Affine Frames and Multiresolution

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Abstract. We use generalized Laurent operators for an extended study of multivariate affine frames. Our main concern are frames which are generated by multiresolution with a single scaling function. No stability constraints are posed on the translates of the scaling function. The connection to the transfer operator is worked out for these families. Moreover, we give a new representation of the lifting scheme. This provides more insight into the problem of controlling the stability of the wavelet bases during the process of lifting. The relation to generalized Toeplitz operators is discussed.

1. Introduction

The interest in wavelet bases over the past two decades has naturally extended to a special type of frames, which are called *wavelet frames* or *affine frames*. Their definition by dilation and translation is similar to wavelet bases, but they allow redundancy and are therefore more flexible with respect to certain applications. Various aspects of such frames are discussed in [1,4,5,9,10,11,17,18,19,20].

Our general setup is as follows. We consider families in $L_2(\mathbb{R}^d)$, which are generated from a finite subset $\Psi = \{\psi_i;\ i \in I\}$ of $L_2(\mathbb{R}^d)$ by means of dilation and translation. These operations are defined by two real and invertible $d \times d$ matrices M and L. M is supposed to be expansive, i.e. all eigenvalues of M have absolute value larger than 1, and L defines a lattice $\mathcal{L} = L\mathbb{Z}^d$. Then we let

$$X = (\Psi, M, \mathcal{L}) := \{|\det M|^{j/2}\psi_i(M^j \cdot -Lk);\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d,\ i \in I\}. \qquad (1)$$

Without further assumptions we refer to X as an *affine family*. Our treatment is based on two operators

$$\mathcal{T}_X : L_2(\mathbb{R}^d) \longrightarrow \ell_2(I \times \mathbb{Z} \times \mathbb{Z}^d), \qquad f \mapsto \left(\langle f, \eta\rangle_{L_2(\mathbb{R}^d)}\right)_{\eta \in X}, \qquad (2)$$

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which will be called the *analysis operator*, and its adjoint, the *synthesis operator*

$$\mathcal{T}_X^* : \ell_2(I \times \mathbb{Z} \times \mathbb{Z}^d) \longrightarrow L_2(\mathbb{R}^d), \qquad \mathbf{d} = (d_\eta)_{\eta \in X} \mapsto \sum_{\eta \in X} d_\eta \, \eta \; . \tag{3}$$

This terminology is used since the coefficients of $\mathcal{T}_X f$ contain localized information of f and its Fourier transform $\widehat{f}$. The affine family X is an *affine frame*, if the operator T_X is bounded from above and below, so there exist two constants $A, B > 0$ such that

$$A||f||_{L_2(\mathbb{R}^d)} \leq ||T_X f||_{\ell_2(I \times \mathbb{Z} \times \mathbb{Z}^d)} \leq B||f||_{L_2(\mathbb{R}^d)} \tag{4}$$

holds. Hence affine frames are total in $L_2(\mathbb{R}^d)$, but can be linearly dependent. As a consequence of the lower estimate in (4), they allow stable representations of any function $f \in L_2(\mathbb{R}^d)$. Finding frame decompositions of a given function f can be viewed as an infinite least squares problem: there always exists a unique solution $\mathbf{d} \in \ell_2(I \times \mathbb{Z} \times \mathbb{Z}^d)$ which has minimal ℓ_2-norm and gives $f = T_X^* \mathbf{d}$. We call X a *Bessel family* if the upper bound in (4) holds.

In this paper we continue our work in [19] and describe several aspects of affine frames. The central technique, which is described in section 2, is a combination of analysis and synthesis operators of two affine frames in such a way that a generalized Laurent operator is obtained. Several new results have been found in this way, see [19,20,21]. In section 3 we deal with the special case, where the family Ψ is constructed by multiresolution. This issue appears in the work of [1,5,17,18,21]. We show that fast pyramidal algorithms can be used for the frame operators instead of numerical integration. Furthermore, the case of compactly supported generators deserves special attention. The connection to the transfer operator, which is one of the main tools for regularity estimates of scaling functions [8,10,12,14,23], is worked out here. The norm estimate of Theorem 3 shows that all generalized Laurent operators which come up in this way have a nice structure. A new representation of the lifting scheme [3,22] is presented in section 4. It amounts to a lower triangular operator matrix whose diagonal is the identity operator. This gives some new insight into the yet unsolved problem under which conditions the lifting scheme produces a new pair of biorthogonal wavelet bases. Although we cannot give simple sufficient conditions, we believe that further research based on techniques from operator algebra might lead to interesting results in this area.

