ can be uniformly approximated on $K$ by a linear combination of fundamental solutions. It is therefore quite useful to approximate a fundamental solution by polynomials. In fact, it is easy to see that a fundamental solution for the Cauchy–Riemann operator at a point $q$ can be approximated on $K$ if and only if $q$ lies in the unbounded complementary component of $K$. For other differential operators, the situation is not so simple. We shall discuss these matters for the Laplace operator.
On the Local Approximation Order of Scattered Data Interpolation by Polyharmonic Splines

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Polyharmonic splines, due to Duchon, are powerful tools for solving multivariate interpolation problems of the form

\[ f(\xi) = s(\xi), \quad \xi \in \Xi, \]

where \( f : \mathbb{R}^d \to \mathbb{R} \) is an unknown function, whose sample values taken at a finite set \( \Xi \subset \mathbb{R}^d \) of scattered locations are given. Due to the interpolation scheme of polyharmonic splines, the interpolant \( s : \mathbb{R}^d \to \mathbb{R} \) in (1) has the form

\[ s = \sum_{\xi \in \Xi} c_{\xi} \phi_{d,k}(\| \cdot - \xi \|) + p, \]

where

\[ \phi_{d,k}(r) = \begin{cases} r^{2k-d} \log(r) & \text{for } d \text{ even}, \\ r^{2k-d} & \text{for } d \text{ odd}, \end{cases} \]

with \( 2k > d \), and where \( p : \mathbb{R}^d \to \mathbb{R} \) in (2) is a polynomial of degree \( k - 1 \).

Pointwise error estimates for polyharmonic spline interpolation are, due to Wu and Schaback, given by

\[ |f(x) - s(x)| \leq C_f \cdot h^{k-d/2}(x), \quad f \in H^k(\mathbb{R}^d), \]

which in turn leads to (global) bounds of the form

\[ \|f - s\|_{L_\infty(\Omega)} \leq C_f \cdot h^{k-d/2}, \quad f \in H^k(\mathbb{R}^d), \]

for some bounded domain \( \Omega \subset \mathbb{R}^d \) comprising the point set \( \Xi \). Hence, the (global) approximation order of polyharmonic spline interpolation is at least \( p = k - d/2 \), for \( f \in H^k(\mathbb{R}^d) \). We remark that the bound in (3) can further be improved by imposing harder restrictions on the regularity of \( f \).

In this talk, we focus on the recent results concerning the local approximation order of polyharmonic spline interpolation. It has been proven that the local approximation order of polyharmonic spline interpolation is \( p = k \), for \( f \in C^k \). To this end, it is shown that the Lebesgue constant of the interpolation scheme is invariant under uniform scalings. This observation implies some useful properties of polyharmonic spline interpolation, e.g., its numerical stability, which are discussed in this talk.
On the Analytic Continuation of the Cauchy Transform

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The Cauchy transform of a function \( g \) in \( L^p(\mathbb{C}) \) and with compact support is

\[
\hat{g}(z) = -\frac{1}{\pi} \int \frac{g(w)}{w - z} dA,
\]

where \( dA \) denotes the two dimensional Lebesgue measure. Let \( \Omega \) be a bounded open set and let \( \chi_\Omega(z) \) denote its characteristic function. We say that \( \hat{\chi}_\Omega(z) \) has an analytic continuation near a point \( z_0 \in \partial \Omega \), if there is an analytic function \( f \) in \( U_{z_0} \), such that \( \hat{\chi}_\Omega \) coincides with \( f \) on \( U_{z_0} \setminus \Omega \), here \( \partial \Omega \) denotes the boundary of \( \Omega \) and \( U_{z_0} \) is a neighborhood of \( z_0 \). This phenomenon occurs in several situations, for example, in the inverse problems of potential theory (See e.g. [3]). However, in many such problems the physical considerations lead to a multivalued analytic continuation, that is, \( \hat{\chi}_\Omega \) coincides with a multivalued analytic function on \( U_{z_0} \setminus \Omega \).

