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Affine Frames, Quasi-affine Frames, and their Duals

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Abstract. The notion of quasi-affine frame was recently introduced by Ron and Shen [9] in order to achieve shift-invariance of the discrete wavelet transform. In this paper, we establish a duality-preservation theorem for quasi-affine frames. Furthermore, the preservation of frame bounds when changing an affine frame to a quasi-affine frame is shown to hold without the decay assumptions in [9]. Our consideration leads naturally to the study of certain sesquilinear operators which are defined by two affine systems. The translation-invariance of such operators is characterized in terms of certain intrinsic properties of the two affine systems.

Keywords: affine frame, quasi-affine frame, dual, sesquilinear operator, translation invariance

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§1. Introduction

Although the standard Mallat (pyramid) wavelet decomposition algorithm is most efficient, it has several limitations, among which is the lack of shift-invariance. The reason for this defect is the downsampling operation. By skipping the downsampling step in a clever way, as suggested by Shensa [10], not only the defect of lack of shift-invariance is eliminated, the decomposition algorithm even yields the (integral) wavelet transform values at the same lattice points as the finest scale (see [2]). For convenience, let us assume that Shensa's algorithm is performed with initial scale level 2^j, $j = 0$. Then the wavelet transform values are obtained at the sample sites $\mathbb{Z}$ for $j \le 0$, but $k/2^j$ for $j > 0$ where $k \in \mathbb{Z}$. Here and throughout, $\mathbb{Z}$ denotes the set of integers.

More recently, Ron and Shen [9] applied this discretization procedure to introduce the notion of *quasi-affine frames*, and proved that, indeed, quasi-affine frames are shift-invariant with respect to integer shifts. The main theorem of Ron and Shen in this direction is Theorem 5.5 in [9], called "fundamental theorem of affine frames". It states that with certain appropriate multiplicative factors for levels 2^j, $j < 0$, an affine system is an affine frame if and only if the quasi-affine system derived from it is also a frame, and furthermore, the two frames have the same frame bounds. Observe that although the original affine system is dilation-invariant, the corresponding quasi-affine system is not. Hence, to gain the property of shift-invariance by using a quasi-affine frame, one loses the dilation-invariance property. This already shows that the approach of Ron and Shen [9] is different from the oversampling approach of Chui and Shi [3], where dilation-invariance is preserved, but the defect of shift-invariance is only diminished but not completely removed. It turns out, however, a portion of the proof in [3] (without the delicate combinatoric argument there) can be applied to give a direct proof of the 'fundamental theorem' mentioned above (Theorem 5.5 in [9]). Additionally, we are able to eliminate the assumptions on the decay of the Fourier transforms which are very mild, but were still needed in the original proof in [9].

The objective of this paper is two-fold. A sesquilinear operator on $L_2(\mathbb{R}^s) \times L_2(\mathbb{R}^s)$ corresponding to two affine systems is introduced. This operator is bounded if the affine systems constitute Bessel families. Our first goal is to characterize, in Theorem 1, the $\mathbf{x}$-shift-invariance, $\mathbf{x} \in \mathbb{R}^s$, of the sesquilinear operator in terms of the intrinsic properties of the affine systems. Our second goal is to establish a criterion, in Theorem 3, when this operator which is defined by two affine systems agrees with its analogue for the two quasi-affine systems. Again, the $\mathbf{x}$-shift-invariance, $\mathbf{x} \in \mathbb{R}^s$, of the original operator plays a crucial role here. As a simple consequence, we deduce that certain duals (which we call a-dual or affine dual, see Definition 2) agree for affine and quasi-affine frames. It turns out that the more delicate part of the fundamental theorem of affine frames of Ron and Shen and the $\mathbf{x}$-shift-invariance of the sesquilinear operator are very much closely related, a circumstance which was not considered in [9]. Finally, we will give an explicit formulation of the duals of certain generalized Meyer wavelets to demonstrate the main results in this paper.

§2. Notations and Definitions

We consider families of functions generated by dilates and shifts of a finite set of functions $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L_2(\mathbb{R}^s)$, $L \in \mathbb{N}$, and follow most of the notions used in [4]. Let M be an $s \times s$ non-singular matrix with

$$\rho(M^{-1}) < 1 \tag{1}$$

where ρ denotes the spectral radius. Later in Section 4 we will assume, in addition, that M has integer entries. The unitary operations of dilation and shift on the Hilbert space $L_2(\mathbb{R}^s)$ are defined by

$$D\,f := |\det M|^{1/2}\, f(M\,.\,),$$

$$T_{\mathbf{x}}\,f := f(\,.\,-\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^s,\ f \in L_2(\mathbb{R}^s).$$

For any ψ_ℓ, $1 \le \ell \le L$, we define, as usual,

$$\psi_{\ell,j,\mathbf{k}} := D^j\, T_{\mathbf{k}}\, \psi_\ell = |\det M|^{j/2}\, \psi_\ell(M^j\,. - \mathbf{k}), \qquad j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^s, \tag{2}$$

and

$$\psi^q_{\ell,j,\mathbf{k}} := \begin{cases} \psi_{\ell,j,\mathbf{k}}, & \text{if } j \ge 0,\ \mathbf{k} \in \mathbb{Z}^s, \\ |\det M|^{j/2} T_{\mathbf{k}}\, D^j\, \psi_\ell = |\det M|^j\, \psi_\ell(M^j(\,. - \mathbf{k})), & \text{if } j < 0,\ \mathbf{k} \in \mathbb{Z}^s, \end{cases} \tag{3}$$

and call

$$X_\Psi := \{\psi_{\ell,j,\mathbf{k}} :\ \ 1 \le \ell \le L,\ j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^s\},$$

$$X^q_\Psi := \{\psi^q_{\ell,j,\mathbf{k}} :\ \ 1 \le \ell \le L,\ j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^s\}$$

the *affine system* (resp. *quasi-affine system*) generated by Ψ.