Let us shortly fix some of our notations. M^* denotes the transpose of M. We identify sequences $\mathbf{c} \in \ell_2(\mathbb{Z}^d)$ and the corresponding Fourier series $C \in L_2(Q)$ without further notice. Here $Q = (-\pi, \pi)^d$. We also use the bracket product

$$[\, f \, , \, g \,] = \sum_{k \in \mathbb{Z}^d} f(\, . + 2\pi k) \, \overline{g}(\, . + 2\pi k)$$

which defines a function in $L_1(Q)$ whenever $f, g \in L_2(\mathbb{R}^d)$. The Fourier transform is defined as $\widehat{f} = \int_{\mathbb{R}^d} f(x) e^{-ix \cdot} \, dx$.

2. Operator Analysis

Given a scaling matrix $M \in \mathbb{R}^{d\times d}$ and two finite families $\Psi = \{\psi_i;\, i \in I\}$ and $\Theta = \{\theta_i;\, i \in I\} \subset L_2(\mathbb{R}^d)$, we define the affine families

$$X = (\Psi, M, \mathcal{L}_X) \qquad \text{and} \qquad Y = (\Theta, M, \mathcal{L}_Y) \tag{5}$$

as in (1). In most cases of interest we will choose $\mathcal{L}_Y = \mathbb{Z}^d$. If both X and Y are Bessel families, their analysis operators (2) are bounded. Our study of affine frames is based on the following principle.

Proposition 1. *If the families X and Y in (5) are Bessel families, then the combination of analysis and synthesis operators*

$$\mathcal{T}_X\, \mathcal{T}_Y^* : \ell_2(I \times \mathbb{Z} \times \mathbb{Z}^d) \longrightarrow \ell_2(I \times \mathbb{Z} \times \mathbb{Z}^d)$$

is a generalized Laurent operator; i.e. it is a bounded linear operator which commutes with the bilateral shift on $\ell_2(H)$ where $H = \ell_2(I \times \mathbb{Z}^d)$.

Let us recall, for the reader's convenience, the definition of the bilateral shift

$$\mathcal{U} : \ell_2(H) \longrightarrow \ell_2(H), \qquad \mathcal{U}\left((\mathbf{c}_j)_{j\in\mathbb{Z}}\right) = (\mathbf{c}_{j-1})_{j\in\mathbb{Z}} \ .$$

Here the biinfinite vector $(\mathbf{c}_j)_{j\in\mathbb{Z}}$ is an element of $\ell_2(H)$, hence its entries $\mathbf{c}_j$ belong to H and the series $\sum_{j\in\mathbb{Z}} \|\mathbf{c}_j\|_H^2$ is finite. The proof of Proposition 1 is given in [19,20] and it relies on the scaling invariance of both families X and Y with respect to the same scaling matrix M.

The choice of two different generating families Ψ and Θ introduces much more freedom for the study of affine frames than the usual approach in the literature, where only one family is considered, see [10,17]. A special form of this can be found in [23] where regularity estimates for scaling functions are investigated.

Two standard techniques from operator theory are useful which are connected to generalized Laurent operators. Their detailed description can be found in [2,13]. First we can consider $\mathcal{T}_X\, \mathcal{T}_Y^*$ as a biinfinite operator matrix

$$\mathcal{T}_X\, \mathcal{T}_Y^* = \begin{pmatrix} \ddots & \ddots & \ddots & \cdots & \\ \ddots & \boxed{S_0} & S_{-1} & S_{-2} & \cdots \\ \ddots & S_1 & \boxed{S_0} & S_{-1} & \ddots \\ \cdots & S_2 & S_1 & \boxed{S_0} & \ddots \\ & \cdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{6}$$

where each S_j, $j \in \mathbb{Z}$, is a bounded linear operator on H. (As usual, we put frames around the entries on the main diagonal of biinfinite matrices. The same will be done for the 0-th entry of biinfinite vectors.) The exact definition of S_j is obtained in the following way. For arbitrary $\mathbf{c} \in H$ we let

$$\bar{\mathbf{c}} := (\ldots, 0, 0, \boxed{\mathbf{c}}, 0, 0, \ldots) \in \ell_2(H).$$

Then the equation

$$\mathcal{T}_X\, \mathcal{T}_Y^*(\bar{\mathbf{c}}) = (\ldots, S_{-2}\mathbf{c}, S_{-1}\mathbf{c}, \boxed{S_0\mathbf{c}}, S_1\mathbf{c}, S_2\mathbf{c}, \ldots)$$

defines all operators S_j, $j \in \mathbb{Z}$. Boundedness of $\mathcal{T}_X\, \mathcal{T}_Y^*$ implies that $\sum_{j\in\mathbb{Z}} S_j^*\, S_j$ is a bounded selfadjoint operator on H. Vice versa, this condition on S_j, $j \in \mathbb{Z}$, is not sufficient for the boundedness of $\mathcal{T}_X\, \mathcal{T}_Y^*$, in general. Instead, the relations

$$\Big\| \sum_{j\in\mathbb{Z}} S_j^*\, S_j \Big\|_{\mathcal{B}(H)}^{1/2} \le \|\mathcal{T}_X\, \mathcal{T}_Y^*\|_{\mathcal{B}(\ell_2(H))} \le \sum_{j\in\mathbb{Z}} \|S_j\|_{\mathcal{B}(H)} \tag{7}$$

can be proven in the same way as for usual Laurent matrices (where $H = \mathbb{C}$). We will see in section 3, however, that the finiteness of the left and the right hand sides in (7) are equivalent for certain frame operators.

