Topics in Scattered Data Interpolation and Non-Uniform Sampling

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Abstract. This paper deals with some basic aspects of scattered data problems. In particular, the following topics are discussed: Self-adjoint scattered data interpolation, sampling sets and interpolation sets, and irregular sampling of shift-invariant spline spaces. Results on band-limited functions are presented as well as results on univariate splines on the real line.

§0. Introduction

Scattered data interpolation problems are encountered in practical applications when the location of measurements is irregularly (or non-uniformly) distributed. It has been one of the major challenges during the past two decades, in Approximation Theory and in Numerical Mathematics, to analyse these problems, to understand the basic structures involved, and to give solutions in terms of efficient numerical algorithms.

Radial basis functions are applied in scattered data interpolation, because they often show surprisingly good approximation properties and because the radial symmetry allows for a quick evaluation of the interpolant. For an account of topics discussed in the radial basis function literature, we may refer to the recent survey paper by Schaback [25] and to the references therein. Our paper does not aim at providing a comparable survey; the reader will recognize that the topics discussed here are more directed into applications in sampling of signals. However, we find it opportune to include at least some basic notions and ideas of radial basis interpolation and approximation; in this way the difference to other recently studied problems of irregular sampling will become more transparent.

Radial basis interpolation and approximation is often put into its variational formulation in order to see the basic structure of spaces involved. This is in complete analogy to the Ritz-Galerkin approach to Finite Elements, but

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the 'energy norm' takes here a very special expression in terms of a quadratic form based on a positive (or conditionally positive) definite matrix; see section 1 below. The equivalence of the usual L_2-norm with this energy norm leads to embedding properties of spaces which have been discussed for quite a long time (see Schoenberg [26], e.g.). The constants appearing in the statement of equivalence of norms also show up when the condition of the linear system of equations appearing in the numerical solution of radial basis interpolation is studied. These topics and others will be addressed in section 1.

Sampling sets and interpolation sets in Hilbert spaces are introduced in section 2 in order to develop and to present the famous results of H. Landau concerning irregular sampling of band-limited functions. It may be interesting to note that, via the Riesz representation of continuous linear functionals, generalized sampling sets can be identified with frames. In connection with oversampling of signals, or redundant signal processing, such frames have been studied to a great extent (see Benedetto [1], e.g.). It is therefore well-known that, given a frame (a generalized sampling set) and its related sampling values, a signal can be represented in terms of the dual frame. The computations involve symmetric operators, and they follow the same lines as solving least squares problems in Linear Algebra.

However, irregular sampling may be approached more directly without appealing to the dual frame. This has been thoroughly demonstrated for Hilbert spaces of band-limited functions, where it is best to refer to a series of papers by Feichtinger, Gröchenig and coworkers, [7–10,13]. Now, band-limited functions are certainly ideal to describe real world signals, due to the compact support in the Fourier transform domain. However, they show a severe deficiency, viz. it is a problem to represent band-limited functions in terms of a basis with good localization properties in the time domain. It is for this reason that we prefer spaces of spline functions over spaces of band-limited functions, for numerical calculations. In this way we are led to the problem of irregular sampling of spline spaces.

We study this problem in section 3 on the Hilbert space of square integrable cardinal splines of order $2m$. Theorem 3.3 will give a first answer to characterizing sampling sets on this space. To our knowledge, this is the first result in the mathematical literature to discuss this problem, and we believe that the result will be useful in order to find more concrete characterizations in the near future. We do hope, though, that this will open a new and promising research area.

A last note may be in order. One may object that deviating from band-limited functions will result in an approximation error, for real world signals. However, depending on the order of splines, this approximation error can be controlled quite satisfactorily in order to meet the needs in practical applications. Even cubic splines will in general do the job.

§1. Self-adjoint Scattered Data Interpolation

The problem of (self-adjoint, d-variate) scattered data interpolation reads as follows: Given

$H \in C(\mathbb{R}^d)$, the *basis function*,

$X_N = \{x_1, \dots, x_N\} \subset \mathbb{R}^d$, the set of *interpolation points*, and

$Y = \{y_1, \dots, y_N\} \subset \mathbb{R}$, the *data set*,

we would like to determine a function

$$s = \sum_{j=1}^{N} c_j \; H(\cdot - x_j) \tag{1.1}$$

solving for the interpolation conditions

$$s(x_i) = y_i \; , \quad i = 1, \dots, N \; . \tag{1.2}$$

For some classes of basis functions and for reasons of approximation powers (which will not be discussed here), one has to add a polynomial of low degree to (1.1), and the coefficients c_j have to fulfil a moment condition, accordingly.

It has been one of the major problems in radial basis function interpolation (where the basis function H is assumed to be radially symmetric) to decide whether the equations (1.2) are uniquely solvable. Quantitative results in this direction bound the spectrum of the interpolation matrix

$$\mathcal{H}_N = (H(x_i - x_j))_{i,j=1}^{N} \tag{1.3}$$

appearing in (1.2). In most cases of applications, $\mathcal{H}_N$ is real symmetric and positive definite; hence the performance of, say, the conjugate gradient method for solving (1.2) relies on the condition number

$$\mathrm{cond}_2 \mathcal{H}_N = \|\mathcal{H}_N\|_2 \|\mathcal{H}_N^{-1}\|_2 = \frac{\Lambda}{\lambda} \; , \tag{1.4}$$

with $\|\cdot\|_2$ denoting the spectral norm, and Λ (λ) the maximal (minimal, resp.) eigenvalue of $\mathcal{H}_N$.