Definition 1. *Given a finite set $\Psi \subset L_2(\mathbb{R}^s)$. Let X_Ψ and X^q_Ψ be defined as above. Then*

(i) X_Ψ is an affine Bessel family, if there exists a constant $B > 0$ such that

$$\sum_{\eta \in X_\Psi} |\langle f , \eta \rangle|^2 \le B\|f\|^2, \qquad f \in L_2(\mathbb{R}^s),$$

(ii) X_Ψ is an affine frame, if there exists, in addition, another positive constant $A \le B$, such that

$$A\|f\|^2 \le \sum_{\eta \in X_\Psi} |\langle f , \eta \rangle|^2 \le B\|f\|^2, \qquad f \in L_2(\mathbb{R}^s). \tag{4}$$

An affine frame is tight, if A and B can be so chosen that $A = B$. Analogously, X^q_Ψ is a quasi-affine Bessel family (or a quasi-affine frame) if corresponding constants $B_q > 0$ (resp. $A_q, B_q > 0$) exist when we sum over all $\eta \in X^q_\Psi$ in the previous inequalities.

Here and throughout, we use the standard notation $\|f\|$ for the L_2-norm of $f \in L_2(\mathbb{R}^s)$ and $\langle f , g \rangle$ for the corresponding inner product. Given a second set $\Theta \subset L_2(\mathbb{R}^s)$ with the same cardinality $L \in \mathbb{N}$, we let $\theta_{\ell,j,\mathbf{k}}$, $\theta^q_{\ell,j,\mathbf{k}}$ be defined as in (2), (3). The operator

$$K_{\Psi,\Theta}(f,g) := \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^s} \langle f , \psi_{\ell,j,\mathbf{k}} \rangle\, \langle \theta_{\ell,j,\mathbf{k}} , g \rangle, \qquad f, g \in L_2(\mathbb{R}^s), \tag{5}$$

plays a crucial role in our investigation. It obviously defines a bounded sesquilinear operator on $L_2(\mathbb{R}^s) \times L_2(\mathbb{R}^s)$, if both X_Ψ and X_Θ are affine Bessel families. Our frame analysis is based on the relation to the operator

$$K^q_{\Psi,\Theta}(f,g) := \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^s} \langle\, f\,,\, \psi^q_{\ell,j,\mathbf{k}} \,\rangle \, \langle\, \theta^q_{\ell,j,\mathbf{k}}\,,\, g \,\rangle, \qquad f,g \in L_2(\mathbb{R}^s), \tag{6}$$

which is associated with the quasi-affine systems. The following terminology will be used.

Definition 2. *Let $\Psi, \widetilde{\Psi} \subset L_2(\mathbb{R}^s)$ be two finite sets in $L_2(\mathbb{R}^s)$ with the same cardinality that generate two Bessel families X_Ψ and $X_{\widetilde{\Psi}}$. Then $\widetilde{\Psi}$ is called an a-dual (or affine dual) of Ψ, if*

$$K_{\Psi,\widetilde{\Psi}}(f,g) = \langle\, f\,,\, g \,\rangle, \qquad f,g \in L_2(\mathbb{R}^s), \tag{7}$$

and a quasi-affine dual if

$$K^q_{\Psi,\widetilde{\Psi}}(f,g) = \langle\, f\,,\, g \,\rangle, \qquad f,g \in L_2(\mathbb{R}^s).$$

In the work [1,3,4] certain a-duals are constructed. It follows by straightforward arguments that being an a-dual (which is a Bessel family by definition) is stronger than being a frame for any affine Bessel family. Note further that the notion of a-dual differs from the standard notion of dual frame which is uniquely determined by a minimization property, see [7]; but a-duals still satisfy the relation

$$f = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mathbf{k}\in\mathbb{Z}^s} \langle\, f\,,\, \widetilde{\psi}_{\ell,j,\mathbf{k}} \,\rangle \, \psi_{\ell,j,\mathbf{k}}, \qquad f \in L_2(\mathbb{R}^s),$$

where the series converges in the L_2-norm.

Let us agree on some further notations. The Fourier transform is denoted by $\widehat{f}(\xi) = \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi}\, dx$. By $\mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s))$, we mean the space of all multipliers of $L_2(\mathbb{R}^s)$; that is

$$m \in \mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s)) \quad \Longleftrightarrow \quad m \in \mathcal{S}',\ \widehat{m} \in L_\infty(\mathbb{R}^s).$$

Here and throughout, $\mathcal{S}'$ denotes, as usual, the space of all tempered distributions on $\mathbb{R}^s$.

§3. Translation-Invariant Sesquilinear Operators

Affine systems X_Ψ are dilation-invariant in the sense that

$$X_\Psi = D(X_\Psi).$$

This is the reason that $K_{\Psi,\Theta}$ in (5) has the property

$$K_{\Psi,\Theta}(Df, Dg) = K_{\Psi,\Theta}(f,g), \qquad f,g \in L_2(\mathbb{R}^s). \tag{8}$$

On the other hand, translation-invariance creates a much more difficult problem, since affine systems do not possess such a structural property. In this section we investigate the special circumstance when $K_{\Psi,\Theta}$ is also translation-invariant. This happens, e.g., for affine frames which have a-duals, since the right-hand side in (7) is translation-invariant. We start with a representation formula for such operators that is well-known for linear operators, see, for example, [12: Ch. 6].

Lemma 1. *Let $K : L_2(\mathbb{R}^s) \times L_2(\mathbb{R}^s) \to \boldsymbol{C}$ be a bounded sesquilinear operator. Then K is translation-invariant, i.e.*

$$K(T_{\mathbf{x}} f, T_{\mathbf{x}} g) = K(f,g) \quad \textit{for all} \quad f,g \in L_2(\mathbb{R}^s),\ \mathbf{x} \in \mathbb{R}^s, \tag{9}$$

if and only if there exists a multiplier $m_K \in \mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s))$ such that

$$K(f,g) = \langle\, m_K * f \,,\, g \,\rangle, \qquad f,g \in L_2(\mathbb{R}^s). \tag{10}$$

m_K will be called the kernel of K.

