Preconditioning of the Frame Algorithm

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Abstract

We present a preconditioning method for the frame operator of affine frames which are oversampled biorthogonal wavelet bases. The frame operator is represented by a self-adjoint biinfinite operator-valued Laurent matrix. Our preconditioner is chosen as an approximate Cholesky factor of a banded submatrix. Filter bank realizations of these operators are given, which are typical for the fast evaluation of frame and wavelet decompositions.

Key words: affine frame, frame operator, oversampling, Laurent operator, Cholesky decomposition, preconditioning, filter bank.
AMS subject classifications: 42C15, 41A58, 65F35, 15A12

1 Introduction

The discrete wavelet transform (DWT) has become an important tool in signal analysis. Its connection to filter banks is described in [9]. The DWT can be understood as a critical sampling of the continuous wavelet transform (CWT)

$$W_\psi(f;a,b) := a^{1/2}\int_{\mathbb{R}} f(t)\,\overline{\psi\left(a(t-b)\right)}\,dt\ , \qquad a>0,\ b\in\mathbb{R},$$

with respect to scale a and position b. More specifically, in the biorthogonal wavelet setting there is a pair of functions $\psi,\widetilde{\psi}\in L_2(\mathbb{R})$ such that

$$f=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f\,,\,\psi_{j,k}\rangle\ \widetilde{\psi}_{j,k} \qquad \text{for any} \qquad f\in L_2(\mathbb{R})\ ,$$

where $\psi_{j,k} := 2^{j/2}\psi(2^j\cdot -k)$, and the sampling values $W_\psi(f;2^j,2^{-j}k)=\langle f\,,\,\psi_{j,k}\rangle$, $j,k\in\mathbb{Z}$, are the unique coefficients in this decomposition of f.

For some applications like pattern recognition, sound analysis or noise removal the critical sampling of the DWT is not enough. One reason for this can be explained as follows. On a fixed scaling level 2^j the sampling frequency of the DWT is 2^j, while the Nyquist frequency of $W(f;2^j,\cdot)$ is larger than 2^{j+1}. Hence the separate treatment (e.g. quantization or thresholding) of each scaling level can have significant aliasing

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as a side-effect. There exist several methods of "oversampling" of the CWT which are discussed in [6]. In this paper we deal with the case, where the scaling is discretized as above, but the sampling frequency is increased by the same factor $N \in \mathbb{N}$ on each scaling level 2^j. The corresponding family

$$\Psi_N := \{\psi_{j,k;N} = 2^{j/2}\psi(2^j \cdot -k/N) \ \mid \ j,k \in \mathbb{Z}\} \tag{1}$$

is called an *affine frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\ \langle f\,,\,\psi_{j,k;N}\rangle\ |^2 \leq B\|f\|^2 \qquad \text{for any} \qquad f \in L_2(\mathbb{R}). \tag{2}$$

In practical examples values of $N \geq 3$ can already give large improvements [1]. On the other hand, the numerical load as compared to the DWT increases at least by a factor N. Nevertheless one wishes to find decompositions of the form

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} d_{j,k;N}\ \psi_{j,k;N}\ , \qquad f \in L_2(\mathbb{R})\ , \tag{3}$$

by real-time methods. For this purpose we need fast algorithms for the inverse of the *frame operator*, which is positive definite and has operator bounds A and B in (2). For $N = 2^n$ we propose a pyramidal scheme which has a filter bank realization as shown in Figure 1. Secondly, we develop a preconditioning method from an incomplete three-diagonal Cholesky factorization of a related operator matrix. Since only one or two steps of an iterative procedure for the inversion of the frame operator should give enough precision, this preconditioning is useful even though the frame operator is well-conditioned in the sense of numerical analysis.

Note that our sampling by the set (1) differs from the CWT algorithm of Holschneider et.al. [7], see also [9]. While our sampling frequency on scaling level 2^j is $N \cdot 2^j$, they use a fixed sampling 2^K on each level $j \leq K-1$. This is done by applying the DWT algorithm without subsampling in each step starting from scaling level 2^K. There are two major differences to our approach. The CWT algorithm has a very large oversampling frequency for scaling levels $j \ll K$, which results in a large computational load if many scaling levels are needed. On the other hand, the upper scaling level 2^{K-1} is still sampled below its Nyquist frequency.