The analytic continuation implies that \( f \) satisfies an over-determined system in \( U_{z_0} \), hence the boundary \( \partial \Omega \cap U_{z_0} \) cannot be arbitrary. In fact, Sakai [2] proved that \( \partial \Omega \cap U_{z_0} \) is a real analytic curve, except for certain types of cusp singularities. The multivalued continuation provides serious difficulties and it is an open problem whether the above result does hold in that case too. Our main result asserts that the free boundary \( \partial \Omega \cap U_{z_0} \) has a zero Lebesgue measure, if \( \hat{\chi}_\Omega(z) \) has a multivalued analytic continuation near a point \( z_0 \), [1]. We discuss also connections to quadrature domains and a related approximation problem.

References:

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Best One–Sided $L^1$–Approximation by $B^{m,1}$–Blending Functions

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Let $f \in C^{m,1}(I^2) := \{ f \in C(I^2) : f^{(\mu,\nu)} := \frac{\partial^{\nu+\mu} f}{\partial x^\mu \partial y^\nu} \in C(I^2); 0 \leq \mu \leq m; 0 \leq \nu \leq 1 \}$, $I^2 := [-1,1]^2$, be a function satisfying $f^{(m,1)} \geq 0$. A function $u^* \in B^{m,1}(I^2) := \{ u \in C^{m,1}(I^2) : u^{(m,1)} = 0 \}$ is called a best one–sided $L^1$–approximation to $f$ from above with respect to $B^{m,1}(I^2)$ if $u^* \geq f$ and

$$
\| f - u^* \|_1 \leq \| f - u \|_1 \quad (u \in B^{m,1}(I^2); u \geq f).
$$

We present a characterization of best one–sided $L^1$–approximation from above (resp. below) with respect to $B^{m,1}(I^2)$ for arbitrary $m \in \mathbb{N}$, thus extending some of the results in [1] and [2].

References:


Wavelet Methods for Linear–Quadratic Elliptic Control Problems

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Wavelet approaches provide a promising potential for the fast numerical solution of control problems governed by linear elliptic boundary value problems with distributed, Dirichlet or Neumann boundary control. A quadratic cost functional involving possibly natural norms of the state and the control is to be minimized subject to linear constraints in weak form. By placing the problem into the framework of (biorthogonal) wavelets, the functional can be equivalently written in terms of $\ell_2$–norms of wavelet expansion coefficients, with constraints in form of an $\ell_2$ automorphism. The resulting first order necessary conditions are then derived as a (still infinite) weakly coupled $\ell_2$ system. This approach allows for the treatment of ‘broken’ Sobolev norms in the functional as well as lays the ground for deriving an asymptotically optimally preconditioned finite system of coupled equations.

For the fast numerical solution, a fully iterative scheme is employed which can be interpreted as an inexact gradient method for the control variable. Combining this with the machinery developed by Cohen, Dahmen and DeVore, a fully adaptive wavelet scheme can be designed. In particular, it can be shown that the adaptive algorithm is asymptotically optimal, that is, the convergence rate achieved for computing the solution up to the target tolerance is asymptotically the same as given by the wavelet–best $N$–term approximation of the solution, and the total computational work is proportional to the number of unknowns.
Splines for Numerical Solution of PDE’s

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We show how to use multivariate splines of any degree $d$ and any smoothness $r$ with $d > r$ for numerical solution of partial differential equations. Our method does not require the construction of locally supported functions. The linear systems associated with our method can be easily assembled which can be done in parallel. The linear systems arising from our method have a special structure. A matrix iterative method for the special structure linear systems is introduced and the convergence will be proven. Several numerical results on 2D and 3D Poisson, biharmonic equations and Navier–Stokes equations will be shown.