Proof: Let K be a bounded translation-invariant sesquilinear operator. Then a bounded linear operator

$$J : L_2(\mathbb{R}^s) \to L_2(\mathbb{R}^s)$$

is defined via the Riesz representation theorem by

$$\langle\, Jf \,,\, g \,\rangle = K(f,g), \qquad g \in L_2(\mathbb{R}^s).$$

It is not difficult to see that J is translation-invariant. Indeed, for any $f, g \in L_2(\mathbb{R}^s)$ and any $\mathbf{x} \in \mathbb{R}^s$, we obtain, from (9),

$$\langle Jf\,,\,g \rangle = K(f,g) = K(T_{\mathbf{x}}f, T_{\mathbf{x}}g) = \langle J\,T_{\mathbf{x}}f\,,\,T_{\mathbf{x}}g \rangle = \langle T_{-\mathbf{x}}J\,T_{\mathbf{x}}f\,,\,g \rangle.$$

Hence $J = T_{-\mathbf{x}}JT_{\mathbf{x}}$, and there exists an $m \in \mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s))$ with $Jf = m * f$ for all $f \in L_2(\mathbb{R}^s)$, which proves (10). The opposite direction of the statement is obvious. ■

As an example we mention that the Dirac distribution δ is the kernel m_K of the "identity operator" $K(f,g) := \langle f\,,\,g \rangle$. In the weak sense, the Riesz representation theorem also gives

$$m_K * f = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^s} \langle f\,,\,\psi_{\ell,j,\mathbf{k}} \rangle\, \theta_{\ell,j,\mathbf{k}}, \qquad f \in L_2(\mathbb{R}^s).$$

The dilation invariance of affine Bessel families directly propagates to a similar property of the kernel m_K.

Lemma 2. *Let $\Psi, \Theta \subset L_2(\mathbb{R}^s)$ both have cardinality L, and X_Ψ, X_Θ be affine Bessel families. Then the following statements hold.*

(i) The operator $K_{\Psi,\Theta}$ in (5) is translation-invariant if and only if it has a kernel $m_K \in \mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s))$ as in (10) which is dilation-invariant; i.e. $|\det M| m_K(M\,.\,) = m_K$, or equivalently,

$$\widehat{m}_K(\xi) = \widehat{m}_K(M^T\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^s. \tag{11}$$

(ii) If $m \in \mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s))$ is dilation-invariant as in (11), then

$$\Xi := \{m * \theta_1,\ m * \theta_2,\ \ldots, m * \theta_L\}$$

is an affine Bessel family. Furthermore, if $K_{\Psi,\Theta}$ is translation-invariant with kernel m_K in (10), then the new operator $K_{\Psi,\Xi}$ is also translation-invariant with kernel $m * m_K$.

The **proof** only makes use of (8) and requires elementary calculus of convolutions, and is therefore omitted here. Our previous result can directly be applied to the case where Ψ generates an affine frame with an a-dual $\widetilde{\Psi}$. Since $K_{\Psi,\widetilde{\Psi}}$ has kernel $m_K = \delta$, a combination of (i) and (ii) in Lemma 2 gives the following.

Corollary 1. *Assume that Ψ generates an affine frame with an a-dual $\widetilde{\Psi}$. Then there is a one-to-one correspondence between the space of all dilation-invariant multipliers in $\mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s))$ and the set of all translation-invariant operators $K_{\Psi,\Theta}$, where Θ varies over all finite subsets of $L_2(\mathbb{R}^s)$ having the same cardinality as Ψ and generating a Bessel family X_Θ.*

Uniqueness of an a-dual has a much stronger consequence. Not only the translation-invariant operators, but even their defining families Θ are parameterized as above.

Theorem 1. *Assume that Ψ generates an affine frame X_Ψ and has an a-dual $\widetilde{\Psi}$. Then the following statements are equivalent:*

(i) The a-dual $\widetilde{\Psi}$ is unique.

(ii) For any $\Theta \subset L_2(\mathbb{R}^s)$ having the same cardinality as Ψ and generating an affine Bessel family, the following holds: the operator $K_{\Psi,\Theta}$ is translation-invariant if and only if there exists an $m \in \mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s))$, which is dilation-invariant, such that

$$\theta_\ell = m * \widetilde{\psi}_\ell, \qquad 1 \le \ell \le L.$$

Proof: It is clear that (i) is equivalent to the condition

$$\text{(i}'\text{)} \qquad K_{\Psi,\Theta} = 0 \iff \Theta = \{0, \ldots, 0\}.$$

Next we show that (i') implies (ii). Recall that the sesquilinear operator $K_{\Psi,\widetilde{\Psi}} = \mathrm{id}$ is translation-invariant and has kernel δ. If $m \in \mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s))$ is dilation-invariant and $\Theta = \{m * \widetilde{\psi}_\ell : 1 \le \ell \le L\}$, then $K_{\Psi,\Theta}$ is translation-invariant by Lemma 2(ii). Conversely, if $K_{\Psi,\Theta}$ is translation-invariant with kernel m_K as in Lemma 2(i), then we may define another family

$$\Sigma = \{m_K * \widetilde{\psi}_\ell : 1 \le \ell \le L\};$$

and by Lemma 2(ii), then the operator $K_{\Psi,\Sigma}$ has the same kernel m_K, and hence

$$K_{\Psi,\Theta} = K_{\Psi,\Sigma}.$$

This and condition (i') yield $\Theta = \Sigma$ which is the desired conclusion.

Finally, let us prove that (ii) implies (i). If Θ is another a-dual, then $K_{\Psi,\Theta} = \mathrm{id}$ has kernel $m_K = \delta$. Condition (ii) and Lemma 2(ii) imply that

$$\theta_\ell = m_K * \widetilde{\psi}_\ell = \widetilde{\psi}_\ell, \qquad 1 \le \ell \le L.$$

This completes the proof of the theorem. ■

§4. Preservation of Sesquilinear Operators in Changing Affine Systems to Quasi-Affine Systems

Several motivations for changing affine systems to quasi-affine systems were given in Section 1. Here, we study their relation to each other and show that there is a close relation with translation-invariance of the associated sesquilinear operators. From now on, we assume that the matrix M not only satisfies (1), but also has integer entries. Our main technique is based on a method adapted from the oversampling results in [3] (see Lemma 3 below).