We will use several helpful notations for operators on $\ell_2(\mathbb{Z})$. Capital X denotes the z-transform of a sequence x, e.g. $X(z) = \sum_{k\in\mathbb{Z}} x_k z^{-k}$. Convolution of sequences and multiplication of their z-transforms are identified. By $X^*(z)$ we denote the z-transform of $x^* := (\overline{x_{-k}})_{k\in\mathbb{Z}}$, which is the reflection and complex conjugation of x. The upsampling operation, which stretches a sequence x and fills in zeros at all odd indices is $U : X(z) \to X(z^2)$. Its adjoint is the subsampling operation $U^* : X(z) \to \sum_{k\in\mathbb{Z}} x_{2k} z^{-k}$. Finally, the notation XU is to be understood as an operator product, first upsampling and then convolution with x. These are the prototypical operators which determine the filter bank of the reconstruction algorithm of wavelet bases. Their adjoints $(XU)^*$, which perform convolution (with x^*) first and then subsampling, appear in the decomposition algorithms. We always think of these linear operators as fast numerical transforms, since the sequence x usually has small support.

2 Filter Bank Realization of the Frame Operator

We start from the general situation in [5] for biorthogonal wavelet bases. Hence there are two scaling functions ϕ, $\widetilde{\phi} \in L_2(\mathbb{R})$ which generate a pair of biorthogonal multiresolution analyses. Their scaling relations are

$$\phi(t) = 2 \sum_{k\in\mathbb{Z}} h_k \phi(2t-k), \qquad \widetilde{\phi}(t) = 2 \sum_{k\in\mathbb{Z}} \widetilde{h}_k \widetilde{\phi}(2t-k). \tag{4}$$

The biorthogonal wavelet bases are generated by the functions

$$\psi(t) = 2 \sum_{k\in\mathbb{Z}} g_k \phi(2t-k), \qquad \widetilde{\psi}(t) = 2 \sum_{k\in\mathbb{Z}} \widetilde{g}_k \widetilde{\phi}(2t-k) \ . \tag{5}$$

In view of numerical applications we will always assume that the sequences h, $\widetilde{h}$, g and $\widetilde{g}$ are finite (giving rise to FIR filters).

For $N \in \mathbb{N}$ we let Ψ_N (and $\widetilde{\Psi}_N$) as in (1). With very mild assumptions on ψ the set Ψ_N is an affine frame, see [2]. We want to consider sampling of the CWT with frequency $N \cdot 2^j$ on scaling level 2^j. This is incorporated into the *analysis operator*

$$\mathcal{T} := \mathcal{T}_{\psi,N} : L_2(\mathbb{R}) \longrightarrow \ell_2(\mathbb{Z}\times\mathbb{Z}), \qquad f \mapsto \big(\langle f\,,\, \psi_{j,k;N}\rangle\big)_{j,k\in\mathbb{Z}} \ .$$

The *frame operator* is then given by $\mathcal{T}^*\mathcal{T}$, and the adjoint $\mathcal{T}^*$ has the explicit form

$$\mathcal{T}^* \left(x_{j,k}\right)_{j,k\in\mathbb{Z}} = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} x_{j,k}\, \psi_{j,k;N} \ .$$

It is clear that the frame operator is positive and has bounds A and B in (2).

