# Norming Sets and Spherical Cubature Formulas

K. Jetter, J. Stöckler and J. D. Ward

*Dedicated to the memory of Professor G. Hämmerlin*

**Abstract.** We investigate the construction of cubature formulas for the unit sphere in $\mathbb{R}^n$ that have almost equal weights. The corresponding knots are taken from equidistributed point sets on the sphere. The notion of norming sets in connection with the Markov inequality of spherical harmonics is used in order to provide a general result on uniformly stable cubature formulas. We also present some numerical evidence that there exist stable and almost equally-weighted cubature formulas, if the number of knots is slightly larger than required by the exactness conditions for spherical harmonics of a certain degree.

## §1. Norming Sets for Spherical Harmonics

In our recent paper [4], we have introduced the notion of norming sets to scattered data problems. In general Banach space notation, we have the following

**Definition 1.** *Let $V$ be a normed linear space with (continuous) dual $V^*$. Given two subspaces $W \subset V$ and $Z \subset V^*$, the set $Z$ is called a* **norming set** *of $W$ if there exists some $c > 0$, the* **norming constant**, *so that*

$$\sup_{z \in Z, \|z\|=1} |z(w)| \geq c\|w\| \qquad \text{for all} \qquad w \in W.$$

The idea behind this notation becomes more transparent if we add the upper estimate

$$\sup_{z \in Z, \|z\|=1} |z(w)| \leq \sup_{z \in V^*, \|z\|=1} |z(w)| = \|w\|,$$

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which holds true for any $w \in W$. This shows that the normed linear space $(W, \|\cdot\|)$ can be endowed with the equivalent norm

$$\|w\|_Z := \sup_{z \in Z, \|z\|=1} |z(w)|.$$

The Banach space considered in our paper is $C(S^{n-1})$, the space of continuous, real-valued functions on the $n$-sphere $S^{n-1} \subset \mathbb{R}^n$, equipped with the usual max-norm (which we denote by $\|\cdot\|_\infty$). As norming sets we will allow the linear span of point evaluation functionals $\delta_x$, $x \in X$, with a **finite** knot set $X \subset S^{n-1}$. In order to be mathematically rigorous, the restriction of such a functional to a subspace $W \subset C(S^{n-1})$ will be denoted by $\delta_x|_W$.

The following result gives a slight generalization of [4, Proposition 2]; it follows from an inspection of the proof given there.

**Proposition 1.** *Let $W$ be a finite dimensional subspace of $C(S^{n-1})$, and let $Z = \text{span}\{\delta_x;\ x \in X\}$, where $X \subset S^{n-1}$ is a finite knot set. Assume that $Z$ is a norming set of $W$ with norming constant $c > 0$. Then $W^*$ can be identified with the space $Z|_W = \text{span}\{\delta_x|_W;\ x \in X\}$, and in particular, any $w^* \in W^*$, with norm $\|w^*\| = 1$, can be identified with some*

$$\sum_{x \in X} a_x \delta_x|_W \qquad \textit{where} \qquad \sum_{x \in X} |a_x| \leq \frac{1}{c}.$$

The existence of norming sets can be verified for spaces $W$ which possess a Markov type inequality. In its integrated form this inequality reads as follows:

$$|w(\mathbf{p}) - w(\mathbf{q})| \leq c_W \ d(\mathbf{p}, \mathbf{q}) \ \|w\|_\infty, \qquad \mathbf{p}, \mathbf{q} \in S^{n-1}, \quad w \in W;$$

here, the constant $c_W$ is generic for the space $W$, and $d(\mathbf{p}, \mathbf{q}) = \arccos(\mathbf{p} \cdot \mathbf{q})$ denotes the spherical distance of the points $\mathbf{p}$ and $\mathbf{q}$ on the sphere. Generally, given a Markov inequality, 'dense' knot sets $X$ generate a norming set $Z$ for the space $W$. In order to be precise, let

$$h(X) := \sup\{d(\mathbf{p}, X);\ \mathbf{p} \in S^{n-1}\}$$

be the (so-called) **mesh norm** of $X$. Then we have

**Proposition 2.** *Suppose that the space $W \subset C(S^{n-1})$ has a Markov inequality with constant $c_W$. Then, given $0 < c < 1$, any knot set $X$ with mesh norm $h(X) \leq \frac{1-c}{c_W}$ generates a norming set $Z = \text{span}\{\delta_x;\ x \in X\}$ of $W$, with constant $c$.*