The second tool for generalized Laurent operators is to build from $\mathcal{T}_X\, \mathcal{T}_Y^*$ the operator-valued function

$$F(z) := \sum_{j\in\mathbb{Z}} z^j S_j, \qquad z \in \mathbb{C} \quad \text{with} \quad |z| = 1. \tag{8}$$

This defines a (weakly) measurable function, which maps the unit circle $\mathbb{T}$ to the Banach algebra $\mathcal{B}(H)$ of bounded linear operators on H. A central theorem about generalized Laurent operators [2,page 235] states that

$$\|\mathcal{T}_X\, \mathcal{T}_Y^*\|_{\mathcal{B}(\ell_2(H))} = \operatorname*{ess\,sup}_{|z|=1} \|F(z)\|_{\mathcal{B}(H)}\ , \tag{9}$$

and that the mapping of generalized Laurent operators to $L_\infty(\mathbb{T}, \mathcal{B}(H))$ is one-to-one. Note that the second inequality in (7) can be easily deduced from (9).

Finally we mention the explicit representation of S_j in (6), which is contained in [20].

Proposition 2. *Let X and Y in (5) be Bessel families where $\mathcal{L}_Y = \mathbb{Z}^d$. Then each operator S_j in (6) is a square operator matrix of size $|I|$ with entries*

$$S_j^{(m,n)}(C) = \frac{|\det M|^{j/2}}{|\det L_X|} \Big[\, C(M^{*j}\, L_X^{*\,-1}\cdot)\, \widehat{\theta}_n(M^{*j}\, L_X^{*\,-1}\cdot)\,,\ \widehat{\psi}_m(L_X^{*\,-1}\cdot)\,\Big]$$

for all $m, n \in I$ and $C \in L_2(Q)$.

3. Affine Frames by Multiresolution

We further deal with a special situation where both families Ψ and Θ are generated by scaling functions in $L_2(\mathbb{R}^d)$. This issue is important for numerical applications of frames. Several papers address this topic, see [1,5,17,18,21]. Let us first give the needed specifications.

The scaling matrix M is an integer matrix. We denote by Γ_j a complete set of representers of the cosets of $M^{*j}\mathbb{Z}^d/\mathbb{Z}^d$. Moreover, we are given two functions ϕ, $\rho \in L_2(\mathbb{R}^d)$ which are compactly supported and satisfy the scaling relations

$$\widehat{\phi}(M^* \cdot) = m_0\widehat{\phi}, \qquad \widehat{\rho}(M^* \cdot) = \widetilde{m}_0\widehat{\rho}$$

a.e. in $\mathbb{R}^d$ where m_0 and $\widetilde{m}_0$ are trigonometric polynomials in $\mathbb{R}^d$. In this setting we do not require stability or existence of frame bounds for the translates of ϕ or ρ. This is in accordance with [17], but differs completely from the frame multiresolution analysis which was introduced in [1]. Finally, the *wavelets* are defined by their corresponding scaling relations

$$\widehat{\psi}_i(M^* \cdot) = m_i\widehat{\phi}, \qquad \widehat{\theta}_i(M^* \cdot) = \widetilde{m}_i\widehat{\rho}, \qquad i \in I,$$

with trigonometric polynomials m_i, $\widetilde{m}_i$. For later use we let $\psi_0 = \phi$ and $\theta_0 = \rho$. Note that we restrict our attention to finite masks for the scaling functions and wavelets. Further investigations are needed for more general cases, see e.g. the framework in [7]. Unlike the construction of wavelet bases from multiresolution analysis, the cardinality of I is not connected to the determinant of the scaling matrix. The affine families to be considered are $X = (\Psi, M, \mathbb{Z}^d)$ and $Y = (\Theta, M, \mathbb{Z}^d)$.