The biorthogonality of the wavelet bases can be expressed as

$$\mathcal{T}^*_{\psi,1}\, \mathcal{T}_{\widetilde{\psi},1} = \mathcal{T}^*_{\widetilde{\psi},1}\, \mathcal{T}_{\psi,1} = \mathrm{id}_{\,L_2(\mathbb{R})} \ , \qquad \mathcal{T}_{\psi,1}\, \mathcal{T}^*_{\widetilde{\psi},1} = \mathcal{T}_{\widetilde{\psi},1}\, \mathcal{T}^*_{\psi,1} = \mathrm{id}_{\,\ell_2(\mathbb{Z}\times\mathbb{Z})} \ .$$

For general $N \in \mathbb{N}$, we therefore have the representation

$$\mathcal{T}^*_{\psi,N}\, \mathcal{T}_{\psi,N} = \mathcal{T}^*_{\psi,1} \left(\mathcal{T}_{\psi,N}\, \mathcal{T}^*_{\widetilde{\psi},1}\right)^* \left(\mathcal{T}_{\psi,N}\, \mathcal{T}^*_{\widetilde{\psi},1}\right) \mathcal{T}_{\psi,1}$$

of the frame operator. The first and last factor on the right hand side can be coded by the usual wavelet decomposition and reconstruction algorithms, hence they have a filter bank realization [8,9]. Without entering more deeply into this well-known area, we next wish to find filter bank realizations for the operators in parentheses.

It is for this purpose that we further restrict to $N = 2^n$, $n \in \mathbb{N}$. The operator $\mathcal{T}_{\psi,N}\, \mathcal{T}^*_{\widetilde{\psi},1}$ is determined by inner products of the form

$$\langle \widetilde{\psi}_{i,\ell}\,,\, \psi_{j,k;N}\rangle = 2^{(i+j)/2} \langle \widetilde{\psi}(2^i(\cdot - 2^{-i}\ell))\,,\, \psi(2^j(\cdot - 2^{-j-n}k))\rangle \ , \qquad i,\, j,\, k,\, \ell \in \mathbb{Z}.$$

These inner products can be evaluated by means of the scaling sequences. The following observation was made in [10, Prop. 5.1].

Lemma 2.1. *Let $N = 2^n$ with $n \in \mathbb{N}$ be given. Then for any $i, j \in \mathbb{Z}$ and any $x = (x_\ell)_{\ell\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ we have*

$$\left(\left\langle \sum_{\ell\in\mathbb{Z}} x_\ell \, \widetilde{\psi}_{i,\ell} \, , \, \psi_{j,k;N} \right\rangle\right)_{k\in\mathbb{Z}} = \begin{cases} 0 & , \text{ if } j \le i-n, \\ S_{j-i}\, x & , \text{ if } j > i-n, \end{cases}$$

where the operators $S_j : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, $j > -n$, are defined in terms of the z-transform by

$$S_j\, X(z) = 2^{n+j/2} \underbrace{G^*\left(z^{2^{n-1}}\right) \cdot \prod_{k=0}^{n-2} H^*\left(z^{2^k}\right)}_{=:\, F(z)} \cdot \left((\widetilde{H}U)^{j+n-1} \circ (\widetilde{G}U)(X)\right)(z) \, . \tag{6}$$

The **proof** can be given in two parts. For $j \le i-n$ one applies the biorthogonality of ψ and $\widetilde{\psi}$ directly, since

$$\langle \widetilde{\psi}_{i,\ell} \, , \, \psi_{j,k;N} \rangle = \langle \widetilde{\psi}_{i,\ell'} \, , \, \psi_{j,0} \rangle \qquad \text{with} \qquad \ell' = \ell - 2^{i-j-n}k \in \mathbb{Z} \, .$$

For $j > i-n$ one uses the scaling relations (4) and (5) consecutively, n-times for the second factor and $(j-i+n)$-times for the first factor. Then the representation of S_{j-i} follows by simple algebraic manipulations and the biorthogonality of the scaling functions ϕ and $\widetilde{\phi}$ on the same scaling level 2^{j+n}.

Let us add some remarks to the previous result. First the multiplication with the function $F(z)$ can be realized recursively, taking into account the increasing number of zero coefficients of each factor of F. This corresponds to n steps of the "algorithme à trous" by Holschneider et.al. [7]. Note, however, that in our case n is fixed (corresponding to the oversampling rate N), while in the algorithme à trous the number of factors of F grows as the total number of scaling levels in the decomposition increases. Secondly, the remaining part of the operator S_j is equal to $j+n$ steps of the reconstruction algorithm for the wavelet $\widetilde{\psi}$. Overall we can conclude, that the operators S_j can be realized by a combination of the filter banks for the reconstruction algorithm (upsampling and convolution) and the algorithme à trous.