**Proof:** Let $w \in W$, $\|w\|_\infty = 1$. Then $|w(\mathbf{p})| = 1$ for some $\mathbf{p} \in S^{n-1}$. Hence we can pick $x \in X$ so that $d(\mathbf{p}, x) < (1+\epsilon)\frac{1-c}{c_W}$, where $\epsilon > 0$ can be arbitrarily small. Markov's inequality implies

$$|w(\mathbf{p}) - w(x)| \leq c_W \ d(\mathbf{p}, x) \ \|w\|_\infty < (1+\epsilon)(1-c).$$

We thus have $|w(x)| \geq |w(\mathbf{p})| - |w(\mathbf{p}) - w(x)| > c - \epsilon(1-c)$, and letting $\epsilon \to 0$ proves the theorem. ■

Usually, one is interested in constants $c$ close to 1. This implies that the packing of the knot set $X$ has to be rather 'dense' on the sphere, and a lot of points will be needed. The best we can do is to look for specific point sets which are equidistributed, in order to keep the number of points at a minimum. For the construction of such points of 'equilibrium' we may refer to [1,3,6,10].

Let us now choose a specific function space $W$ for our present investigation of cubature formulas. The degree of exactness of such formulas is usually defined in terms of graded spaces of spherical harmonics. Let $W = \widetilde{V}_\ell$ denote the space of spherical harmonics up to order $\ell$. The fundamental properties of these functions are described in [2,7,9]. Furthermore, it is known that $W$ satisfies a Markov inequality with constant $c_W = \ell$ [4, Section 2.3]. (Note that the related inequality in the survey paper by W. Freeden et al. [2, Lemma 2.1] does not reproduce the best possible result as far as the power of the polynomial degree $\ell$ is concerned.) Thus our results can be applied in this special case. Combining our findings, we arrive at the following fundamental result:

**Theorem 1.** *Let $X \subset S^{n-1}$ be a finite knot set with mesh norm $h(X) \leq \frac{1-c}{\ell}$, for some $0 < c < 1$ and $\ell \in \mathbb{N}$. Then, for any linear functional $\lambda$ on $C(S^{n-1})$ with norm $\|\lambda\| = 1$, there exist constants $a_x$, $x \in X$, such that*

$$\sum_{x \in X} |a_x| \leq \frac{1}{c} \quad \text{and} \quad \lambda(Y) = \sum_{x \in X} a_x \, Y(x), \quad Y \in \widetilde{V}_\ell.$$

We point to the central message in this theorem, namely that the functional $\lambda$ can be represented on the subspace $\widetilde{V}_\ell$ by the functional $\sum_{x \in X} a_x \delta_x$ in such a way that the norm of the latter functional is bounded by a constant, independent of the order $\ell$. However, the number of points in this representation and their position on the sphere has to be chosen according to the condition on the mesh size. This number will be rather large in general, even for a good positioning of the knots.

Let us briefly compare the result in Theorem 1 with the interpolatory representation of $\lambda$. In this case, the number of points in $X$ equals the dimension of $\widetilde{V}_\ell$. But the norms $\sum_{x \in X} |a_x|$ of the representing functionals go to infinity, as the order of the spherical harmonics increases [9].