The characterization of tight affine frames X which are defined by multiresolution from a scaling function ϕ was recently obtained by Ron and Shen [17,Theorem 6.5] under mild assumptions on the Fourier transform of ϕ. In this section we present a connection of the frame operator and the *transfer operator* in [6,8,12,14]. This is the mapping T_u on $L_2(Q)$ with

$$T_u(f) = \sum_{k\in\Gamma_1} u(M^{*-1}(\cdot + 2\pi k))\ f(M^{*-1}(\cdot + 2\pi k))$$

where u is a given function in $L_\infty(Q)$. Note that its adjoint is given by

$$T_u^*(f) = |\det M|\,\overline{u} \cdot f(M^* \cdot), \qquad f \in L_2(Q),$$

which corresponds to upsampling and convolution of the corresponding sequences of Fourier coefficients. The representation of S_j in Proposition 2 can be transformed by means of the scaling relations. With $\mu = |\det M|$ and by simple algebraic manipulations we obtain

$$S_0^{(i,n)}(C)(\xi) = [\,\widehat{\theta}_n\,,\ \widehat{\psi}_i\,](\xi) \cdot C(\xi),$$

$$S_j^{(i,n)}(C)(\xi) = \mu^{j/2} \widetilde{m}_n(M^{*j-1}\xi) \prod_{\nu=0}^{j-2} \widetilde{m}_0(M^{*\nu}\xi)\, [\widehat{\rho}\,,\, \widehat{\psi}_i](\xi) \cdot C(M^{*j}\xi)$$

for $j > 0$, and

$$\begin{aligned} S_j^{(i,n)}(C)(\xi) &= \mu^{j/2} \sum_{k \in \Gamma_{|j|}} \left[\widehat{\theta}_n\,,\, \widehat{\psi}_i(M^{*|j|}\cdot) \right] (M^{*j}(\xi + 2\pi k)) \cdot C(M^{*j}(\xi + 2\pi k)) \\ &= \mu^{j/2} T_{\overline{m_i}} \circ T_{\overline{m_0}}^{|j|-1} \left([\widehat{\theta}_n\,,\, \widehat{\phi}] \cdot C \right)(\xi) \end{aligned}$$

for $j < 0$.

3.1 Pyramidal algorithm

The above representation of the entries of the square operator matrix S_j gives rise to efficient numerical algorithms. For this purpose we recall that the functions m_i, $\widetilde{m}_i$, $i \in I' = I \cup \{0\}$, are trigonometric polynomials. Moreover, we can conclude, by means of the Poisson Summation Formula, that all functions $[\widehat{\theta}_n, \widehat{\psi}_i]$, $i, n \in I'$, are trigonometric polynomials, too. Hence S_0 represents a matrix filter which acts on a vector $\mathbf{c} \in H = \ell_2(I \times \mathbb{Z}^d)$ and has finite filters in each entry. S_j for $j > 0$ consists of FIR filters and upsampling, while S_j for $j < 0$ uses downsampling instead. The overall operator

$$\mathcal{T}_X\, \mathcal{T}_Y^*(\mathbf{c}_j)_{j\in\mathbb{Z}} = \left(\sum_{k\in\mathbb{Z}} S_k\, \mathbf{c}_{j-k} \right)_{j\in\mathbb{Z}}, \qquad (\mathbf{c}_j)_{j\in\mathbb{Z}} \in \ell_2(H),$$

can be decomposed into three parts. The first part is the filtering

$$\left(\mathbf{c}_j\right)_{j\in\mathbb{Z}} \mapsto \left(S_0 \mathbf{c}_j\right)_{j\in\mathbb{Z}}.$$

The second part

$$\left(\mathbf{c}_j\right)_{j\in\mathbb{Z}} \mapsto \left(\sum_{k>0} S_k\, \mathbf{c}_{j-k} \right)_{j\in\mathbb{Z}}$$

corresponds to Mallat's pyramidal algorithm [15] with filters $\widetilde{m}_i$, $i \in I'$, which is often called the reconstruction algorithm for the wavelets $\widetilde{\psi}_i$. The only difference consists in the final filtering with FIR filters $[\widehat{\rho}, \widehat{\psi}_i]$ in each component. The third part

$$\left(\mathbf{c}_j\right)_{j\in\mathbb{Z}} \mapsto \left(\sum_{k<0} S_k\, \mathbf{c}_{j-k} \right)_{j\in\mathbb{Z}}$$

uses $[\widehat{\theta}_n, \widehat{\phi}]$ as initial filters in each component $\mathbf{c}_j$, $j \in \mathbb{Z}$, and then applies the decomposition algorithm with filters $\overline{m_i}$, $i \in I'$. Hence the computational complexity for the evaluation of $\mathcal{T}_X\, \mathcal{T}_Y^*$ is comparable to the sum of complexities for the wavelet reconstruction and decomposition with the specified filters.