On Almost Interpolation with Radial Basis Functions

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Error Estimates for Radial Basic Function Approximation

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The title is a little bit general. Let me try to explain what I intend to tackle. Given any radial basic function, there is a certain space which it generates naturally – the native space as it is styled by Schaback and his co–workers. If the function being interpolated by linear combinations of radial basic functions on a scattered data set lies in this smoothness space, then the asymptotic rate of approximation, as the density of points increases is well–known. Recently, there has been quite a lot of interest in results where the function does not possess the correct smoothness, but rather some lower order smoothness. Yoon has made a significant number of contributions in this direction. Work by Johnson is also interesting and relevant. We will describe briefly some of this work, and also give a presentation of the results I am developing with my student Rob Brownlee.
Common Zeros of Orthogonal Polynomials on the Three-dimensional Sphere

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In the univariate case the zeros of orthogonal polynomials show very pleasant properties with respect to interpolation. Looking for similar points in the multivariate case one is led to the concept of common zeros of orthogonal polynomials as introduced by Xu, for example. Following this idea, polynomials $A_m$ (generated by the univariate polynomials $C_{\mu^{1/2}}$) are inspected in this talk. Separation of the polynomials $A_m$ according to the multiindex $m$ gives polynomials $p_n$ that turn out to have common zeros on the sphere in some special cases. These results will be presented for the sphere $S^2$.

Reversible Integer DCT Algorithms

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Integer DCTs have important applications in lossless coding. An integer DCT of radix–2 length $n$ is a nonlinear, invertible mapping which acts on $\mathbb{Z}^n$ and approximates the classical discrete cosine transform (DCT) of length $n$. In image compression, the DCT of type II (DCT–II) is of special interest.

We present two new approaches to construct reversible integer DCT–II. Our methods are based on a factorization of the cosine matrix of type II into a product of sparse, orthogonal matrices (see [2]). Up to some permutations, each matrix factor is a block–diagonal matrix, where every block is an orthogonal matrix of order 2. Hence one has to construct only integer transforms of length 2.

The first approach uses expansion factors and rounding–off for the construction of integer transforms of length 2. Here we have adapted a method for the Haar wavelet transform for integers presented in [1]. The second approach works with lifting steps and rounding–off. This allows the construction of new (one– and two–dimensional) integer DCT–II algorithms. For simplicity, the most interesting case $n = 8$ is considered in detail. Explicit estimates for the truncation error are given for all integer DCT–II algorithms proposed.

References:

Localization of Functions on the $d$–variate Sphere and Torus

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In this talk we present some ideas of time–frequency–localization for functions on the sphere and on the torus. Especially we will focus on uncertainty principles for these multivariate functions. Particular attention will be given to polynomials, in a certain sense, already localized in frequency. Using such uncertainty inequalities we look for polynomials best localized in space. We discuss the construction of such kernel functions and describe how bases and frames can be computed out of these kernels.

Finally we show the connection between the Heisenberg principle for functions on the Euclidean space and the uncertainty principles on the torus and on the sphere.

Approximation of Density Functions and Reconstruction of the Approximants

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We consider functions $F \in C(B^r), r \geq 3$, together with their Appell projections $F_\nu$ of index $s$. If the subdiagonal summation matrix $A = (a_{\mu,\nu})$ makes the Laplace series of $\hat{F} \in C(S^{r+s})$, defined by $\hat{F}(x_1, \ldots, x_{r+s+1}) := F(x_1, \ldots, x_r)$, uniformly convergent, then the corresponding holds for the Appell series of $F$. A best possible choice of $A$ is gained from the Newman–Shapiro kernels in $r + s + 1$ dimensions. The resulting approximation operators are positive and provided with the best possible saturation order. We show how in the particular case where $s = 1$ the $A$–partial sums from the $F_\nu$, with all their favorable properties, can be (re–)constructed from the Radon transform of $F$ itself, and we discuss the stability problem belonging to the method.
Subdivision with finitely supported masks is an efficient method to create discrete multiscale representations of smooth surfaces for CAGD applications. Recently a new subdivision scheme for triangular meshes, called $\sqrt{3}$–subdivision, has been studied. In comparison to dyadic subdivision which is based on the dilation matrix $2I$, $\sqrt{3}$–subdivision is based on a dilation $M$ with $\det M = 3$. This has certain advantages, for example, a slower growth for the number of control points.