As before, affine systems always produce dilation-invariant operators $K_{\Psi,\Theta}$. On the other hand, it can be seen easily from the definition of quasi-affine systems, that

$$K^q_{\Psi,\Theta}(T_{\mathbf{k}}f\,,\,T_{\mathbf{k}}g) = K^q_{\Psi,\Theta}(f,g), \qquad f,g \in L_2(\mathbb{R}^s),\ \mathbf{k} \in \mathbb{Z}^s.$$

Hence, $K^q_{\Psi,\Theta}$ is invariant with respect to multi-integer shifts of both arguments. The two main results of this section describe the relation between affine systems and quasi-affine systems in the following way.

Theorem 2. *Let Ψ, $\Theta \subset L_2(\mathbb{R}^s)$ be finite sets with the same cardinality. Then the following statements hold.*

(i) *X_Ψ is a Bessel family if and only if X_Ψ^q is a Bessel family. Furthermore, their exact upper bounds are equal.*

(ii) *X_Ψ is an affine frame if and only if X_Ψ^q is a quasi-affine frame. Furthermore, the constants*

$$A = A_q \qquad \text{and} \qquad B = B_q$$

can be chosen as their exact upper and lower bounds.

Theorem 3. *Assume that Ψ and Θ have the same cardinality and generate two affine Bessel families. Then $K_{\Psi,\Theta}$ is translation-invariant if and only if*

$$K_{\Psi,\Theta} = K_{\Psi,\Theta}^q. \tag{12}$$

We mention that Theorem 2 is called the Fundamental Theorem of affine frames in Ron and Shen [9]. Note, however, that they need a mild decay condition of the Fourier transform of $\psi \in \Psi$ for their proof, while our result covers the most general case. Also, their method of proof is totally different from ours. Some new results which stem from our development are obtained as direct consequences of Theorem 3.

Corollary 2. *Let $\Psi \subset L_2(\mathbb{R}^s)$ be a finite family that generates an affine frame with an a-dual $\widetilde{\Psi}$. Then $\widetilde{\Psi}$ is also a quasi-affine dual; i.e.*

$$K_{\Psi,\widetilde{\Psi}}^q(f,g) = \langle\, f\,,\, g\,\rangle, \qquad f,g \in L_2(\mathbb{R}^s).$$

Consequently, if X_Ψ is a tight frame, then X_Ψ^q is also a tight frame.

Corollary 3. *Let Ψ, $\widetilde{\Psi} \subset L_2(\mathbb{R}^s)$ be as in Corollary 2, and assume that the a-dual $\widetilde{\Psi}$ is unique. For any family $\Theta \subset L_2(\mathbb{R}^s)$ which has the same cardinality as Ψ and generates an affine Bessel sequence, the following statements are equivalent.*

(i) $K_{\Psi,\Theta} = K_{\Psi,\Theta}^q$.

(ii) $K_{\Psi,\Theta}$ *is translation-invariant.*

(iii) *There exists a dilation-invariant multiplier* $m \in \mathcal{M}(L_2(\mathbb{R}^s), L_2(\mathbb{R}^s))$ *such that* $\theta_\ell = m * \widetilde{\psi}_\ell$ *holds for all* ℓ, $1 \le \ell \le L$.

We break the proof of Theorems 2 and 3 into several parts which we state as separate lemmas. The first describes our main technique which we adapt from [3]. For this we need the following notations. For $J \ge 0$, we let Q_J denote a complete set of representatives of $\mathbb{Z}^s/(M^J\mathbb{Z}^s)$, so that

$$\# Q_J = |\det M|^J.$$

The special choice of representatives is not important except for Lemma 4(ii) below, where we use

$$Q_J := \mathbb{Z}^s \cap M^J([1,2)^s). \tag{13}$$

We further use the notations

$$\begin{aligned} K_j(f,g) &:= \sum_{\ell=1}^{L} \sum_{\mathbf{k}\in\mathbb{Z}^s} \langle f , \psi_{\ell,j,\mathbf{k}} \rangle \langle \theta_{\ell,j,\mathbf{k}} , g \rangle, \\ K_j^q(f,g) &:= \sum_{\ell=1}^{L} \sum_{\mathbf{k}\in\mathbb{Z}^s} \langle f , \psi^q_{\ell,j,\mathbf{k}} \rangle \langle \theta^q_{\ell,j,\mathbf{k}} , g \rangle \end{aligned} \tag{14}$$

for any $j \in \mathbb{Z}$. Note that Ψ and Θ are usually defined by the context and, in particular, $K_j = K_j^q$ holds for all $j \ge 0$.

Lemma 3. *Let* Ψ, $\Theta \subset L_2(\mathbb{R}^s)$ *be finite and have the same cardinality* L, *and let* $J > 0$ *be an integer. Then for all* $j \ge -J$ *and* $f, g \in L_2(\mathbb{R}^s)$,

$$K_j^q(f,g) = |\det M|^{-J} \sum_{\nu\in Q_J} K_j(T_\nu f , T_\nu g).$$

Proof: For any $j \ge 0$ the sesquilinear operator K_j is invariant with respect to multiinteger shifts, so that

$$K_j^q(f,g) = K_j(f,g) = K_j(T_\nu f, T_\nu g), \qquad \nu \in Q_J.$$

This gives the desired formula. For any $-J \le j < 0$, we first observe that K_j is invariant with respect to shifts $\mathbf{k} \in M^{-j}\mathbb{Z}^s$. Furthermore, $H := \mathbb{Z}^s/M^{-j}\mathbb{Z}^s$ is a normal subgroup of $G := \mathbb{Z}^s/M^J\mathbb{Z}^s$ and there is a group isomorphism of G/H and $M^{-j}\mathbb{Z}^s/M^J\mathbb{Z}^s$. Hence, $|G/H| = |\det M|^{J+j}$, and

$$\begin{aligned}|\det M|^{-J} \sum_{\nu \in Q_J} K_j(T_\nu f\,,\, T_\nu g) &= |\det M|^{-J} \sum_{\lambda \in Q_{-j}} \sum_{\substack{\nu \in Q_J \\ \nu-\lambda \in M^{-j}\mathbb{Z}^s}} K_j(T_\nu f\,,\, T_\nu g) \\ &= |\det M|^{j} \sum_{\lambda \in Q_{-j}} K_j(T_\lambda f\,,\, T_\lambda g).\end{aligned}$$

By inserting the definition (3) of $\psi^q_{\ell,j,\mathbf{k}}$ we see that the last expression is equal to $K^q_j(f,g)$. ■

The above result is purely algebraic and is independent of the special choice of representatives of Q_J. This technique is used in the following proof.