For a general discussion of this topic it is opportune to bound the condition number independently of the number N of interpolation points, *i.e.*, one interpretes $\mathcal{H}_N$ as a finite (symmetric) section of a bi-infinite matrix

$$\mathcal{H} = (H(x_i - x_j))_{i,j=-\infty}^{+\infty} \; . \tag{1.5}$$

Associated with $\mathcal{H}$ we have the quadratic form

$$(\mathcal{H}c, c) = \sum_{i,j \in \mathbb{Z}} c_i c_j H(x_i - x_j) \; ,$$

which is well-defined and symmetric on the dense subspace $\ell_2^0(\mathbb{Z})$ of all finitely supported sequences. Hence our basic problem is to bound this quadratic form in terms of

$$A\ \|c\|^2 \le (\mathcal{H}c, c) \le B\ \|c\|^2\ , \quad c \in \ell_2^0(\mathbb{Z})\ , \tag{1.6}$$

with constants $0 < A \le B < \infty$, where $\|c\|^2 := \sum_{j\in\mathbb{Z}} |c_j|^2$. It is then clear that we have the uniform bound $\mathrm{cond}_2 \mathcal{H}_N \le \frac{B}{A}$, for any finite symmetric section of $\mathcal{H}$. An account of estimates in this direction as well as references up to 1995 can be found in R. Schaback's recent survey paper [25].

In most applications, the basis function H increases at infinity thus leading to an unbounded operator $\mathcal{H}$. It is one of the central objects of the paper [3] to study this phenomenon from an abstract, operator theoretic point of view. While the notation in [3] is chosen so as to hold for the multivariate case as well, the concrete examples are still univariate. The rough idea is to *modify the operator symmetrically* in terms of

$$\mathcal{H}' := C\ \mathcal{H}\ C^*$$

with C a banded bi-infinite matrix, and then to provide estimates (1.6) for the *preconditioned* operator $\mathcal{H}'$. This can be worked out subject that certain assumptions for the Fourier transform of H hold; for more details we refer to [3,18].

For an extension of radial basis interpolation it is useful to rewrite the entries of the matrix (1.5) in terms of *distributional notation*,

$$H(x_i - x_j) = < H * \delta_{x_j} | \delta_{x_i} > \ .$$

Here, δ_{x_i} stands for evaluation at the point x_i, $*$ is the convolution of tempered distributions, and $< \cdot | \cdot >$ denotes the canonical bilinear form of the underlying dual pairing. With this notation it is clear that radial basis interpolation is naturally imbedded into a much wider class of problems

$$\mathcal{H} = (< H * \mu_j | \mu_i >)_{i,j=-\infty}^{+\infty}\ ,$$

viz. H can be assumed to be a tempered function, and μ_i may be any compactly supported distributions. In interesting cases of applications, these distributions are finite linear combinations of point evaluations (representing consistent discretizations of differential operators). The spline example below will show some of the typical features in this approach.

There is still another way to look at (1.6) in terms of a *Riesz basis property*. Namely, if H is a positive definite function with square root $h \in L_2(\mathbb{R}^d)$, *i.e.*,

$$H = h * h^*\ ,$$

with $h^*(x) = \overline{h(-x)}$ the involution of h, then

$$< H * \mu_j | \mu_i > = < h * \mu_j | h * \mu_i >$$

and (1.6) tells us that the system

$$\varphi_i := h * \mu_i \ , \quad i \in \mathbb{Z} \ ,$$

provides a Riesz basis for $V := \text{clos}_{L_2} \text{ span } < \varphi_i \ ; \ i \in \mathbb{Z} >$. In other words: V is an isomorphic copy of $\ell_2(\mathbb{Z})$. This is an important aspect when interpreting radial basis interpolation as the Ritz-Galerkin equations for a related variational problem. Concerning this variational approach we refer to Madych and Nelson [24] or to the more recent papers [25,27]; see also the spline example below.

We now turn to some examples whose interest arises from problems of irregular sampling of signals.

1.1. Self-adjoint sinc-interpolation in $\mathbb{R}^1$

Here,

$$H(x) := \frac{\sin \pi x}{\pi x} = \text{sinc}(x)$$

is an $L_2(\mathbb{R})$-function with corresponding Fourier transform

$$H^\wedge = \chi_{[-\pi,+\pi]} \ .$$

This shows that the associated quadratic form can be expressed via Parseval's theorem through

$$(\mathcal{H}c, c) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left| \sum_{k=-\infty}^{+\infty} c_k \ e^{-i\xi x_k} \right|^2 \ d\xi \ , \quad c \in \ell_2(\mathbb{Z}) \ .$$

Bounding this form leads to the interesting question whether the set of exponentials $\varphi_k(\xi) = e^{-i\xi x_k}$, $k \in \mathbb{Z}$, forms a Riesz basis for the space $L_2([-\pi, +\pi])$. An answer is given by the famous Kadec 1/4-Theorem [22], see also [28, p. 42]. In this connection we also refer to the complete characterization of frames of complex exponentials by Jaffard [15]. Multivariate research in this direction includes Favier and Zalik [6], or Chui and Shi [4].