## §2. Norming Sets and Cubature Formulas

In order to apply our results to cubature formulas, we just have to specify the functional $\lambda$ as the (normalized) integral

$$\lambda(f) := \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} f(\mathbf{p}) \, d\mathbf{p}, \quad f \in C(S^{n-1}),$$

where $\omega(S^{n-1})$ denotes the surface area of the $n$-sphere. A representation of this integral on the spaces of spherical harmonics, as a linear combination of point evaluations as in Theorem 1, will be called a **cubature formula for the sphere, of order** $\ell$. Hence, $\sum_{x\in X}|a_x|$ is the norm of the cubature formula, and Theorem 1 tells that there exist cubature formulas of order $\ell$, for all $\ell \in \mathbb{N}$, with **uniformly bounded** norms. In other words, there exist knot sets $X = X^{(\ell)}$ and constants $a_x^{(\ell)}$ such that

$$\frac{1}{\omega(S^{n-1})}\int_{S^{n-1}} f(\mathbf{p})\, d\mathbf{p} = \sum_{x\in X^{(\ell)}} a_x^{(\ell)}\, f(x), \quad f \in \widetilde{V}_\ell, \tag{CF}$$

and

$$\sum_{x\in X} |a_x^{(\ell)}| \leq \frac{1}{c}$$

for any integer $\ell$. Here, $c$ is again the constant of Theorem 1.

Since the spaces $\widetilde{V}_\ell$, $\ell \in \mathbb{N}$, are dense in $C(S^{n-1})$, i.e.,

$$\operatorname{clos}_{\|\cdot\|_\infty} \cup_{\ell=1}^{\infty} \widetilde{V}_\ell = C(S^{n-1}),$$

a standard application of the Banach-Steinhaus Theorem yields the following existence result of convergent cubature schemes:

**Theorem 2.** *For the cubature formulas defined as before and for any* $f \in C(S^{n-1})$ *we have that*

$$\sum_{x\in X^{(\ell)}} a_x^{(\ell)}\, f(x) \;\to\; \frac{1}{\omega(S^{n-1})}\int_{S^{n-1}} f(\mathbf{p})\, d\mathbf{p} \quad \text{as} \quad \ell\to\infty.$$

While this result only gives the existence of a convergent cubature scheme, it nevertheless seems to be a first step towards the successful construction of stable cubature formulas for $n$-spheres. Of course, it seems to be more desirable to have positive cubature weights, as they appear in [5, Proposition 1] in connection with isometric imbeddings, or in the construction of cubature formulas with equal weights (so-called spherical designs). We shall see later in our numerical examples, that the additional degrees of freedom can be used for obtaining formulas with positive weights, even if the number of knots is only slightly larger than the dimension of $\widetilde{V}_\ell$.

## §3. Numerical Examples for $S^2$

On the sphere $S^2 \subset \mathbb{R}^3$ we have constructed cubature formulas of various orders, based on (almost) equidistributed point sets. Here, equidistribution means that the point set $X$ is extremal with respect to a potential $U$ in the following way: Given $N$, the number of points, we try to minimize the energy

$$E(X) := \sum_{x\in X}\sum_{y\in X\setminus\{x\}} U(d(x,y)).$$

This is consistent with the approach by Kuijlaars and Saff [3], except that we prefer the spherical distance rather than the Euclidean distance. The reason is that this leads to a formulation of the minimization problem in terms of the Riemannian geometry of the sphere and therefore avoids any type of parametrization. A more detailed description is under preparation [1]. Of course, the potential functions in [3] can all be converted to the above form.

It was observed in [3] and by the authors that various potentials lead to similar energy minimizing point distributions. We have chosen the potential function

$$U(t) := e^{-t} + e^{-(2\pi - t)}, \quad 0 \le t \le \pi,$$

which is strictly decreasing, convex and assigns an equilibrium position to two antipodal points on the sphere (since $U'(\pi) = 0$).

For the construction of (almost) equidistributed point sets, we gratefully acknowledge the help by U. Depczynski. We have used an iterative procedure based on an intrinsic gradient method. The theoretical basis of this and some quasi-Newton algorithm is discussed in the forthcoming paper [1].

Given the (almost equidistributed) point set $X = X^{(\ell)}$, we have constructed cubature formulas for the 2-sphere, of order $\ell$, as in $(CF)$. The number $N = N^{(\ell)}$ of points in $X$ was chosen according to the restriction

$$N^{(\ell)} \ge \dim \widetilde{V}_\ell = (\ell + 1)^2.$$

Due to the (almost) equidistribution of the points, we expect to get a cubature formula with (almost) equal weights, i.e.,

$$a_x \approx \frac{1}{\#X}, \quad x \in X.$$

This expectation was fully verified in our numerical tests (see Table 1), if there were sufficiently many degrees of freedom $N^{(\ell)} - \dim \widetilde{V}_\ell$. As a surprising result of our computations we found that only a small number of additional points (much less than twice the dimension of the space) are needed in order to achieve almost equally-weighted cubature formulas.