3.2 Operator norms

In order to find exact expressions for the operator norm of S_j, $j \in \mathbb{Z}$, we further build the selfadjoint operators $S_j^* S_j$ for $j > 0$ and $S_j S_j^*$ for $j < 0$. The products are chosen in this way in order to obtain pure multiplication operators; more precisely, the computation with adjoints as in [19] shows that these are square operator matrices of size $|I|$ with entries

$$(S_j^* S_j)^{(\nu,n)}(C)(\xi) = T_{\overline{\widetilde{m}_\nu} \widetilde{m}_n} \circ T^{j-1}_{|\widetilde{m}_0|^2} \Big(\sum_{i \in I} \Big| [\,\widehat{\rho}\,,\, \widehat{\psi}_i\,] \Big|^2 \Big)(\xi) \cdot C(\xi) \tag{10}$$

for $j > 0$, and

$$(S_j S_j^*)^{(\nu,n)}(C)(\xi) = T_{\overline{m_\nu} m_n} \circ T^{|j|-1}_{|m_0|^2} \Big(\sum_{i \in I} \Big| [\,\widehat{\theta}_i\,,\, \widehat{\phi}\,] \Big|^2 \Big)(\xi) \cdot C(\xi)$$

for $j < 0$, each of them acting on $L_2(Q)$. Note that the product of transfer operators in these formulas is applied to the functions

$$g := \sum_{i \in I} \Big| [\,\widehat{\rho}\,,\, \widehat{\psi}_i\,] \Big|^2, \qquad h := \sum_{i \in I} \Big| [\,\widehat{\theta}_i\,,\, \widehat{\phi}\,] \Big|^2,$$

but not to C. The same argument as in the previous section shows that g and h are trigonometric polynomials. Hence the vector spaces

$$\begin{aligned} E_g &= \mathrm{Span}\,\{T^j_{\widetilde{u}}(g);\ j \geq 0\} \qquad &&\text{with} \qquad \widetilde{u} = |\widetilde{m}_0|^2, \\ E_h &= \mathrm{Span}\,\{T^j_u(h);\ j \geq 0\} \qquad &&\text{with} \qquad u = |m_0|^2 \end{aligned}$$

are finite dimensional spaces of trigonometric polynomials, see [8,Prop. 3.1] for more details. The Perron-Frobenius theory for non-negative operators on finite dimensional space leads to the following result.

Theorem 3. *Let X and Y be Bessel families defined by multiresolution with compactly supported scaling functions and wavelets. Then the operators S_j in (6) satisfy*

$$\sum_{j \in \mathbb{Z}} \|S_j\|_{\mathcal{B}(H)} < \infty. \tag{11}$$

In particular, formal matrix calculus is allowed for multiplication and transposition of the biinfinite operator matrices in (6).

Proof: We show a stronger result, namely that (11) holds if and only if

$$\Big\| \sum_{j > 0} S_j^* S_j \Big\|_{\mathcal{B}(H)} < \infty \qquad \text{and} \qquad \Big\| \sum_{j < 0} S_j S_j^* \Big\|_{\mathcal{B}(H)} < \infty. \tag{12}$$

Then the assertion follows from (7), since boundedness of $\mathcal{T}_X \mathcal{T}_Y^*$ implies that the first series in (12) is finite, and boundedness of $\mathcal{T}_Y \mathcal{T}_X^*$ gives finiteness of the second series.

Let us now prove the equivalence of (11) and (12). Obviously, (11) implies (12) by the triangle inequality and since $\ell_1(\mathbb{Z}) \subset \ell_2(\mathbb{Z})$. For the converse implication we only deal with the first series in (12), since the second series is treated analogously. We also omit the trivial case where all $\widetilde{m}_n$, $n \in I$, are identically zero. Let $\nu = n \in I$ be chosen such that $\widetilde{m}_n$ is a nontrivial trigonometric polynomial. Then (10) and (12) imply that

$$\left\| \sum_{j>0} (S_j^* S_j)^{(n,n)}(C) \right\|_{L_2(Q)} = \left\| \sum_{j>0} T_{|\widetilde{m}_n|^2} \circ T_{\widetilde{u}}^{j-1}(g) \cdot C \right\|_{L_2(Q)} < \infty$$

for all $C \in L_2(Q)$ with norm 1. Since the last operator is a multiplication on $L_2(Q)$, we further conclude that

$$\left\| \sum_{j>0} T_{|\widetilde{m}_n|^2} \circ T_{\widetilde{u}}^{j-1}(g) \right\|_{L_\infty(Q)} < \infty .$$

Let r denote the spectral radius of the transfer operator $T_{\widetilde{u}}$ when restricted to its invariant subspace E_g. Since E_g is finite dimensional and the operator is non-negative on E_g, we know that r is an eigenvalue of $T_{\widetilde{u}}$ with a non-negative eigenfunction $f \in E_g$, $f \neq 0$. If we write f in terms of the basis of E_g, which is constituted by the Krylov sequence g, $T_{\widetilde{u}}(g)$, $T_{\widetilde{u}}^{\dim E_g - 1}(g)$, then the triangle inequality leads directly to

$$\left(\sum_{j>0} r^{j-1} \right) \cdot \| T_{|\widetilde{m}_n|^2}(f) \|_{L_\infty(Q)} = \left\| \sum_{j>0} T_{|\widetilde{m}_n|^2} \circ T_{\widetilde{u}}^{j-1}(f) \right\|_{L_\infty(Q)} < \infty .$$