1.2. Self-adjoint spline interpolation on the real line

For given $m \in \mathbb{N}$, we consider the fundamental solution for the differential operator $D^m \ : \ f \mapsto f^{(m)}$, *i.e.*, the truncated power function

$$h(x) := h_m(x) := x_+^{m-1}/(m-1)! \ ;$$

its Fourier transform (in the sense of Gel'fand and Vilenkin [11]) is given by

$$h_m^\wedge(\xi) = (i\xi)^{-m} + \frac{i^{m-1}\pi}{(m-1)!}\delta^{(m-1)} \ .$$

Here, the order of polynomial growth of h is reflected by the order of the singularity of $h^\wedge$ at the origin. Hence, in order to build $L_2(\mathbb{R})$-functions from h, we have to take differences as is done when writing m-th order B-splines as combinations of translated versions of h_m.

In order to be more precise, we consider an *(extended) knot sequence of order* m

$$X = \{\cdots \le x_k \le x_{k+1} \le \cdots\} \ , \quad x_k < x_{k+m} \ , \tag{1.7}$$

subject to $\lim_{k\to-\infty} x_k = -\infty$ and $\lim_{k\to+\infty} x_k = +\infty$ being satisfied. Based on this knot sequence the divided differences of order m, with proper normalization, are given by

$$\mu_{j,m} := m! \sqrt{\frac{x_{j+m} - x_j}{m}} \ [x_j, x_{j+1}, \ldots, x_{j+m}|\cdot] \ , \quad j \in \mathbb{Z} \ . \tag{1.8}$$

Then

$$\varphi_{j,m} := h_m * \mu_{j,m} \ , \quad j \in \mathbb{Z} \ , \tag{1.9}$$

are B-splines of order m with respect to the knot sequence X, and (with the given normalization) they are well-known to form a Riesz basis on the subspace of $L_2(\mathbb{R})$ which they span. A more recent result by Jetter and Stöckler [19, Theorem 2.3] estimates Sobolev norms

$$\|f\|_{H^s(\mathbb{R})} := \left(\frac{1}{2\pi} \int_{\mathbb{R}} (1 + |\xi|^2)^s \ |f^\wedge(\xi)|^2 \ d\xi\right)^{\frac{1}{2}}$$

rather than $L_2(\mathbb{R})$-norms (case $s = 0$). We give the result for later use.

Theorem 1.1. *Let* $\mu \in \{1, \ldots, m\}$ *such that*

$$q_\mu := \inf_j (x_{j+\mu} - x_j) > 0 \ , \tag{1.10}$$

and let $0 \le t < \frac{1}{2}$. *Then for any* $c \in \ell_2(\mathbb{Z})$ *and the corresponding spline* $s = \sum_{j\in\mathbb{Z}} c_j \varphi_{j,m}$ *of order* m*, the Sobolev norm* $\|s\|_{H^{m-\mu+t}(\mathbb{R})}$ *is bounded above and below by a constant multiple of* $\|c\|_{\ell_2(\mathbb{Z})}$*, where the constants can be chosen to depend only on* m*,* t*, and* q_μ.

With some efforts, involving imbedding arguments and the Bramble-Hilbert lemma, this theorem can be used to derive a stability result for self-adjoint Lagrange interpolation on the real line by splines of even order $2m$, say. In order to define $2m$-th order splines in $L_2(\mathbb{R})$, we may refer to (1.7) – (1.9) with h_m, $\mu_{j,m}$, $\varphi_{j,m}$ replaced by h_{2m}, $\mu_{j,2m}$ and $\varphi_{j,2m}$, respectively.

Theorem 1.2. *Let (1.10) be satisfied for* $\mu = 1$ *and in addition, let* X *have bounded global mesh ratio, i.e.,*

$$\sup_{j,k} \frac{x_{k+1} - x_k}{x_{j+1} - x_j} < \infty \ .$$

Then, for any data vector $c \in \ell_2(\mathbb{Z})$, *there is a unique spline function* $\sigma = \sum_{j\in\mathbb{Z}} b_j \varphi_{j,2m} \in L_2(\mathbb{R})$ *solving for*

$$\sigma(x_i) = c_i \ , \quad i \in \mathbb{Z} \ .$$

Moreover, there exist constants $0 < \gamma_1 \le \gamma_2 < \infty$ *(which may be chosen to depend on* m *and* X *only) such that*

$$\gamma_1 \ \|c\|_{\ell_2(\mathbb{Z})} \le \|\sigma\|_{L_2(\mathbb{R})} \le \gamma_2 \ \|c\|_{\ell_2(\mathbb{Z})} \ . \tag{1.11}$$

A more general result, dealing with regularized splines, and an extension dealing with Hermite interpolation is given in [19], where also references can be looked up. The possibility of interpolating data of power growth (without treating the problem of stability) was already considered by Jakimovski and others, see [16] and the references therein.