The full operator $\mathcal{T}_{\psi,N}\, \mathcal{T}^*_{\widetilde{\psi},1}$ is now obtained by superposition of operators S_j in Lemma 2.1. More precisely, we let $(x_{j,k})_{j,k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}\times\mathbb{Z})$ be the input to the operator, and denote by X_j the individual z-transform for fixed j. Then the above operator produces the output $(y_{j,k})_{j,k\in\mathbb{Z}}$, where each sequence $y_j \in \ell_2(\mathbb{Z})$ is given by its z-transform

$$Y_j(z) = \sum_{i<j+n} S_{j-i}\, X_i(z) = F(z) \cdot \sum_{i<j+n} 2^{n+(j-i)/2} \left((\widetilde{H}U)^{j+n-i-1} \circ (\widetilde{G}U)(X_i)\right)(z) \, .$$

We can thus see, that n steps of the algorithme à trous are needed for each output sequence y_j, while the second factor is the part of the reconstruction algorithm for the wavelet $\widetilde{\psi}$ running between scaling levels 0 to 2^{j+n-1}. Figure 1 demonstrates this relation in terms of the corresponding filter bank.

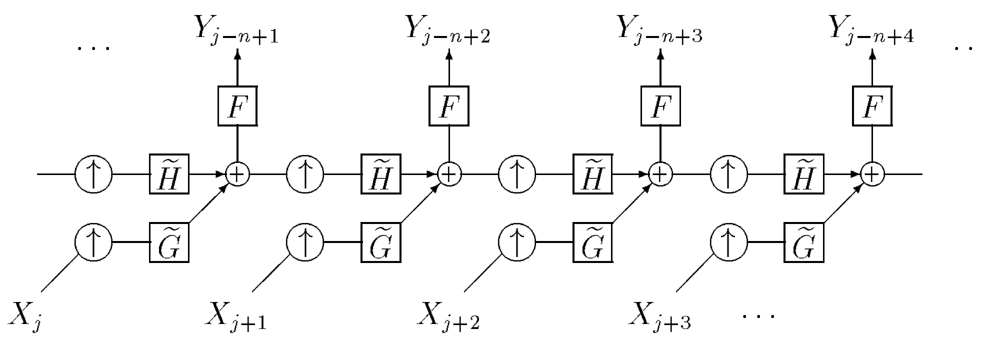

Figure 1. The filter bank realization of the operator $\mathcal{T}_{\psi,N}\,\mathcal{T}^*_{\widetilde{\psi},1}$.

For a simplified analysis of the computational complexity we assume that all filters G, H etc. have the same length ℓ, and that the number of data in X_j is $2^{j-J}M$, with J the highest scaling level. We also truncate the filter bank at level Y_J. Then the convolution of sequences gives a total of $4\ell M$ operations (additions or multiplications) which are needed for the filters $\widetilde{G}$ in Figure 1; furthermore, for the filters $\widetilde{H}$ and F one needs $2^{n+1}\ell M$ and $n2^{n+2}\ell M$ operations, thus leading to an overall complexity of

$$(n2^{n+2} + 2^{n+1} + 4) \cdot \ell M$$

operations for the filter bank. The complexity of related filter banks and faster convolution techniques are described in [11, Chapter 6].

3 Preconditioning of the Frame Operator

The frame decomposition (3) of a function $f \in L_2(\mathbb{R})$ is not unique, in general. The coefficient sequence with minimal ℓ_2-norm, which is a solution of (3), is defined by

$$d = (d_{j,k})_{j,k\in\mathbb{Z}} = \mathcal{T}_{\psi,N}\left(\mathcal{T}^*_{\psi,N}\,\mathcal{T}_{\psi,N}\right)^{-1} f. \qquad (7)$$