Since f is non-negative, we have $T_{|\widetilde{m}_n|^2}(f) \neq 0$, and hence $r < 1$ follows. From here it is easy to prove (11). Note that by (10)

$$\|S_j\|_{\mathcal{B}(H)}^2 = \|S_j^* S_j\|_{\mathcal{B}(H)} \leq \left(\sum_{\nu, n \in I} \| T_{\overline{\widetilde{m}_\nu} \widetilde{m}_n} \circ T_{\widetilde{u}}^{j-1}(g) \|_{L_\infty(Q)}^2 \right)^{1/2}$$

where the summation enters because $S_j^* S_j$ is a multiplication operator on vectors in $H = L_2(Q)^I$. The spectral radius formula assures that there exists a constant $K = K(\epsilon)$ such that

$$\| T_{\widetilde{u}}^{j-1}(g) \|_{L_\infty(Q)} \leq K (r + \epsilon)^{j-1} \|g\|_{L_\infty(Q)} \qquad \text{for all} \qquad j > 0 .$$

If we let $\epsilon > 0$ be small enough such that $r + \epsilon < 1$, then

$$\sum_{j>0} \|S_j\|_{\mathcal{B}(H)} \leq \widetilde{K} \sum_{j>0} (r + \epsilon)^{(j-1)/2} < \infty ,$$

where the constant $\widetilde{K}$ includes the remaining terms $\|g\|_{L_\infty(Q)}^{1/2}$ and the sum over the operator norms of $T_{\overline{\widetilde{m}_\nu} \widetilde{m}_n}$. Thus we have shown that the part of the series in (11) for $j > 0$ is finite. As mentioned above, the remaining part is treated in exactly the same way. ■

The proof contains a result which is interesting by itself.

Corollary 4. *Under the same assumptions as in Theorem 3 the transfer operator $T_{\widetilde{u}}$ restricted to E_g has spectral radius strictly less than 1. The same is true for the transfer operator T_u restricted to E_h.*

Remark. A similar technique for the proof has been applied in several articles, see e.g. [8,12].

4. The Lifting Scheme

There is a nice application of the generalized Laurent operator in connection with the lifting scheme which was introduced in [3,22]. As in [22] we confine ourselves to the univariate case and scaling by $M = (2)$. The starting point are two biorthogonal multiresolution analyses and associated compactly supported wavelet bases. Hence with the same notation as in section 3 the functions ϕ and ρ are generators of two multiresolution analyses of $L_2(\mathbb{R})$, with respect to scaling by 2 and translates by $\mathbb{Z}$, and they satisfy the biorthogonality relation

$$[\widehat{\rho}, \widehat{\phi}] = 1 \qquad \text{a.e. in } \mathbb{R}\,. \tag{13}$$

Furthermore, the functions ψ and θ are the generators of two biorthogonal wavelet bases. (We can drop the index set I, since only one wavelet is needed for each family. We still use m_0, m_1 etc. for the scaling filters.) This means that the affine families $X = (\psi, 2, \mathbb{Z})$ and $Y = (\theta, 2, \mathbb{Z})$ are biorthogonal Riesz bases of $L_2(\mathbb{R})$. The orthogonality relations can be expressed by the identities for the bracket products

$$[\widehat{\theta}, \widehat{\phi}] = 0, \qquad [\widehat{\rho}, \widehat{\psi}] = 0, \qquad \text{and} \qquad [\widehat{\theta}, \widehat{\psi}] = 1 \tag{14}$$

which hold a.e. in $\mathbb{R}$. The representation of the operators S_j in section 3 gives

$$\mathcal{T}_X \mathcal{T}_Y^* = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 0 & \boxed{\mathrm{id}_H} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{\mathrm{id}_H} & 0 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = \mathrm{id}_{\ell_2(H)}\ ,$$

which is another way to express the biorthogonality. Here H is simply $\ell_2(\mathbb{Z})$.