It may be worth-while to add the remark that the spline solution σ of Theorem 1.2 is the unique element from the affine subspace

$$H_c := \{f \in H^m(\mathbb{R}) \ ; \ f(x_i) = c_i \ , \ i \in \mathbb{Z}\}$$

minimizing the Sobolev seminorm of order m. This is well-known from the variational theory of splines (which originated from the classical paper by Golomb and Weinberger [12]), and can be seen as follows: Given $f \in H_c$ and the spline solution σ, we find that $(f - \sigma)(x_i) = 0$ for $i \in \mathbb{Z}$, whence

$$\mu_{i,m}(f - \sigma) = 0 \ , \quad i \in \mathbb{Z} \ .$$

By Peano's Theorem this tells that $(f-\sigma)^{(m)}$ is orthogonal to the spline space

$$S_{m,X} = \mathrm{clos}_{L_2} \ \mathrm{span} < \varphi_{i,m} \ ; \ i \in \mathbb{Z} > \ ,$$

and in particular is orthogonal to $\sigma^{(m)} \in S_{m,X}$. Therefore,

$$\|f^{(m)}\|^2_{L_2(\mathbb{R})} = \|(f - \sigma)^{(m)}\|^2_{L_2(\mathbb{R})} + \|\sigma^{(m)}\|^2_{L_2(\mathbb{R})} \ge \|\sigma^{(m)}\|^2_{L_2(\mathbb{R})} \ ,$$

showing that σ is a minimal solution. For another minimal solution, we must have that $f-\sigma$ is a polynomial of order m which vanishes on X, hence vanishes identically.

1.3. Shift-invariant subspaces of $L_2(\mathbb{R}^d)$

For shift-invariant spaces, self-adjoint interpolation means cardinal interpolation, and this is equivalent to orthogonal L_2-projection. Here, the L_2-projector refers to a basis function $h \in L_2(\mathbb{R}^d)$, while the interpolation projector refers to its auto-correlation function $H = h * h^*$. Stability heavily relies on the fact that the symbol of the auto-correlation function,

$$H^\sim(\xi) := \sum_{\alpha\in\mathbb{Z}^d} H^\wedge(\xi + 2\pi\alpha) = \sum_{\alpha\in\mathbb{Z}^d} |h^\wedge(\xi + 2\pi\alpha)|^2 \ , \tag{1.12}$$

be bounded above and below, *i.e.*, there exist constants

$$0 < A := \text{ess inf}\, H^{\sim}(\xi) \le \text{ess sup}\, H^{\sim}(\xi) =: B < \infty \ . \tag{1.13}$$

The notation here is consistent with our earlier one if we change to multi-indices, viz. if we consider the Laurent operator $\mathcal{H}$ on $\ell_2(\mathbb{Z}^d)$ represented by the matrix

$$\mathcal{H} = (H(\alpha - \beta))_{\alpha,\beta\in\mathbb{Z}^d} \ .$$

Thus, A and B are bounds for the spectrum of $\mathcal{H}$ showing that the convolution equations

$$\sum_{\beta\in\mathbb{Z}^d} a_\beta \ H(\alpha-\beta) = c_\alpha \ , \quad \alpha \in \mathbb{Z}^d \ ,$$

can be deconvolved in a stable way. The constants A and B are also Riesz bounds for the basis

$$\varphi_\alpha := h(\cdot - \alpha) \ , \quad \alpha \in \mathbb{Z}^d \ ,$$

of the space

$$S_h := \text{clos}_{L_2(\mathbb{R}^d)} \ \text{span} < \varphi_\alpha \ ; \ \alpha \in \mathbb{Z}^d > \ ,$$

which is shift-invariant with respect to multi-integer shifts.

Shift-invariant spaces were intensively studied in the last decade, in particular in connection with multiscale methods such as subdivision algorithms and wavelet analysis. Their approximation power is now well understood. We refer to the fundamental paper by de Boor, DeVore and Ron [2] studying the orthogonal L_2-projector on S_h and scaled versions thereof. More general approximation processes and related approximation orders (including the order of simultaneous approximation) are considered by Jetter and Zhou [17,20,21].

§2. Sampling Sets and Interpolation Sets

Irregular sampling and scattered data interpolation can be treated in terms of (generalized) interpolation sets and sampling sets in Hilbert spaces. In case of self-adjoint interpolation, the underlying Hilbert space appears in the variational formulation of the problem, and the corresponding norm of an element is the so-called energy norm. Estimates of type (1.6) then show an imbedding property for the space.

We allow for more general situations and consider an arbitrary (separable) Hilbert space Y with norm $\|\cdot\|_Y$ related to the inner product $(\cdot,\cdot)$, and with canonical bilinear form $< \cdot,\cdot >$. If $\lambda \in Y'$ (*i.e.*, if λ is a continuous linear functional on Y) and if $\varphi \in Y$ denotes its representer, we have

$$< f, \lambda > = (f, \varphi) \ , \quad f \in Y \ .$$

Such functionals can be used for coding elements from Y. A set

$$\Lambda := \{\lambda_k \in Y' \; ; k \in \mathbb{Z}\} \tag{2.1}$$

is called an *interpolation set* if, for any sequence $d = (d_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, the 'interpolation problem'

$$< f, \lambda_k >= d_k \;, \quad k \in \mathbb{Z} \;, \tag{2.2}$$

has a unique solution $f \in Y$ depending continuously on the data d, *i.e.*,

$$\|f\|_Y \leq \text{const} \; \|d\|_{\ell_2(\mathbb{Z})} \; .$$

On the other hand, Λ is called a *sampling set* if there exist constants $0 < A \leq B < \infty$ such that

$$A \; \|f\|_Y^2 \leq \sum_{k\in\mathbb{Z}} | < f, \lambda_k > |^2 \leq B \; \|f\|_Y^2 \;, \quad f \in Y \; . \tag{2.3}$$