Our computations were performed with MATLAB 5.0 on an IBM workstation RS6000, model 3CT, with a Power2-processor. Here are the details:

**Basis of spherical harmonics:**

Expressed in spherical coordinates $(\varphi, \theta) \in [0, 2\pi) \times [0, \pi]$, the functions

$$Y_k^n(\varphi, \theta) = \sqrt{2k+1}\; S_k^{|n|}(\cos\theta) \times \begin{cases} \cos |n|\varphi, & \text{for } n = 0, 1, \ldots, k, \\ \sin |n|\varphi, & \text{for } n = -1, -2, \ldots, -k, \end{cases}$$

$k = 0, 1, \ldots, \ell,$

| $\#X^{(\ell)} \diagdown \ell$ | 4 | 7 | 10 | 13 | 16 | 19 |
|---|---|---|---|---|---|---|
| 25 | 1.110 | | | | | |
| 32 | 1.001 | | | | | |
| 50 | 1.027 | | | | | |
| 64 | 1.017 | 1.143 | | | | |
| 72 | 1.012 | 1.080 | | | | |
| 81 | 1.013 | 1.074 | | | | |
| 100 | 1.009 | 1.042 | | | | |
| 121 | 1.004 | 1.023 | 1.532 | | | |
| 128 | 1.005 | 1.026 | 1.091 | | | |
| 144 | 1.005 | 1.023 | 1.063 | | | |
| 169 | 1.003 | 1.018 | 1.039 | | | |
| 196 | 1.002 | 1.014 | 1.040 | −1.735 | | |
| 225 | 1.001 | 1.010 | 1.029 | 1.106 | | |
| 242 | 1.001 | 1.007 | 1.029 | 1.082 | | |
| 289 | 1.001 | 1.006 | 1.018 | 1.054 | −13.51 | |
| 324 | 1.001 | 1.006 | 1.018 | 1.067 | 1.123 | |
| 361 | 1.001 | 1.004 | 1.015 | 1.032 | 1.085 | |
| 392 | 1.001 | 1.005 | 1.020 | 1.039 | 1.097 | |
| 400 | 1.001 | 1.004 | 1.009 | 1.028 | 1.072 | −2.129 |
| 450 | 1.000 | 1.003 | 1.009 | 1.021 | 1.060 | 1.132 |
| 512 | 1.000 | 1.002 | 1.006 | 1.020 | 1.037 | 1.075 |
| 578 | 1.000 | 1.002 | 1.005 | 1.013 | 1.028 | 1.060 |
| 648 | 1.000 | 1.001 | 1.005 | 1.011 | 1.028 | 1.053 |
| 722 | 1.000 | 1.001 | 1.004 | 1.008 | 1.019 | 1.047 |
| 800 | 1.000 | 1.001 | 1.003 | 1.008 | 1.017 | 1.033 |

**Table 1.** Ratios of the maximal/minimal cubature weights $a_x$ for various degrees $\ell$ and almost equidistributed knot sets $X^{(\ell)}$.

form an orthonormal basis of $\widetilde{V}_\ell$. Here, $S_k^{|n|}$ are the (seminormalized) associated Legendre functions of degree $k$ (accessible through MATLAB by the command 'S=legendre(k,X,'sch')'). The basis is essentially the same as in Potts et al. [8] except for the proper normalization, *i.e.*,

$$\frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} Y_k^n(\varphi,\theta)\; Y_{k'}^{n'}(\varphi,\theta)\;\; \sin\theta\; d\varphi d\theta = \delta_{k,k'}\delta_{n,n'}.$$