This talk concerns the problem of achieving maximal sum rule orders for stationary $\sqrt{3}$–subdivision schemes with given mask support. This is important because the sum rule order characterizes the order of the polynomial reproduction, and provides an upper bound on the Sobolev smoothness of the surface. We study both interpolating and approximating schemes for a natural family of symmetric mask support sets related to squares of sidelength $2n$ in $\mathbb{Z}^2$, and obtain exact formulas for the maximal sum rule order for arbitrary $n$. For approximating schemes, the solution is simple, and schemes with maximal sum rule order are realized by an explicit family of schemes based on repeated averaging.

On Generalized Shift–Invariant Systems (GSI)

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On some Multivariate Quadratic Spline Quasi–Interpolants on Bounded Domains

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The aim of this talk is to present some properties and applications of $C^1$ quadratic spline discrete quasi-interpolants (dQIs) on bounded domains $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. These operators are of the form $Qf(x) = \sum_{k \in K(\Omega)} \mu_k(f) B_k(x)$, where $K(\Omega)$ is the set of indices of B-splines $B_k$ whose support is included in the domain $\Omega$ and $\mu_k(f)$ is a discrete linear functional $\sum_{i \in I(r)} \lambda_k(i) f(x_i+k)$, with $I(r) = \mathbb{Z}^d \cap [-r,r]^d$ for $r \in \mathbb{N}$ fixed (and small). The data points $x_j$ are vertices of a uniform or nonuniform partition of the domain $\Omega$, where the function $f$ is to be approximated.

Such operators have been widely studied in recent years, but in general, except in the univariate or multivariate tensor-product cases, they are defined on the whole space $\mathbb{R}^d$: here we restrict our study to bounded domains and to quadratic spline dQIs. Their main attraction lies in the fact that they provide approximants having the best approximation order and small norms while being easy to compute. They are particularly useful as initial approximants at the first step of a multiresolution analysis.

First we define and determine near-best (NB) dQIs on uniform and non-uniform meshes of a bounded interval of the real line, or on bounded domains of the plane with a non-uniform criss-cross triangulation. We say that $Q$ is NB if the sums $\sum_{i \in I(r)} |\lambda_k(i)|$ are minimal for the coefficients ensuring that $Q$ has a given approximation order (for quadratics, it is equal to one or two). In particular, we prove that the infinite norms of these dQIs are bounded independently of the mesh. We give, then, some results on boolean sums of univariate and multivariate dQIs: infinite norms, error estimates, and superconvergence results. Finally, if there is enough time, we will show that such operators are useful for obtaining approximate formulas for integration and differentiation, for approximating zeros of polynomials and for solving boundary value problems using collocation or variational methods.
The talk considers stationary vector subdivision operators in several variables, that is, subdivision operators obtained by “convolving” a finitely supported matrix valued sequence with vector or matrix valued sequences. Moreover, scaling is done not only by a factor of two but by an arbitrary expanding matrix which means a matrix all of whose eigenvalues are greater than one in modulus.

In the scalar univariate case an important property of such subdivision schemes, which is crucial for polynomial reproduction by the subdivision operator as well as smoothness and approximation order of the associated refinable function, is that the associated symbol (the Laurent polynomial whose coefficients are the coefficients of the subdivision operator, in other words, the $z^{-1}$ transform of this sequence) has a zero of a certain order at $-1$; generalizations of that to matrix factorizations are known also for univariate vector subdivision schemes. The multivariate counterpart of this property is containment in a certain natural quotient ideal which is in turn related to factorizing certain difference operators.