Sampling sets and frames can be identified, viz. if $\Phi := \{\varphi_k \in Y \; ; \; k \in \mathbb{Z}\}$ denotes the set of representers for the sampling set Λ, then Φ is a frame of Y, the *frame operator*

$$S \; : \; Y \to Y \; ; f \mapsto \sum_{k\in\mathbb{Z}} < f, \lambda_k > \; \varphi_k$$

satisfying the estimates $A \; \mathrm{id}_Y \leq S \leq B \; \mathrm{id}_Y$. Moreover, elements $f \in Y$ can be represented by

$$f = \sum_{k\in\mathbb{Z}} < f, \lambda_k > \; S^{-1}(\varphi_k) \; . \tag{2.4}$$

Therefore, the data $(< f, \lambda_k >)_{k\in\mathbb{Z}}$ represent a stable coding of f, and this coding is efficient if the *dual frame*

$$\tilde{\Phi} := \{\tilde{\varphi}_k := S^{-1}(\varphi_k) \; ; \; k \in \mathbb{Z}\}$$

is easily available, and if the sum in (2.4) can be evaluated through a fast algorithm.

Several results from the literature can now be compactly reformulated in terms of these new notions. For example, Theorem 1.2 tells that, with the assumptions given there, the point evaluation functionals $\lambda_k \; : \; f \mapsto f(x_k)$, $k \in \mathbb{Z}$, form a sampling set and also an interpolation set for the space of spline functions

$$Y = S_{2m,X} = \mathrm{clos}_{L_2} \; \mathrm{span} \; < \varphi_{i,2m} \; ; \; i \in \mathbb{Z} > \; .$$

Here, Y is a Hilbert space itself (as a closed subspace of $L_2(\mathbb{R})$). We mention that point evaluation is a continuous operation on the space; this follows, e.g., from the fact that the L_2-norm and the Sobolev norm of order m are equivalent on Y.

In order to have another example, we quote H. Landau's famous results on coding of multivariate band-limited signals [23]; see also Gröchenig and Razafinjatovo [14]. Let $\emptyset \neq \Omega \subset \mathbb{R}^d$ be a *compact* set, and let

$$Y := B_\Omega := \{f \in L_2(\mathbb{R}^d) \ ; \ \text{supp}\ f^\wedge \subset \Omega\} \ . \tag{2.5}$$

We define $s \in B_\Omega$ through its Fourier transform

$$s^\wedge := \chi_\Omega \ . \tag{2.6}$$

Then

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f^\wedge(\xi)\ \chi_\Omega(\xi)\ e^{+ix\cdot\xi}\ d\xi \\ &= \frac{1}{(2\pi)^d} (f^\wedge, s(\cdot - x)^\wedge)_{L_2(\mathbb{R}^d)} = (f, s(\cdot - x))_{L_2(\mathbb{R}^d)} \ , \end{aligned}$$

showing that B_Ω, as a subspace of $L_2(\mathbb{R}^d)$, is a reproducing kernel Hilbert space. Hence point evaluations are continuous linear functionals, and we may test a given knot set

$$X = \{x_k \in \mathbb{R}^d \ ; k \in \mathbb{Z}\}$$

on the property of being a sampling set or an interpolation set.

To this end, Landau (refering to Arne Beurling) introduces *the upper and the lower density of the point set* X assuming that X is *minimally separated, i.e.,*

$$\inf_{k\neq\ell} \|x_k - x_\ell\|_\infty =: 2\delta > 0 \ . \tag{2.7}$$

With $B_r(x) \subset \mathbb{R}^d$ denoting the ℓ_∞-cube of radius r centered at the point x, and $\#(X \cap B_r(x))$ the number of points from X inside $B_r(x)$, the upper density of X is defined by

$$D^+(X) := \limsup_{r\to\infty} \frac{\max_{x\in\mathbb{R}^d} \#(X \cap B_r(x))}{(2r)^d} \ ,$$

while the lower density of X is given by

$$D^-(X) := \liminf_{r\to\infty} \frac{\min_{x\in\mathbb{R}^d} \#(X \cap B_r(x))}{(2r)^d} \ .$$

With this notation H. Landau's results can be formulated as follows:

Theorem 2.1. *Let $X = (x_k)_{k\in\mathbb{Z}} \subset \mathbb{R}^d$ satisfy the property (2.7) of minimal separation, and let $\Omega \subset \mathbb{R}^d$ be compact with Lebesgue measure $|\Omega|$. Then:*

(a) If X is a set of interpolation for B_Ω, then $D^+(X) \leq |\Omega|/(2\pi)^d$.

(b) If X is a set of sampling for B_Ω, then $D^-(X) \geq |\Omega|/(2\pi)^d$.

In particular, in case of uniform sampling of band-limited functions this result shows the correct Nyquist rate.

§3. Irregular Sampling of Shift-Invariant Spline Spaces

This section is devoted to a preliminary study of irregular sampling of shift-invariant spline spaces. To our knowledge, this is the first paper to discuss this problem. Our interest stems from various apparent facts which are important for the representation of signals. First, splines allow a basis of compact support with reasonable decay in the Fourier transform domain (while spaces of band-limited functions rely on a basis with poor localization properties in the time domain). Second, a cardinal spline series can be efficiently evaluated on fine uniform grids (as is the case for any principal shift-invariant space with refinable basis function). Third, there is no problem to approximate band-limited functions by scaled cardinal splines at a reasonable approximation order. Therefore, shift-invariant spline spaces seem to be a perfect tool in order to approximate real world signals, and irregular or non-uniform sampling originates from practical needs where the location of sampling points will usually not be at our disposal.