Proof of Theorem 2(i): Let $\Theta = \Psi$ in (14). First we note that all summands of $K_{\Psi,\Psi}$ and $K^q_{\Psi,\Psi}$ are nonnegative. If Ψ generates an affine Bessel family with upper bound $B \ge 0$, then

$$\begin{aligned}K^q_{\Psi,\Psi}(f,f) &= \lim_{J\to\infty} \sum_{j \ge -J} K^q_j(f,f) \\ &= \lim_{J\to\infty} |\det M|^{-J} \sum_{\nu \in Q_J} \sum_{j \ge -J} K_j(T_\nu f\,,\, T_\nu f) \\ &\le \lim_{J\to\infty} |\det M|^{-J} \sum_{\nu \in Q_J} K_{\Psi,\Psi}(T_\nu f\,,\, T_\nu f) \\ &\le \lim_{J\to\infty} |\det M|^{-J} \sum_{\nu \in Q_J} B\|T_\nu f\|^2 = B\|f\|^2\end{aligned}$$

holds for all $f \in L_2(\mathbb{R}^s)$. Here, we used the translation invariance of the norm. We thus have shown that the quasi-affine frame X^q_Ψ is also a Bessel family with the same upper bound B.

Conversely, let us assume that X^q_Ψ is a Bessel family with upper bound $B_q \ge 0$. Let us further assume that there exists an $f \in L_2(\mathbb{R}^s)$, such that

$$\|f\| = 1 \quad \text{and} \quad K_{\Psi,\Psi}(f,f) > B_q.$$

Then by the dilation invariance of X_Ψ, we can find $N \in \mathbb{N}$ such that

$$\sum_{j=-N}^{\infty} K_j(f,f) = \sum_{j=0}^{\infty} K_j(D^N f, D^N f) > B_q.$$

But this contradicts with the definition of B_q, since

$$K^q_{\Psi,\Psi}(D^N f, D^N f) \geq \sum_{j=0}^{\infty} K^q_j(D^N f, D^N f) = \sum_{j=0}^{\infty} K_j(D^N f, D^N f),$$

and the dilation D is an isometry. Thus, we can conclude that X_Ψ must be a Bessel family with upper bound B_q. This completes the proof of part (i) in Theorem 2. ■

Before we can complete the proof of Theorem 2, we first need a technical lemma on the cutoff of the series that define $K_{\Psi,\Psi}(f,f)$ and $K^q_{\Psi,\Psi}(f,f)$ for negative summation index j.

Lemma 4. *Let $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L_2(\mathbb{R}^s)$ and let $f \in L_2(\mathbb{R}^s)$ be a function with compact support. Then*

$$\lim_{N\to\infty} \sum_{j<0} K^q_j(D^N f, D^N f) = 0 \tag{15}$$

and

$$\lim_{N\to\infty} |\det M|^{-N} \sum_{j<-N} \sum_{\nu \in Q_N} K_j(T_\nu f, T_\nu f) = 0 \tag{16}$$

where the special choice of representatives (13) is used in (16) for any $N \in \mathbb{N}$.

Proof: We invoke Lebesgue's dominated convergence theorem for both limits. Let Ω denote the support of f and let us first choose $N_0 > 0$ so large that $D^{-N}\Omega$ is contained in a ball of radius $1/2$ around the origin for all $N \geq N_0$. In order to prove

(15), we consider

$$
\begin{aligned}
\sum_{j<0} K_j^q(D^N f, D^N f) &= \sum_{j<0} |\det M|^j \sum_{\ell=1}^{L} \sum_{\mathbf{k}\in\mathbb{Z}^s} |\langle f\,,\, D^{-N} T_{\mathbf{k}} D^j \psi_\ell \rangle|^2 \\
&= \sum_{j<0} |\det M|^j \sum_{\ell=1}^{L} \sum_{\mathbf{k}\in\mathbb{Z}^s} |\langle T_{M^N\mathbf{k}} f\,,\, D^{j-N} \psi_\ell \rangle|^2 \\
&\leq \sum_{j<0} |\det M|^j \sum_{\ell=1}^{L} \sum_{\mathbf{k}\in\mathbb{Z}^s} \|f\|^2 \int_{\Omega+M^N\mathbf{k}} |D^{j-N}\psi_\ell(\mathbf{x})|^2\, d\mathbf{x} \\
&= \|f\|^2 \cdot \sum_{j<0} |\det M|^j \sum_{\mathbf{k}\in\mathbb{Z}^s} \int_{M^{j-N}\Omega+M^j\mathbf{k}} \sum_{\ell=1}^{L} |\psi_\ell(\mathbf{x})|^2\, d\mathbf{x}.
\end{aligned} \tag{17}
$$

By our previous choice of N_0, we obtain that

$$
\sum_{j<0} K_j^q(D^N f, D^N f) \leq \|f\|^2 \int_{\mathbb{R}^s} w_N(\mathbf{x}) \sum_{\ell=1}^{L} |\psi_\ell(\mathbf{x})|^2\, d\mathbf{x}
$$

holds for all $N \geq N_0$, where

$$
w_N(\mathbf{x}) = \sum_{j<0} |\det M|^j \chi_{M^j(\mathbb{Z}^s+M^{-N}\Omega)}(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^s. \tag{18}
$$