An important part of the work presented has been done jointly with H. M. Möller.
Approximation from Sparse Grids and Sobolev Spaces of Dominating Mixed Smoothness

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For a given $2\pi$-periodic and continuous function $f$ we denote by $I_n f$ the trigonometric polynomial of order less than or equal to $n$ which interpolates $f$ at the points $t_k = 2\pi k / (2n + 1)$, $k \in \mathbb{Z}$. Approximation of $f$ by $I_j f$ is of the same order as the best approximation by trigonometric polynomials of degree less than or equal to $n$ as long as $f$ belongs to a periodic Besov space $B^s_{p,q}$ with $s > 1/p$ and $1 < p < \infty$. Here we are discussing the quality of approximation of a given function $f = f(x_1, x_2)$ by

$$B_m f = \sum_{j+k \leq m} \Delta_{j,k} f, \quad m = 1, 2, \ldots,$$

where $\Delta_{j,k} = (I_{2^j} - I_{2^j-1}) \otimes (I_{2^k} - I_{2^k-1})$, $j, k \geq 1$ (modification if $\min(j, k) = 1$). This is Smolyak’s construction with respect to the $I_{2^j}$ in the bivariate situation. Let $S^r_p W$ denote the Sobolev space of dominating mixed smoothness of fractional order $r$. Our main result consists in the exact determination of the asymptotic behaviour of $\|I - B_m : S^r_p W \hookrightarrow L_p(\mathbb{R}^2)\|$. Indeed, it holds

$$\|I - B_m : S^r_p W \hookrightarrow L_p\| \sim m^{1/2} 2^{-mr}$$

if $1 < p < \infty$ and $r > 1/\min(2, p)$. The tools we use to prove the above estimate are Fourier multiplier and Littlewood-Paley assertions (due to Lizorkin in this context), the tensor product structure of $S^r_p W$, and complex interpolation of Lizorkin-Triebel spaces of dominating mixed smoothness.
Split–Radix Algorithms for Discrete Trigonometric Transforms

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Discrete trigonometric transforms are widely used in processing and compression of signals and images. Examples of such transforms are discrete cosine transforms (DCT) and discrete sine transforms.

In this talk, we present new split–radix DCT–algorithms of radix–2 length, which are based on real factorization of the corresponding cosine matrices into products of sparse, orthogonal matrices. These algorithms use only permutations, scaling with $\sqrt{2}$, butterfly operations, and plane rotations. They can be seen by analogy with the well–known split–radix FFT. Our new algorithms have a very low arithmetical complexity which compares with the best known fast DCT–algorithms. Further, a detailed analysis of the roundoff errors for the new split–radix DCT–algorithm shows its excellent numerical stability which outperforms the real fast DCT–algorithms based on polynomial arithmetic. These split–radix DCT–algorithms can be applied to the construction of reversible integer DCT.

Some Open Problems in Multivariate Approximation

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The main goal of this talk is to discuss the state of the art in approximation of classes of multivariate functions with bounded mixed derivative. As a motivation of studying these classes we demonstrate connections between the following two big areas of research: the theory of cubature formulas (numerical integration) and the discrepancy theory. In particular, we will show how standard in the theory of cubature formulas settings can be translated into the discrepancy problem and into a natural generalization of the discrepancy problem. This leads to a concept of the $r$-discrepancy. One of the important messages of this discussion is that the theory of discrepancy is closely connected with the theory of cubature formulas for the classes of functions with bounded mixed derivative. We will formulate some open problems in the theory of cubature formulas, in the theory of widths, and in the theory of nonlinear approximation for the classes of functions with bounded mixed derivative.
Probabilistic Results on the Discrepancy of Point Sets

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We give a survey on recent results on the discrepancy of point sequences and their applications. Applying methods from approximation theory we present lower bounds for the discrepancy. Furthermore we establish upper bounds as well as constructions of low discrepancy sequences. We focus on probabilistic results and we present applications of the design of random number generators. Various quality parameters are studied. Finally, applications to Mathematical Finance are presented.