We will concentrate here on the shift-invariant space $Y = S_{2m,\mathbb{Z}}$ of univariate splines of order $2m$, with simple knots at the integers. Considered as a subspace of $L_2(\mathbb{R})$, this is a Hilbert space carrying further norms which are equivalent to the $L_2(\mathbb{R})$-norm, viz. the Sobolev norm $\|f\|_m$ of order m, the *discrete norm* $\|f|_{\mathbb{Z}}\|_{\ell_2(\mathbb{Z})}$, and the *mixed norm* $\|\cdot\|_{m,\mathbb{Z}}$ defined by

$$\|f\|_{m,\mathbb{Z}}^2 := \|f|_{\mathbb{Z}}\|_{\ell_2(\mathbb{Z})}^2 + \|f^{(m)}\|_{L_2(\mathbb{R})}^2 ,$$

where $f|_{\mathbb{Z}} := (f(k))_{k\in\mathbb{Z}}$ stands for evaluating f on the knot sequence $\mathbb{Z}$. This equivalence of norms can be readily seen from a proper application of Theorem 1.1 and Theorem 1.2. Another result will be used as well:

Lemma 3.1. *$U := \{f^{(m)}\ ;\ f \in S_{2m,\mathbb{Z}}\}$ is a dense subspace of $S_{m,\mathbb{Z}}$, with respect to the $L_2(\mathbb{R})$-norm.*

Proof: The result follows from the more general assertion that

$$U_k := \{f'\ ;\ f \in S_{2m-k,\mathbb{Z}}\} \subset S_{2m-k-1,\mathbb{Z}}$$

is dense in $S_{2m-k-1,\mathbb{Z}}$, for $k = 0,\ldots,m-1$.

In order to see this, let $g \in S_{2m-k-1,\mathbb{Z}}$ and let $\varepsilon > 0$. We choose $\tilde{g} \in S_{2m-k-1,\mathbb{Z}}$, with compact support, such that

$$\|g - \tilde{g}\|_{L_2(\mathbb{R})} < \frac{\varepsilon}{2} .$$

Putting $I(f) := \int_{\mathbb{R}} f(x)dx$, we find another $\hat{g} \in S_{2m-k-1,\mathbb{Z}}$, with compact support, satisfying

$$I(\hat{g}) = 1 \quad \text{and} \quad \|\hat{g}\|_{L_2(\mathbb{R})} < \frac{\varepsilon}{2|I(\tilde{g})|} .$$

Now $g^* := \tilde{g} - I(\tilde{g})\hat{g} \in S_{2m-k-1,\mathbb{Z}}$ has compact support, and $I(g^*) = 0$, whence $g^* \in U_k$. Since $\|g^* - \tilde{g}\|_{L_2(\mathbb{R})} < \frac{\varepsilon}{2}$, we find $\|g^* - g\|_{L_2(\mathbb{R})} < \varepsilon$. The lemma is proved. ∎

We would like to characterize sampling sets for Y based on an extended knot sequence of order m as given by (1.7). The corresponding functionals are described by $f|_X$, *i.e.*, evaluating $f \in Y$ on the knot sequence X. Evaluation here means taking the function value at each knot x_k, or considering a block $f(x_k), f'(x_k), \ldots, f^{(\mu-1)}(x_k)$ of consecutive derivatives if the multiplicity of x_k is $\mu > 1$. Thus only derivatives up to order $m-1$ will be involved. Therefore, the functionals considered by $f|_X$ are continuous on the Hilbert space Y.

Based on X, we will refer to the divided differences $\mu_{j,m}$ in (1.8) and the corresponding B-splines $\varphi_{j,m}$ in (1.9), for $j \in \mathbb{Z}$. One further property of X will be needed: *We require that the reduced knot set* $X^- = (y_k)_{k\in\mathbb{Z}}$ *has bounded global mesh ratio,*

$$\sup_{j,k} \frac{y_{k+1} - y_k}{y_{j+1} - y_j} < \infty \; ; \tag{3.1}$$

here, the *reduced set* is derived from X by counting each knot x_k with multiplicity 1 only.

Lemma 3.2. *With the given assumptions, the following are equivalent:*

(a) There exist constants $0 < A \le B < \infty$ *such that*

$$A \, \|f\|^2_{L_2(\mathbb{R})} \le \|f|_X\|^2_{\ell_2(\mathbb{Z})} \le B \, \|f\|^2_{L_2(\mathbb{R})} \, , \quad f \in S_{2m,\mathbb{Z}} \, . \tag{3.2}$$

(b) There exist constants $0 < \tilde{A} \le \tilde{B} < \infty$ *such that*

$$\tilde{A} \, \|g\|^2_{L_2(\mathbb{R})} \le \sum_{j\in\mathbb{Z}} |(g, \varphi_{j,m})|^2 \le \tilde{B} \, \|g\|^2_{L_2(\mathbb{R})} \, , \quad g \in S_{m,\mathbb{Z}} \, . \tag{3.3}$$