In the terminology of linear least squares problems, the operator on the right hand side is the generalized inverse of $\mathcal{T}^*_{\psi,N}$. In order to use our filter bank realization of Section 2, we reformulate (7) as

$$d = \left(\mathcal{T}_{\psi,N}\,\mathcal{T}^*_{\widetilde{\psi},1}\right)\,\left(\left(\mathcal{T}_{\psi,N}\,\mathcal{T}^*_{\widetilde{\psi},1}\right)^*\,\left(\mathcal{T}_{\psi,N}\,\mathcal{T}^*_{\widetilde{\psi},1}\right)\right)^{-1}\,\mathcal{T}_{\widetilde{\psi},1}\,(f).$$

Hence an iterative procedure for the inversion of the positive definite operator

$$\left(\mathcal{T}_{\psi,N}\,\mathcal{T}^*_{\widetilde{\psi},1}\right)^*\,\left(\mathcal{T}_{\psi,N}\,\mathcal{T}^*_{\widetilde{\psi},1}\right):\ \ell_2(\mathbb{Z}\times\mathbb{Z}) \to \ell_2(\mathbb{Z}\times\mathbb{Z}) \qquad (8)$$

is needed. It is important to note that each step of the iteration has a computational complexity which is a multiple of N times the number of (discretized) data of f, by the result of Section 2. Therefore fast algorithms can only be expected if the condition number of the operator in (8) is close to 1 or if very good preconditioners can be found, such that only 1 or 2 steps of the iteration provide sufficient accuracy. We will construct such preconditioners in the special situation of Section 2. This can be done only after a precise analysis of the operator in (8).

We first recall some properties of the *transfer operator* which serves as a tool in wavelet theory in order to characterize convergence and smoothness of solutions of the so-called refinement equation [4]. Let $(w_k)_{k\in\mathbb{Z}}$ be a finite sequence with z-transform $W(z)$. With our notation of operators in the z-transform domain we define

$$T_W(X)(z) = U^*\ (W(z)X(z) + W(-z)X(-z))\ , \qquad x \in \ell_2(\mathbb{Z}).$$

This operator maps the set E of all Laurent polynomials (which are the z-transforms of all sequences with finite support) into itself. The following result is contained in [4].

Proposition 3.1. *Let ϕ be a compactly supported function in $L_2(\mathbb{R})$ which satisfies (4) where $h_k = 0$ for all $|k| > K$, and let us assume that the translates $(\phi(\cdot - k);\ k \in \mathbb{Z})$ define a Riesz basis. Then the transfer operator $T := T_{H^*H}$ has the invariant subspace*

$$E_{2K} := \left\{ X(z) = \sum_{k=-2K}^{2K} x_k z^{-k};\ x_k \in \mathbb{C} \text{ for } |k| \le 2K,\ X(1) = 0 \right\}\ ,$$

and all eigenvalues of $T|_{E_{2K}}$ have modulus strictly less than 1. Furthermore, for any $X \in E$ there exists $\nu \in \mathbb{N}$ such that $T^\nu X \in E_{2K}$. In particular, the operator $(\mathrm{id} - T)|_E$ is invertible.

The transfer operator naturally appears in the representation of $S_j^* S_j$, where S_j is defined in (6). By straightforward computations one can find that [10, Sect. 1.3]

$$S_j^* S_j\ X(z) = T_{\widetilde{G}^*\widetilde{G}}\, T_{\widetilde{H}^*\widetilde{H}}^{j+n-1}\, (2^n F^* F)(z) \cdot X(z), \qquad j > -n.$$

Note that F (hence F^*F) in (6) belongs to E, since the sequences h and g have finite supports and $G(1) = 0$. By Proposition 3.1 there exists a constant $0 < \rho < 1$ such that

$$\|S_j\| = \|S_j^* S_j\|^{1/2} \le \text{const}\ \rho^{j+n-1},$$

which shows that $\sum_{j>-n} \|S_j\|$ is finite. Hence the formal calculus of series is allowed in order to represent the operator (8) as

$$\left(T_{\psi,N}\, T_{\widetilde{\psi},1}^*\right)^* \left(T_{\psi,N}\, T_{\widetilde{\psi},1}^*\right)\ (x_{j,k})_{j,k\in\mathbb{Z}} = \left(\sum_{i\in\mathbb{Z}} \Big(\underbrace{\sum_{k>-n} S_k^*\, S_{i+k}}_{=:\ R_i}\Big) X_{j-i}\right)_{j\in\mathbb{Z}} . \qquad (9)$$

The same algebraic manipulations as in [10, Sect. 1.3] lead to the following representation of the operators R_i on the z-transform domain.