The lifting scheme tries to change the scaling filters m_i and $\widetilde{m}_i$, $i \in \{0,1\}$, in a specific way such that new biorthogonal wavelet bases are generated. Let us call the new filters p_i and $\widetilde{p}_i$. Then we define in analogy with [22]

$$\begin{aligned} p_0 &= m_0, & \widetilde{p}_0 &= \widetilde{m}_0 + \widetilde{m}_1\, \overline{t(2\cdot)}, \\ p_1 &= m_1 - m_0\, t(2\cdot), & \widetilde{p}_1 &= \widetilde{m}_1 \end{aligned} \tag{15}$$

where t is a trigonometric polynomial. This choice of new filters implies that orthogonality still holds for the filters, i.e.

$$\begin{pmatrix} p_0 & p_0(\cdot+\pi) \\ p_1 & p_1(\cdot+\pi) \end{pmatrix} \begin{pmatrix} \widetilde{p}_0 & \widetilde{p}_0(\cdot+\pi) \\ \widetilde{p}_1 & \widetilde{p}_1(\cdot+\pi) \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is still unanswered under which (explicit) conditions on t the new filters give rise to biorthogonal multiresolution analyses and wavelet bases. While we were not yet able to find precise conditions on t, we still hope that the following results provide more insight into the subject.

Let us denote by $\widetilde{X} = (\eta, 2, \mathbb{Z})$ the new family with

$$\widehat{\eta}(2\cdot) = p_1 \widehat{\phi}, \qquad \text{hence} \qquad \widehat{\eta} = \widehat{\psi} - t\, \widehat{\phi}.$$

The following is an auxiliary result which can be proved by the usual integration techniques (Parseval's identity and periodization), with applications of the scaling relations for η, θ and the orthogonality relations (13), (14).

Lemma 5. *Let $\eta_{j,k} := 2^{j/2}\eta(2^j \cdot - k)$ and $\theta_{j,k}$ be analogously defined for any $j,k \in \mathbb{Z}$. Then taking inner products in $L_2(\mathbb{R})$ we obtain*

$$\langle \theta_{\ell,\kappa}, \eta_{j,k} \rangle = \begin{cases} 0 & ,\ \text{if } \ell > j, \\ \delta_{\kappa,k} & ,\ \text{if } \ell = j, \end{cases}$$

where δ denotes the Kronecker symbol, and if $\ell < j$ we have

$$\langle \theta_{\ell,\kappa}, \eta_{j,k} \rangle = -\frac{2^{(j-\ell)/2}}{2\pi} \int_Q e^{ik\xi} \left(\overline{t(\xi)}\widetilde{m}_1(2^{j-\ell-1}\xi) \prod_{\nu=0}^{j-\ell-2} \widetilde{m}_0(2^\nu \xi) e^{-i2^{j-\ell}\kappa\xi} \right) d\xi.$$

This representation of inner products is needed in order to find the following.

Theorem 6. *Let $X = (\psi, 2, \mathbb{Z})$ and $Y = (\theta, 2, \mathbb{Z})$ be biorthogonal wavelet bases of $L_2(\mathbb{R})$, which are generated from multiresolution analyses with finite filters m_i, $\widetilde{m}_i$, $i \in \{0,1\}$. Furthermore, let t be a trigonometric polynomial defining the new filter p_1 in (15) and $\widetilde{X} = (\eta, 2, \mathbb{Z})$. Then $\widetilde{X}$ is a Bessel family if and only if $t(0) = 0$.*

Proof: Note that ϕ and η are compactly supported, hence their Fourier transforms are entire functions. In particular, $\widehat{\phi}(0) \neq 0$, $m_0(0) = 1$ and $m_1(0) = 0$ are implications from the assumptions on X and Y, see [10]. The condition $t(0) = 0$ is necessary, since otherwise $\widehat{\eta}(0) \neq 0$ would contradict the general admissibility condition [4,10]. The more difficult part is to prove the sufficiency of the condition. For this we define operators S_j, $j > 0$, on $L_2(Q)$ by

$$S_j(C)(\xi) = -2^{j/2}\overline{t(\xi)}\widetilde{m}_1(2^{j-1}\xi) \prod_{\nu=0}^{j-2} \widetilde{m}_0(2^\nu \xi) \cdot C(2^j \xi). \tag{16}$$

Their connection to a generalized Laurent operator will become clear later. The same computations as in section 3 give

$$\|S_j\|^2 = \|S_j^* S_j\| = \|T_{|\widetilde{m}_1|^2} \circ T_{|\widetilde{m}_0|^2}^{j-1}(|t|^2)\|_{L_\infty(Q)}.$$

If $t(0) = 0$, then the spectral radius of the transfer operator $T_{|\widetilde{m}_0|^2}$ when restricted to the invariant subspace $E_{|t|^2}$ is less than 1 by [8,Theorem 3.1]. Here, the condition was used that the translates of ρ define a Riesz basis of their closed linear span. Hence we obtain that

$$\sum_{j>0} \|S_j\| < \infty. \tag{17}$$

In order to show that $\widetilde{X}$ is a Bessel family, we take an arbitrary function $f \in L_2(\mathbb{R})$. Its expansion with respect to the wavelet basis Y is given by

$$f = T_Y^*(\mathbf{c}_j)_{j\in\mathbb{Z}} \qquad \text{with} \qquad (\mathbf{c}_j)_{j\in\mathbb{Z}} \in \ell_2(H).$$