Proof: 1. Both upper estimates always hold true: The reduced set X^- has minimal separation distance $\inf_j (y_{j+1} - y_j) =: 2\delta > 0$, and for any $j \in \mathbb{Z}$ the evaluation functionals $f^{(\mu)}(y_j)$, $\mu = 0, \ldots, m-1$, are continuous on the local Sobolev space $H^m([y_j - \delta, y_j + \delta])$. Therefore,

$$\begin{aligned} \|f|_X\|^2_{\ell_2(\mathbb{Z})} &\le \sum_{j\in\mathbb{Z}} \sum_{\mu=0}^{m-1} |f^{(\mu)}(y_j)|^2 \\ &\le \mathrm{const}_{m,\delta} \sum_{j\in\mathbb{Z}} \|f\|^2_{H^m([y_j-\delta, y_j+\delta])} \\ &\le \mathrm{const}_{m,\delta} \, \|f\|^2_{H^m(\mathbb{R})} \, . \end{aligned}$$

This yields the upper bound in (3.2), since the Sobolev norm and the L_2-norm are equivalent on $S_{2m,\mathbb{Z}}$. Moreover, the B-splines $\varphi_{j,m}$, $j \in \mathbb{Z}$, form a Riesz basis on the space which they span, whence are a Bessel sequence in $L_2(\mathbb{R})$; this gives the upper bound in (3.3).

2. In order to deal with the lower bounds we note that, by Lemma 3.1, (b) is equivalent to

(b') There exist constants $0 < \tilde{A} \leq \tilde{B} < \infty$ *such that*

$$\tilde{A}\ \|f^{(m)}\|^2_{L_2(\mathbb{R})} \leq \sum_{j\in\mathbb{Z}} |(f^{(m)}, \varphi_{j,m})|^2 \leq \tilde{B}\ \|f^{(m)}\|^2_{L_2(\mathbb{R})}\ , \quad f \in S_{2m,\mathbb{Z}}\ . \tag{3.4}$$

We also use the fact that the bi-infinite vector $\mu_X(f) := ((f^{(m)}, \varphi_{j,m}))_{j\in\mathbb{Z}}$ can be written as

$$\mu_X(f) = C_X\ f|_X \tag{3.5}$$

with the bi-infinite, banded matrix C_X having uniform bounded entries (the bound depending on the global mesh ratio and m only). This can be seen as in [19, section 4] where the Hermite case is also discussed.

3. In order to see that $(b') \Rightarrow (a)$, we rely on the fact that the L_2-norm and the mixed norm $\|\cdot\|_{m,X}$ defined by

$$\|f\|^2_{m,X} := \|f|_X\|^2_{\ell_2(\mathbb{Z})} + \|f^{(m)}\|^2_{L_2(\mathbb{R})}$$

are equivalent on $S_{2m,\mathbb{Z}}$, by the considerations in 1. Therefore, given $f \in S_{2m,\mathbb{Z}}$, we find

$$\begin{aligned}\|f\|^2_{L_2(\mathbb{R})} &\leq \mathrm{const}_{m,X}\ \{\|f|_X\|^2_{\ell_2(\mathbb{Z})} + \|f^{(m)}\|^2_{L_2(\mathbb{R})}\} \\ &\leq \mathrm{const}_{m,X}\ \{\|f|_X\|^2_{\ell_2(\mathbb{Z})} + \|\mu_X(f)\|^2_{\ell_2(\mathbb{Z})}\} \\ &\leq \mathrm{const}_{m,X}\ \|f|_X\|^2_{\ell_2(\mathbb{Z})}\end{aligned}$$

where we have used the lower estimate in (3.4) and the properties of the matrix C_X in the representation (3.5). The constants in these estimates change, of course, from line to line, and the indices point to the parameters where the constants may be chosen to depend on. This verifies the lower estimate in (3.2).

Let us now start from the lower estimate in (3.2), and let us assume that the lower estimate in (3.4) does not hold, *i.e.*, for any $\varepsilon > 0$ there exists $f \in S_{2m,\mathbb{Z}}$ with $\|f^{(m)}\|_{L_2(\mathbb{R})} = 1$ such that $\|\mu_X(f)\|^2_{\ell_2(\mathbb{Z})} < \varepsilon$. In order to arrive at a contradiction, we construct

$$u \in S_{2m,X} \quad \text{satisfying} \quad u|_X = f|_X \tag{3.6}$$

(note that $f|_X$ is square summable, by the upper estimate in (3.2), so that [19, Theorem 4.7] can be applied) and

$$v \in S_{2m,\mathbb{Z}} \quad \text{satisfying} \quad v|_{\mathbb{Z}} = u|_{\mathbb{Z}}\ . \tag{3.7}$$

These are both self-adjoint interpolation problems. By the minimal property of spline interpolation as discussed in section 1.2, we have

$$\|v^{(m)}\|_{L_2(\mathbb{R})} \leq \|u^{(m)}\|_{L_2(\mathbb{R})} \ ,$$

and by the Riesz basis property of $\{\varphi_{j,m}\}_{j\in\mathbb{Z}}$,

$$\|u^{(m)}\|_{L_2(\mathbb{R})} \leq \text{const}_m \ \|\mu_X(u)\|_{\ell_2(\mathbb{Z})} = \text{const}_m \ \|\mu_X(f)\|_{\ell_2(\mathbb{Z})} \ .$$

Therefore, the spline function $\tilde{f} := f - v \in S_{2m,\mathbb{Z}}$ satisfies

$$\|\tilde{f}\|_{H^m(\mathbb{R})} \geq \|f^{(m)}\|_{L_2(\mathbb{R})} - \|v^{(m)}\|_{L_2(\mathbb{R})} > 1 - \text{const}_m \ \varepsilon \ . \tag{3.8}$$