Lemma 3.2. *The operators R_i in (9) have the form*

$$R_0\, X(z) \;=\; \underbrace{T_{\widetilde{G}^*\widetilde{G}}\ \left((\mathrm{id} - T_{\widetilde{H}^*\widetilde{H}})|_E\right)^{-1}\ (2^n F^* F)\ (z)}_{=:\ C(z)} \cdot X(z),$$

$$R_i\, X(z) \;=\; 2^{i/2}\ \underbrace{T_{\widetilde{G}^*\widetilde{H}}\ \left((\mathrm{id} - T_{\widetilde{H}^*\widetilde{H}})|_E\right)^{-1}\ (2^n F^* F)\ (z)}_{=:\ D(z)} \cdot \left((\widetilde{H}U)^{i-1} \circ (\widetilde{G}U)(X)\right)(z)$$

for $i > 0$ *and* $R_i = R_{-i}^*$ *for all* $i < 0$.

This is an explicit form of the operator in (9), since the polynomials C and D can be computed as follows. First we let

$$\begin{aligned}\widetilde{D}(z) &= \left((\mathrm{id} - T_{\widetilde{H}^*\widetilde{H}})|_E\right)^{-1}(2^n F^*F)(z) \\ &= \left((\mathrm{id} - T_{\widetilde{H}^*\widetilde{H}})|_{E_{2K}}\right)^{-1}\left(T^{\nu}_{\widetilde{H}^*\widetilde{H}}(2^n F^*F)\right)(z) + \sum_{j=0}^{\nu-1}\left(T^{j}_{\widetilde{H}^*\widetilde{H}}(2^n F^*F)\right)(z)\end{aligned}$$

where ν is so large that $T^{\nu}_{\widetilde{H}^*\widetilde{H}}(F^*F) \in E_{2K}$. Finding the inverse of $(\mathrm{id} - T_{\widetilde{H}^*\widetilde{H}})|_{E_{2K}}$ is a finite dimensional problem of size $\dim E_{2K} = 4K$. Finally, we have that

$$C = T_{\widetilde{G}^*\widetilde{G}}(\widetilde{D}), \qquad D = T_{\widetilde{G}^*\widetilde{H}}(\widetilde{D}).$$

Equation (9) shows that the operator in (8) is a generalized Laurent operator; i.e. it is given by the biinfinite operator-valued matrix

$$\mathcal{R} = \begin{pmatrix} \ddots & \ddots & \ddots & \cdots & \\ \ddots & \boxed{R_0} & R_1^* & R_2^* & \cdots \\ \ddots & R_1 & \boxed{R_0} & R_1^* & \ddots \\ \cdots & R_2 & R_1 & \boxed{R_0} & \ddots \\ & \cdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{10}$$

where the box denotes the diagonal of the biinfinite matrix. This representation acts on the sequence of z-transforms $(X_j)_{j\in\mathbb{Z}}$ of an input sequence $(x_{j,k})_{j,k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}\times\mathbb{Z})$. Since $\mathcal{R}$ is self-adjoint and positive definite, the operator R_0 is also self-adjoint and positive definite. By Lemma 3.2 this means that $C(z)$ is real for $|z| = 1$ and

$$0 < \lambda = \inf_{|z|=1} C(z) \le \sup_{|z|=1} C(z) = \Lambda < \infty.$$

Moreover, the condition number of $\mathcal{R}$ is bounded from above and below by

$$\frac{\Lambda}{\lambda} \le \mathrm{cond}(\mathcal{R}) \le \frac{\Lambda + 2\sum_{i>1}\|R_i\|}{\lambda - 2\sum_{i>1}\|R_i\|}, \tag{11}$$

if the denominator on the right hand side is still positive.