Reconstruction of Functions from Large Data Sets

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The efficient visualization of large data sets is a key problem in modern numerical analysis with roots in the field of computer graphics. In more than one dimension, the structure of the current data plays an important role and has to be considered in the employed method. In general, one distinguishes between regular data, where the points form (part of) a regular grid, and scattered data, where no such information on their location is known. While there exist some efficient methods in case of regular data much less is known in case of scattered point sets.

In this talk I want to discuss the advantages and drawbacks of a method based on radial basis functions and a partition of unity. After reviewing the theoretical background and recent developments, the numerical realization is presented. The method’s strength will be demonstrated on several different examples, starting with simple terrain reconstructions and data sets on the sphere. Other applications come from coupling problems in the field of aeroelasticity.

Here, different models are used to describe the same object, for example an aero-plane, and information such as deformation, pressure, forces have to be exchanged between these models.
Lagrange Interpolation by Bivariate Splines with Optimal Approximation Order

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There exists a vast literature on local Hermite interpolation by smooth splines on triangulations. On the other hand, since recently no results of this type were known for Lagrange interpolation by bivariate splines. Such interpolation schemes are advantageous in applications since they do not require (estimates of) derivatives at prescribed points. Jointly with Günther Nürnberger and Larry L. Schumaker, we developed the first local Lagrange interpolation methods for bivariate splines on triangulations and triangulated quadrangulations. The methods are based on a fast coloring (with two colors) of the triangles and quadrilaterals, respectively. Due to the locality of the schemes the Lagrange interpolating splines yield optimal approximation order. The splines can be efficiently computed with linear complexity. For the evaluation and visualization of the splines Bernstein-Bézier techniques from CAGD can be applied. We give numerical tests for smooth functions and real world data with up to $10^6$ points which show the efficiency of the interpolation methods.

References:

Approximation Theory Problems Arising from Learning Theory

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Learning Theory studies learning objects from random samples. The main question is: How many samples do we need to ensure an error bound with certain confidence? To answer this question, we need to solve some approximation theory problems including estimates for the approximation error and the confidence. For kernel machine learning such as the support vector machine, a reproducing kernel Hilbert space associated with a Mercer kernel is often used as the hypothesis space.

In this talk we shall discuss some approximation theory problems arising from learning theory concerning the reproducing kernel Hilbert spaces: approximation error, covering number, sampling theory and more. Some results are based on joint work with Steve Smale.

Three Scale versus Matrix Refinement Equations

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We shall work with the dilation matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$; note that $M^2 = 2I$.

A function $\varphi : \mathbb{R}^2 \to \mathbb{C}$ satisfies a three scale refinement equation, if there exist coefficient sequences $(a_1[\alpha])_{\alpha \in \mathbb{Z}^2}$ and $(a_2[\alpha])_{\alpha \in \mathbb{Z}^2}$ such that

$$
\varphi(x) = \sum_{\alpha} a_1[\alpha] \varphi(Mx - \alpha) + \sum_{\alpha} a_2[\alpha] \varphi(M^2x - \alpha).
$$

On the other hand, we consider vector valued functions $\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{C}^2$ satisfying a matrix refinement equation

$$
\Phi(x) = \sum_{\alpha} A[\alpha] \Phi(2x - \alpha) \quad \text{with} \quad \varphi_2(x) = \varphi_1(Mx - e_1),
$$

where $(A[\alpha])_{\alpha \in \mathbb{Z}^2}$ is a sequence of matrix coefficients in $M_{2 \times 2}(\mathbb{C})$, and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We give conditions on the masks $a_1$ and $a_2$ on the one hand and $A$ on the other hand ensuring that (1) is equivalent to (2) with $\varphi = \varphi_1$. We also ask under which conditions the family $\{ \varphi_1(\cdot - \alpha), \varphi_2(\cdot - \alpha) \}_{\alpha \in \mathbb{Z}^2}$ interpolates on $M^{-1}\mathbb{Z}^2$.

Furthermore, we present a few examples.