## Determining the cubature weights:

Inserting the basis into equation $(CF)$ gives a linear system of equations for the weights $\mathbf{a} = (a_x)_{x\in X}$ of the following type:

$$\mathbf{A}\;\mathbf{a} = \mathbf{e}\;. \qquad (EQ)$$

Here, $\mathbf{e} = (1, 0, \ldots, 0)^T$ is the unit vector of length $(\ell+1)^2$, and $\mathbf{A}$ is the $(\ell+1)^2 \times N^{(\ell)}$-matrix with the rows built from the values of a basis element $Y_k^n$ on the knot set $X$. The first row corresponds to the basis element $Y_0^0 \equiv 1$.

| $\#X^{(\ell)}$ \ $\ell$ | 4 | 7 | 10 | 13 | 16 | 19 |
|---|---|---|---|---|---|---|
| 25 | 2.375 | | | | | |
| 32 | 1.054 | | | | | |
| 50 | 1.038 | | | | | |
| 64 | 1.021 | 48.74 | | | | |
| 72 | 1.019 | 1.493 | | | | |
| 81 | 1.013 | 1.231 | | | | |
| 100 | 1.011 | 1.107 | | | | |
| 121 | 1.007 | 1.045 | 50.79 | | | |
| 128 | 1.006 | 1.042 | 6.028 | | | |
| 144 | 1.005 | 1.030 | 1.530 | | | |
| 169 | 1.004 | 1.024 | 1.155 | | | |
| 196 | 1.003 | 1.022 | 1.084 | 2250. | | |
| 225 | 1.002 | 1.019 | 1.078 | 2.223 | | |
| 242 | 1.001 | 1.016 | 1.064 | 1.434 | | |
| 289 | 1.001 | 1.009 | 1.039 | 1.102 | 272.2 | |
| 324 | 1.001 | 1.010 | 1.037 | 1.110 | 2.538 | |
| 361 | 1.001 | 1.006 | 1.025 | 1.066 | 1.413 | |
| 392 | 1.001 | 1.007 | 1.023 | 1.063 | 1.233 | |
| 400 | 1.001 | 1.005 | 1.018 | 1.058 | 1.132 | 1703. |
| 450 | 1.001 | 1.004 | 1.013 | 1.042 | 1.100 | 13.38 |
| 512 | 1.000 | 1.004 | 1.011 | 1.032 | 1.082 | 1.257 |
| 578 | 1.000 | 1.002 | 1.010 | 1.024 | 1.056 | 1.127 |
| 648 | 1.000 | 1.002 | 1.007 | 1.018 | 1.042 | 1.096 |
| 722 | 1.000 | 1.001 | 1.005 | 1.018 | 1.035 | 1.072 |
| 800 | 1.000 | 1.001 | 1.004 | 1.016 | 1.034 | 1.053 |

**Table 2.** Condition numbers of the extended collocation matrices **A**.

In case **A** is quadratic, the solution of $(EQ)$ is unique for all knot distributions that we encountered. For larger knot sets we have a linear variety of solutions, since **A** has full rank $(\ell+1)^2$ in every considered case, and we choose the (least-squares-) solution of minimal $\ell_2$-norm (using the MATLAB command '$\mathbf{a} = \text{pinv}(\mathbf{A}) * \mathbf{e}$' with $\text{pinv}(\mathbf{A})$ the Moore-Penrose pseudoinverse of the matrix **A**).

### Condition of the matrix:

As a characteristic number for the stability of our computations we look at the $\ell_2$-condition number of the matrix **A**, given by the quotient of the maximal and the minimal non-zero singular value. The results are given in Table 2, for various values of $N$ and $\ell$. Apparently, even moderate degrees $\ell$ create large condition numbers in case of a quadratic matrix (this is the case where the cubature formula is of interpolatory type). We also noticed that the condition number itself is very unstable with respect to the point set $X$, if the number of points is still close to the dimension of the space. But doubling the number of knots brings the condition number down to almost 1 for all degrees $2 \leq \ell \leq 19$.

**Test for equidistribution:**

In Table 1 we have listed the range of the cubature weights $a_x$, $x \in X$, as a quotient $\max a_x / \min a_x$. The negativity points to the fact that the solution contains negative weights. A result close to 1 corresponds to an almost equally weighted cubature formula where $a_x \approx \frac{1}{N}$. The size of this ratio may be taken as a test for the quality of the equidistribution of the points.

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