Preconditioning of the operator (8) is therefore the same as preconditioning of the biinfinite matrix $\mathcal{R}$. This problem is quite subtle, since we are only interested in preconditioners $\mathcal{M}$ which can be fast evaluated, e.g. which have a filter bank realization with short filters. We propose a triangular preconditioner

$$\mathcal{M} = \begin{pmatrix} \ddots & & & & \\ \ddots & \boxed{M_0} & & \mathbf{0} & \\ \ddots & M_1 & \boxed{M_0} & & \\ \cdots & M_2 & M_1 & \boxed{M_0} & \\ & \cdots & \ddots & \ddots & \ddots \end{pmatrix}$$

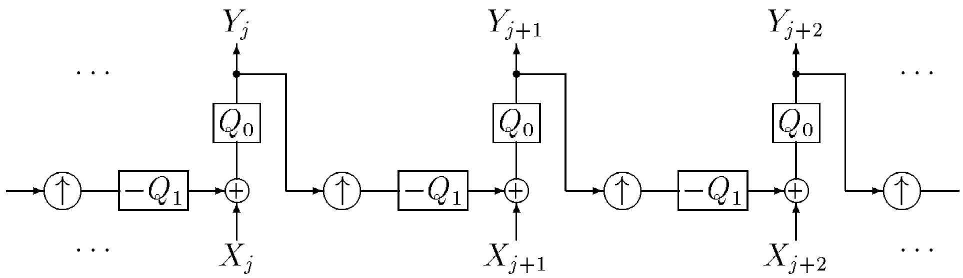

Figure 2. The filter bank realization of the operator $\mathcal{L}^{-1}$.

where each operator M_i has the form

$$M_i X(z) = 2^{i/2}\, Q_i(z)\, U^i\, X(z), \qquad i \geq 0, \tag{12}$$

with short sequences $q_i = (q_{i,k})_{k\in\mathbb{Z}}$. Furthermore, the q_i should satisfy certain properties which are needed for a filter bank realization of $\mathcal{M}$ and $\mathcal{M}^*$.

In order to find good preconditioners of this type we use an idea for the Cholesky factorization of banded real Laurent matrices in [3]. Certain modifications are needed for our operator-valued matrix. First we chop off diagonals R_i, R_i^* with $i \geq 2$ in (10), since the norm of these operators is negligible in most practical examples of biorthogonal wavelet bases. This is explained more precisely in [10, Sect. 3.2] and is due to the fact, that scaling levels j and $j+i$ are nearly orthogonal if $i \geq 2$. Then we compute the biinfinite Cholesky factors

$$\mathcal{L} = \begin{pmatrix} \ddots & & & & \\ \ddots & \boxed{L_0} & & \mathbf{0} & \\ & L_1 & \boxed{L_0} & & \\ & & L_1 & \boxed{L_0} & \\ & \mathbf{0} & & \ddots & \ddots \end{pmatrix}$$

and $\mathcal{L}^*$ of the reduced matrix. It turns out, that such factors (which are not unique in contrast to the finite dimensional case, see [3]) can be constructed which are diagonally dominant and with operators L_0, L_1 as in (12). Finally, for a fast evaluation of the inverse of $\mathcal{L}$ (and of $\mathcal{L}^*$) we use short filters Q_0 for L_0^{-1} and Q_1 for L_1 and work through the usual back- and forward substitution for solving triangular systems of linear equations. This corresponds to a filter bank algorithm which is described in Figure 2.

Note that there is a major difference between the filter banks in Figures 1 and 2. The data sequences X_j and Y_j in Figure 2 correspond to the usual sampling in the time domain, while Y_j in Figure 1 is related to oversampling by 2^n. A similar analysis as in the previous section and truncation at level Y_J give a total of

$$(4k_0 + 2k_1) \cdot M$$

operations; here k_0, k_1 denote the lengths of the filters Q_0, Q_1 respectively, and the number of data in X_j is $2^{j-J}M$. Hence, assuming that k_0, k_1 and ℓ are equal, the complexity of the filter bank in Figure 1 exceeds the preconditioning by a factor $n2^{n+1}/3$. This shows that the additional work spent on preconditioning is small compared to the computation of the frame operator.