Since

$$
w_N(\mathbf{x}) \leq \sum_{j<0} |\det M|^j = |\det M|/(|\det M|-1), \qquad N \geq N_0,
$$

and since $\sum_{\ell=1}^{L} |\psi_\ell|^2 \in L_1(\mathbb{R}^s)$, the dominated convergence theorem can be applied to the above integral. It thus suffices to show that w_N converges to 0 a.e. as N tends to infinity, or more precisely

$$
\lim_{N\to\infty} w_N(\mathbf{x}) = 0 \quad \text{for all} \quad \mathbf{x} \in U := \mathbb{R}^s \setminus \bigcup_{j<0} M^j\mathbb{Z}^s.
$$

This last assertion can be shown as follows. By (1) and the compactness of Ω the sequence of numbers $r_j := \sup\{\|M^j\mathbf{y}\| : \mathbf{y}\in\Omega\}$ tends to zero as $j\to-\infty$. If we fix $\mathbf{x}\in U$, then all terms in (18) which satisfy

$$
r_{j-N} < \operatorname{dist}(x, M^j\mathbb{Z}^s) := d_j(\mathbf{x}), \qquad j<0,
$$

vanish. In other words,

$$w_N(\mathbf{x}) \le \sum_{\substack{j<0 \\ r_{j-N} \ge d_j(\mathbf{x})}} |\det M|^j \to 0 \quad \text{as} \quad N \to \infty.$$

In the second relation (16), we let $N \ge N_0$ and use similar transformations as in (17) in order to obtain

$$\begin{aligned} &|\det M|^{-N} \sum_{\nu \in Q_N} \sum_{j<-N} K_j(T_\nu f, T_\nu f) \\ &\quad \le \|f\|^2 \cdot |\det M|^{-N} \sum_{\nu \in Q_N} \sum_{j<-N} \sum_{\mathbf{k} \in \mathbb{Z}^s} \int_{M^j(\Omega+\nu)+\mathbf{k}} \sum_{\ell=1}^{L} |\psi_\ell(\mathbf{x})|^2 \, d\mathbf{x}. \end{aligned}$$

We now define

$$v_N(\mathbf{x}) := |\det M|^{-N} \sum_{\nu \in Q_N} \sum_{j<-N} \sum_{\mathbf{k} \in \mathbb{Z}^s} \chi_{M^j(\Omega+\nu)+\mathbf{k}}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^s. \tag{19}$$

Our remaining task is to show that v_N is uniformly bounded in N, as N tends to infinity, and converges to zero pointwise a.e. We will even show that v_N tends to zero uniformly. Recall that Q_N is assumed to be of the form (13), and hence

$$M^{-N}(\Omega+\nu) \subset \Xi := (1/2, 3/2)^s \quad \text{for all} \quad N \ge N_0,\ \nu \in Q_N.$$

In order to resolve the various summations in (19), we fix j and $\mathbf{k}$ and first observe that

$$\sum_{\nu \in Q_N} \chi_{M^j(\Omega+\nu)+\mathbf{k}} \le c_1 \chi_{M^{j+N}\Xi+\mathbf{k}}$$

holds for all $N \ge N_0$, $j < -N$, and $\mathbf{k} \in \mathbb{Z}^s$; and the constant $c_1 > 0$ only depends on the compact set Ω. Next the properties of the scaling matrix M (and the set Ξ where the special choice of Q_N enters) give the estimate

$$\sum_{j<-N} \chi_{M^{j+N}\Xi+\mathbf{k}} \le c_2 \chi_{B_r(0)+\mathbf{k}}$$

for all $N \geq N_0$, $\mathbf{k} \in \mathbb{Z}^s$, where the constants $c_2, r > 0$ only depend on M. Finally, summation over $\mathbf{k} \in \mathbb{Z}^s$ gives another constant $c_3 > 0$ which only depends on r (hence on M) such that

$$\sum_{\mathbf{k} \in \mathbb{Z}^s} \chi_{B_r(0)+\mathbf{k}} \leq c_3 .$$

Combining all these relations yields

$$v_N(\mathbf{x}) \leq c_1\, c_2\, c_3\, |\det M|^{-N}, \qquad \mathbf{x} \in \mathbb{R}^s,$$

and hence the uniform convergence of v_N to zero. This completes the proof of the lemma. ■

We can now complete the proof of Theorem 2.

Proof of Theorem 2(ii): It only remains to consider the lower frame bounds A and A_q. The proof follows the same argument as the proof of Theorem 2(i). The only differences are the use of Lemma 4 at certain places. E.g., the relation $A_q \geq A$ follows by using (16) in the third line of

$$\begin{aligned} K^q_{\Psi,\Psi}(f,f) &= \lim_{J\to\infty} \sum_{j\geq -J} K^q_j(f,f) \\ &= \lim_{J\to\infty} |\det M|^{-J} \sum_{\nu\in Q_J} \sum_{j\geq -J} K_j(T_\nu f\,,\, T_\nu f) \\ &= \lim_{J\to\infty} |\det M|^{-J} \sum_{\nu\in Q_J} \sum_{j\in\mathbb{Z}} K_j(T_\nu f\,,\, T_\nu f) \\ &= \lim_{J\to\infty} |\det M|^{-J} \sum_{\nu\in Q_J} K_{\Psi,\Psi}(T_\nu f\,,\, T_\nu f) \\ &\geq \lim_{J\to\infty} |\det M|^{-J} \sum_{\nu\in Q_J} A\|T_\nu f\|^2 = A\|f\|^2 \end{aligned}$$

for all $f \in L_2(\mathbb{R}^s)$ which have compact support. Since this is a dense subset of $L_2(\mathbb{R}^s)$, the relation $K^q_{\Psi,\Psi}(f,f) \geq A\|f\|^2$ holds for all $f \in L_2(\mathbb{R}^s)$. The opposite relation $A_q \leq A$ is shown by assuming the contrary, so that

$$K_{\Psi,\Psi}(f,f) \leq A_q - \epsilon \quad \text{for some} \quad f \in L_2(\mathbb{R}^s),\ \|f\| = 1,$$

and some $\epsilon > 0$. Without loss of generality, we can assume that f has compact support. The dilation-invariance of the operator $K_{\Psi,\Psi}$ gives

$$K_{\Psi,\Psi}(D^N f, D^N f) \leq A_q - \epsilon \quad \text{for all} \quad N \in \mathbb{N}.$$