The norm of the sequence $(\mathbf{c}_j)_{j\in\mathbb{Z}}$ is equivalent to the L_2-norm of f, since Y is a Riesz basis. Let us define

$$\mathbf{d}_j := (\langle f, \eta_{j,k}\rangle)_{k\in\mathbb{Z}}, \qquad j \in \mathbb{Z}.$$

If we insert $T_Y^*(\mathbf{c}_j)_{j\in\mathbb{Z}}$ for f, then a direct application of Lemma 5 gives

$$D_j = C_j + \sum_{k>0} S_k(C_{j-k}). \tag{18}$$

The boundedness of the series in (17) implies that there exist constants K and $\widetilde{K}$ with

$$\|(\mathbf{d}_j)_{j\in\mathbb{Z}}\|_{\ell_2(H)} \leq K\|(\mathbf{c}_j)_{j\in\mathbb{Z}}\|_{\ell_2(H)} \leq \widetilde{K}\|f\|_{L_2(\mathbb{R})}.$$

This proves that $\widetilde{X}$ is a Bessel family and completes the proof of the theorem. ■

Note that equation (18) defines a generalized Laurent operator on $\ell_2(H)$, since shifts of the index j on the left and right hand sides of (18) are equivalent. Hence we can summarize Theorem 6 by the equation

$$\mathcal{T}_{\widetilde{X}}\mathcal{T}_Y^* = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & S_1 & \boxed{\mathrm{id}_H} & 0 & 0 & \cdots \\ \cdots & S_2 & S_1 & \boxed{\mathrm{id}_H} & 0 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{19}$$

where the operators S_j on $L_2(Q)$ are given in (16) and satisfy (17).

Let us shortly describe some further consequences for the lifting scheme. By definition $\widetilde{X}$ is a wavelet basis if and only if $\mathcal{T}_{\widetilde{X}}$ is an invertible operator which maps $L_2(\mathbb{R})$ onto $\ell_2(\mathbb{Z}\times\mathbb{Z})$. This is equivalent to the condition that $\mathcal{T}_{\widetilde{X}}\mathcal{T}_Y^*$ in (19) is an invertible generalized Laurent operator, since Y is a wavelet basis by our starting assumptions. However, the inverse of this operator need not be of the same form as in (19). Indeed, if the inverse has this form, then the generalized Toeplitz operator

$$\mathcal{V}(C_j)_{j\geq 0} = \left(C_j + \sum_{k=0}^{j} S_k C_{j-k}\right)_{j\geq 0}$$

is also invertible, see [16,Lemma 4.2]. This is much stronger than invertibility of the generalized Laurent operator, in general. One can explain the difference with a simple example in the scalar case. Let A be the Laurent matrix

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 0 & 2 & \boxed{1} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 2 & \boxed{1} & 0 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}$$

which is lower triangular and has absolutely summable columns. Its symbol is the function $F(z) = 1 + 2z$, $z \in \mathbb{T}$. The operator is invertible on $\ell_2(\mathbb{Z})$ since the symbol does not vanish. The corresponding Toeplitz operator, however, is not invertible, since the winding number of $F(z)$ around 0 is 1. It turns out that the inverse of A on $\ell_2(\mathbb{Z})$ is no longer a lower triangular matrix, but the Laurent matrix with symbol $1/F(z) = \sum_{k=1}^{\infty}(-1)^{k-1}(2z)^{-k}$.

This explains how it might happen, that the new family $\widetilde{X}$ defines a wavelet basis, while the lifting scheme does not give the biorthogonal wavelet basis. On the other hand, it is likely (but only conjectured here), that the lifting scheme produces a new pair of biorthogonal wavelet bases if and only if the inverse of the operator matrix (19) is again lower triangular with finite sum of the norms in each column. This last condition can then be studied by means of the transfer operator in the following way. The adjoint of $\mathcal{T}_{\widetilde{X}} \mathcal{T}_Y^*$ defines the operator-valued function

$$F^*(z) = \mathrm{id}_H - \sum_{j>0} z^{-j} S_j^*$$

in (8). Taking adjoints in (16) leads to

$$F^*(z)(C) = \mathrm{id}_H - \sum_{j>0} 2^{-j/2} z^{-j} T_{\overline{\widetilde{m}_1}} \circ T_{\overline{\widetilde{m}_0}}^{j-1}(t \cdot C).$$

Hence the problem is transferred to finding an inverse of $F^*(z)$ on $L_2(Q)$ for all z in the closed unit disc, and the inverse must correspond to an upper triangular Laurent matrix. Since the above series is a Neumann series for $T_{\overline{\widetilde{m}_0}}$, we hope that further techniques from operator algebras can help to solve this problem completely.

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