Let us now present the main result for the computation of the Cholesky factor $\mathcal{L}$.

Theorem 3.3. *Let R_0, R_1 be as in Lemma 3.2, and let us assume that*

$$\|R_0^{-1/2}\,R_1\,R_0^{-1/2}\| \le q < 1/2. \tag{13}$$

Then the sequence $p^{(\nu)} \in \ell_1(\mathbb{Z})$, $\nu \ge 0$, which is defined by the z-transforms

$$P^{(0)} = C, \qquad P^{(\nu+1)} = C - T_{\widetilde{G}^*\widetilde{G}}\left(D^* D/P^{(\nu)}\right), \qquad \nu \ge 0, \tag{14}$$

converges in $\ell_1(\mathbb{Z})$. Furthermore, the functions $P^{(\nu)}$ are real-valued and satisfy

$$(1-2q^2)C(z) \le P^{(\nu)}(z) \le C(z), \qquad \nu \ge 0, \quad |z| = 1. \tag{15}$$

The above result is a reformulation of [10, Theorem 5.13] in terms of the transfer operator. The proof in [10] consists of showing that the iteration satisfies the assumptions of Banach's fixed point theorem. Note also that $p^{(\nu)} \in \ell_1(\mathbb{Z})$ holds by Wiener's theorem, as a consequence of the estimate (15).

The limit $p \in \ell_1(\mathbb{Z})$ of the iteration (14) satisfies

$$P = C - T_{\widetilde{G}^*\widetilde{G}}\left(D^* D/P\right) .$$

This can be used in order to construct the Cholesky factorization $\mathcal{L}^*\mathcal{L}$ of the tridiagonal band of $\mathcal{R}$. We let

$$\begin{aligned} L_0\,X(z) &:= \sqrt{P(z)}\,X(z), \\ L_1\,X(z) &:= L_0^{-1}\,R_1\,X(z) = \frac{2^{1/2}D(z)\,\widetilde{G}(z)}{\sqrt{P(z)}}\cdot X(z^2). \end{aligned}$$

Then the product $\mathcal{L}^*\mathcal{L}$ is a biinfinite tridiagonal Laurent matrix. The operator on its diagonal is

$$(L_0^2 + L_1^*\,L_1)X(z) = \left(P(z) + T_{\widetilde{G}^*\widetilde{G}}\left(D^* D/P\right)(z)\right)\cdot X(z) = C(z)\,X(z).$$

Moreover, the lower and upper diagonals are $L_0\,L_1 = R_1$ and $L_1^*\,L_0 = R_1^*$, respectively. This shows that the iteration (14) leads to a Cholesky decomposition of a tridiagonal band of $\mathcal{R}$.

As our last step we wish to explain how the filter bank realization of $\mathcal{L}^{-1}$ in Figure 2 and of its adjoint are obtained. We can choose short filters q_0 and q_1 by trigonometric interpolation or approximation such that

$$\epsilon_0 := \sup_{|z|=1}\left|Q_0(z) - 1/\sqrt{P(z)}\right|, \qquad \epsilon_1 := \sup_{|z|=1}\left|Q_1(z) - D(z)\widetilde{G}(z)/\sqrt{P(z)}\right|$$

are both small. In our practical examples we performed only 2 steps of the iteration in Theorem 3.3, thus used $P^{(2)}$ instead of the limit P for the definition of the preconditioner. Then we found q_0, q_1 by trigonometric interpolation at 128 data points on the unit circle and chopping off high order coefficients.

4 Example

As an example we choose the semi-orthogonal cubic spline wavelet. (The dual wavelet and scaling function have infinite supports, but finite approximations of the sequences $\widetilde{h}$, $\widetilde{g}$ of lengths 25 are used.) The filters Q_0 and Q_1 are found by the technique of Theorem 3.3. They are filters of lengths 25 and 19, respectively. With preconditioning we obtain a condition number of 1.01, while the operator $\mathcal{R}$ itself has a condition number ≥ 5.3 by (11).

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