By (15), there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} K^q_{\Psi,\Psi}(D^N f, D^N f) &< \sum_{j=0}^{\infty} K^q_j(D^N f, D^N f) + \frac{\epsilon}{2} \\ &= \sum_{j=0}^{\infty} K_j(D^N f, D^N f) + \frac{\epsilon}{2} \\ &\leq K_{\Psi,\Psi}(D^N f, D^N f) + \frac{\epsilon}{2} \leq A_q - \frac{\epsilon}{2} \end{aligned}$$

which contradicts with the definition of the lower frame bound A_q of X^q_Ψ. Thus the result stated in the theorem is obtained. ■

The proof of Theorem 3 gives another application of Lemmas 3 and 4.

Proof of Theorem 3: First we assume that $K_{\Psi,\Theta}$ is translation-invariant. Then, as in the previous proof, we have

$$K^q_{\Psi,\Theta}(f,g) = \lim_{J\to\infty} |\det M|^{-J} \sum_{\nu\in Q_J} K_{\Psi,\Theta}(T_\nu f\ ,\ T_\nu g)$$

for all $f, g \in L_2(\mathbb{R}^s)$ with compact support. Since $K_{\Psi,\Theta}$ is assumed to be translation-invariant, the right-hand side equals

$$\lim_{J\to\infty} |\det M|^{-J} \sum_{\nu\in Q_J} K_{\Psi,\Theta}(f,g) = K_{\Psi,\Theta}(f,g).$$

The equality extends to all functions in $L_2(\mathbb{R}^s)$ by density and boundedness of both operators. On the other hand, as a consequence of (12), the operator $K_{\Psi,\Theta}$ is invariant with respect to shifts $T_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^s$. Its dilation invariance implies that it is invariant with respect to shifts

$$T_{\mathbf{x}}, \qquad \mathbf{x} \in \bigcup_{j\in\mathbb{Z}} M^j \mathbb{Z}^s.$$

This union of sets is dense in $\mathbb{R}^s$, since M is a scaling matrix. Since translation is a continuous operation on $L_2(\mathbb{R}^s)$ and $K_{\Psi,\Theta}$ is bounded, it is invariant with respect to any shift $T_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^s$. ■

§5. A Class Related to the Meyer Wavelets

As a specific example for our results we look at a class of univariate functions ψ, namely

$$\mathcal{C} := \{\psi \in L_2(\mathbb{R}) : \ \operatorname{supp} \widehat{\psi} \subset [-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3]\}.$$

Functions in this subspace have been considered before by Dai and Larson [5: chapter 5], Hernandez et.al. [8] and Stöckler [11]. These works contain a complete characterization of functions $\psi \in \mathcal{C}$ which generate Riesz bases of $L_2(\mathbb{R})$ with scaling by 2.

Let us introduce some useful notation. If we let $I_1 := [2\pi/3, 4\pi/3]$, then the support of $\widehat{\psi}$ can also be written as

$$\Omega := [-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3] = I_1 \cup 2I_1 \cup (I_1 - 2\pi) \cup 2(I_1 - 2\pi),$$

where the sets on the right are essentially disjoint. Hence any element $\psi \in \mathcal{C}$ is uniquely defined by the 2×2-matrix

$$Z_\psi(\xi) := \begin{pmatrix} \widehat{\psi}(\xi) & \widehat{\psi}(\xi - 2\pi) \\ \widehat{\psi}(2\xi) & \widehat{\psi}(2\xi - 4\pi) \end{pmatrix}, \qquad \xi \in I_1. \tag{20}$$

This matrix plays a crucial role for the description of Riesz bases which are generated by a function in $\mathcal{C}$. Its matrix norm in ℓ_2 is denoted by $|Z_\psi(\xi)|$ and its adjoint is $Z_\psi(\xi)^*$.

The following result was established in [11].

Theorem 4. *Let $\psi \in \mathcal{C}$ and $X_\psi = \{\psi_{j,k} : \ j,k \in \mathbb{Z}\}$ be the family obtained by dyadic scaling and shift. Then X_ψ is a Riesz basis of $L_2(\mathbb{R})$ if and only if the matrix norms $|Z_\psi(\xi)|$ and $|Z_\psi(\xi)^{-1}|$ are essentially bounded on I_1. The exact Riesz bounds A, B are then given by*

$$B = \operatorname*{ess\,sup}_{\xi \in I_1} |Z_\psi(\xi)|^2, \qquad A^{-1} = \operatorname*{ess\,sup}_{\xi \in I_1} |Z_\psi(\xi)^{-1}|^2.$$

Furthermore, there exists no inexact frame X_ψ of this type; i.e. any affine frame X_ψ which is generated by $\psi \in \mathcal{C}$ and scaling by 2 is a Riesz basis.

The result in Theorem 2 shows that the quasi-affine system X_ψ^q is a frame if and only if the matrix-valued function Z_ψ in (20) satisfies the same condition as above. While X_ψ is a Riesz basis, the quasi-affine system X_ψ^q is an inexact frame. In our present work, we wish to give an explicit representation for the dual Riesz basis of X_ψ which is also the a-dual and quasi-affine dual in our terminology. By Theorem 4, the a-dual is unique, so that all equivalences of Corollary 3 hold in this case.

Theorem 5. *Let $\psi \in \mathcal{C}$ be such that X_ψ is a Riesz basis of $L_2(\mathbb{R})$. Then the dual Riesz basis is the family X_θ, where $\theta \in \mathcal{C}$ is defined by the matrix*

$$Z_\theta(\xi) := (Z_\psi(\xi)^*)^{-1}, \qquad \xi \in I_1.$$

Proof: For a moment, let $\psi, \theta \in \mathcal{C}$ be arbitrary elements such that $|Z_\psi(\xi)|$ and $|Z_\theta(\xi)|$ are essentially bounded on I_1. Then we can give an explicit representation of the sesquilinear forms K_j, $j \in \mathbb{Z}$, in (14). For any $f, g \in L_2(\mathbb{R})$, we define the vectors

$$f_j(\xi) := 2^{j/2}(\widehat{f}(2^j\xi), \widehat{f}(2^j(\xi - 2\pi)))$$

and $g_j(\xi)$ analogously. Hence, by Parseval's identity, we have

$$\langle f , g \rangle = \frac{1}{2\pi} \int_{I_1} \sum_{j \in \mathbb{Z}} f_j(\xi)\, g_j(\xi)^* \, d\xi,$$

since the dilates $2^j(I_1 \cup (I_1 - 2\pi))$ are an essentially disjoint covering of the real line. By another application of Parseval's identity and an obvious substitution, we obtain

$$K_j(f,g) = \frac{2^j}{2\pi}\int_0^{2\pi} \sum_{k\in\mathbb{Z}} \widehat{f}(2^j(\xi+2\pi k))\overline{\widehat{\psi}}(\xi+2\pi k)\cdot \sum_{k\in\mathbb{Z}} \overline{\widehat{g}}(2^j(\xi+2\pi k))\widehat{\theta}(\xi+2\pi k)\,d\xi.$$

By periodicity, we can change the domain of integration to $[2\pi/3, 8\pi/3] = I_1 \cup 2I_1$. The support property of $\widehat{\psi}$, $\widehat{\theta}$ leaves only two nonzero terms in the series for each ξ. More precisely,

$$2^{j/2}\sum_{k\in\mathbb{Z}} \widehat{f}(2^j(\xi+2\pi k))\overline{\widehat{\psi}}(\xi+2\pi k) = \begin{cases} f_j(\xi)\begin{pmatrix} \overline{\widehat{\psi}}(\xi) \\ \overline{\widehat{\psi}}(\xi-2\pi)\end{pmatrix} & \text{for } \xi\in I_1, \\ 2^{-1/2} f_{j+1}(\xi/2)\begin{pmatrix} \overline{\widehat{\psi}}(\xi) \\ \overline{\widehat{\psi}}(\xi-4\pi)\end{pmatrix} & \text{for } \xi\in 2I_1, \end{cases}$$

and a similar relation holds for the second series. A further substitution for $\xi \in 2I_1$ leads to

$$K_j(f,g) = \frac{1}{2\pi}\int_{I_1}\Bigg[f_j(\xi)\begin{pmatrix} \overline{\widehat{\psi}}(\xi) \\ \overline{\widehat{\psi}}(\xi-2\pi)\end{pmatrix}\ (\widehat{\theta}(\xi),\ \widehat{\theta}(\xi-2\pi))\ g_j(\xi)^*$$
$$+ f_{j+1}(\xi)\begin{pmatrix} \overline{\widehat{\psi}}(2\xi) \\ \overline{\widehat{\psi}}(2\xi-4\pi)\end{pmatrix}\ (\widehat{\theta}(2\xi),\ \widehat{\theta}(2\xi-4\pi))\ g_{j+1}(\xi)^*\Bigg]\,d\xi.$$

Our assumptions on ψ and θ are strong enough in order to ensure summability in $L_1(I_1)$ of both terms when the summation runs over all $j \in \mathbb{Z}$. By combining the terms with j and $j+1$, we obtain, from elementary matrix calculus, that

$$K_{\psi,\theta}(f,g) = \frac{1}{2\pi}\int_{I_1}\sum_{j\in\mathbb{Z}} f_j(\xi)Z_\psi(\xi)^* Z_\theta(\xi) g_j(\xi)^*\,d\xi.$$

Now, if $\theta \in \mathcal{C}$ is chosen as in the theorem, then we have

$$K_{\psi,\theta}(f,g) = \frac{1}{2\pi}\int_{I_1}\sum_{j\in\mathbb{Z}} f_j(\xi) g_j(\xi)^*\,d\xi = \langle\, f\,,\,g\,\rangle.$$

This gives the desired conclusion that X_θ is the dual Riesz basis of X_ψ. ■

We thus have shown that any affine frame which is generated by $\psi \in \mathcal{C}$ has a unique a-dual. We end our presentation with two examples. The first is a nontrivial example, i.e. not identity, of a translation-invariant operator $K_{\psi,\theta}$. The second example gives an operator which is not translation-invariant, so that none of the equivalent conditions of Corollary 3 hold.

Example 1. Let ψ be a Meyer wavelet, see e.g. [6, p. 119]. Then X_ψ is an orthonormal basis of $L_2(\mathbb{R})$. We take any $m \in \mathcal{M}(L_2(\mathbb{R}), L_2(\mathbb{R}))$ which is dilation-invariant as in (11) and $\theta := m * \psi$. Then the operator $K_{\psi,\theta}$ is translation-invariant with kernel m according to Theorem 1, since ψ is its own a-dual.

Example 2. We define $\psi \in \mathcal{C}$ by

$$\widehat{\psi} = \chi_{I_1} + \chi_{(2I_1 - 4\pi)} - \chi_{2I_1},$$

so that

$$Z_\psi(\xi) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \xi \in I_1.$$

Obviously, this matrix satisfies the condition in Theorem 4, and hence X_ψ is a Riesz basis of $L_2(\mathbb{R})$ with bounds $A = (3-\sqrt{5})/2$, $B = (3+\sqrt{5})/2$. The dual Riesz basis is defined by θ with

$$Z_\theta(\xi) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \xi \in I_1,$$

by our Theorem 5. There is no (dilation-invariant) multiplier $m \in \mathcal{M}(L_2(\mathbb{R}), L_2(\mathbb{R}))$ with the property $\psi = m * \theta$, since $\widehat{\psi}(\xi)/\widehat{\theta}(\xi)$ is not essentially bounded. Hence, the operator $K_{\psi,\psi}$ cannot be translation-invariant by Theorem 1.

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