On the other hand,

$$\|(u-v)|_X\|^2_{\ell_2(\mathbb{Z})} \leq \text{const}_{m,\delta} \ \|u-v\|^2_{H^m(\mathbb{R})}$$

by the same argument as in 1., and

$$\|u-v\|^2_{H^m(\mathbb{R})} \leq \text{const}_m \ \|(u-v)^{(m)}\|^2_{L_2(\mathbb{R})}$$

by the Bramble-Hilbert Lemma, since $(u-v)|_{\mathbb{Z}} = 0$. From this

$$\begin{aligned} \|\tilde{f}|_X\|_{\ell_2(\mathbb{Z})} &= \|(u-v)|_X\|_{\ell_2(\mathbb{Z})} \\ &\leq \ \text{const}_{X,m} \ \{\|u^{(m)}\|_{L_2(\mathbb{R})} + \|v^{(m)}\|_{L_2(\mathbb{R})}\} \\ &\leq \text{const}_{X,m}\|u^{(m)}\|_{L_2(\mathbb{R})} < \text{const}_{X,m} \ \varepsilon \ . \end{aligned} \tag{3.9}$$

Since ε was arbitrary, (3.8) and (3.9) contradict the lower bound in (3.2). ■

Theorem 3.3. *For a given extended knot sequence X of order m, such that the reduced sequence X^- has bounded global mesh ratio, the following are equivalent.*

(a) X provides a set of sampling for $S_{2m,\mathbb{Z}}$, i.e., (3.2) holds true.

(b) The orthogonal projector $\mathcal{P}_V \ : \ L_2(\mathbb{R}) \to V$ onto the spline space

$$V := S_{m,X} = clos_{L_2(\mathbb{R})} \ span \ \{\varphi_{j,m} \ ; \ j \in \mathbb{Z}\}$$

satisfies the estimate

$$\|\mathcal{P}_V(g)\|_{L_2(\mathbb{R})} \geq const \ \|g\|_{L_2(\mathbb{R})} \ , \quad g \in S_{m,\mathbb{Z}} \ , \tag{3.10}$$

with a positive constant.

Proof: We show that (3.10) is equivalent to (3.3). But this is clear, since $\|\mathcal{P}_V(g)\|_{L_2(\mathbb{R})}$ is always bounded by $\|g\|_{L_2(\mathbb{R})}$, and $\|\mathcal{P}_V(g)\|^2_{L_2(\mathbb{R})}$ is bounded above and below by some multiple of $\sum_{j\in\mathbb{Z}} |(g,\varphi_{j,m})|^2$, since $\{\varphi_{j,m}\ ;\ j\in\mathbb{Z}\}$ is a Riesz basis of $V = S_{m,X}$. ■

This theorem gives a characterization of sampling sets X for cardinal spline spaces of even order $2m$ in terms of L_2-projection operators onto the space of splines of order m with respect to the knot sequence X, namely that this projector is coercive for cardinal splines of order m. In our future work, we aim at establishing criteria for the knot set X itself which are either necessary or sufficient for (3.10) to be satisfied. For example, it is clear that any type of "oversampling" with $\mathbb{Z} \subset X$ yields condition (3.10) trivially.

An equivalent formulation can be given by means of the matrix

$$A = \Big((N_{i,m}\ ,\ \varphi_{j,m}) \Big)_{i,j\in\mathbb{Z}}\ , \tag{3.11}$$

with $N_{i,m}$, $i\in\mathbb{Z}$, the basis of cardinal B-splines of $S_{m,\mathbb{Z}}$ and $\varphi_{j,m}$, $j\in\mathbb{Z}$, the Riesz basis of $S_{m,X}$. Then A has rows and columns of finite support, and (3.10) is equivalent to

$$AA^* \geq \text{const id}_{\ell_2(\mathbb{Z})}\ . \tag{3.12}$$

A necessary condition can be given in terms of the lower density $D^-(X)$.

Theorem 3.4. *Let X be an extended knot sequence of order m, such that the reduced sequence X^- has bounded global mesh ratio. If X is a set of sampling for $S_{2m}(\mathbb{Z})$, then $D^-(X) \geq 1$.*

Proof: Let us assume, to the contrary, that $D^-(X) < 1$ holds. Then there exist $r > 0$ and an interval $I_r = [a, a+r]$ such that

$$\#(X \cap I_r) < [r] - 2m\ . \tag{3.13}$$

I_r contains at least $[r]$ integers, which we denote by $i, \ldots, i+[r]-1$. Let B denote the submatrix of A which contains only rows i through $i+[r]-m-1$. If we can show that the rows of B are linearly dependent, then A cannot satisfy (3.12), since there exists a non-trivial solution $c \in \ell_2^0(\mathbb{Z})$ to the homogeneous linear system $c^*A = 0$. Note that the support of each $N_{i+\nu,m}$, $0 \leq \nu \leq [r]-m-1$, is contained in I_r. On the other hand, (3.13) implies that

$$\#\{\varphi_{j,m};\ \operatorname{supp}\varphi_{j,m} \cap I_r \neq \emptyset\} < [r] - m\ .$$

Hence the rows of B must be linearly dependent and A does not satisfy (3.12). ■

A general approach in order to find further sufficient conditions is to establish (3.12) by constructing another bounded operator B on $\ell_2(\mathbb{Z})$ such that $AB = \text{id}_{\ell_2(\mathbb{Z})}$. For the special case $\mathbb{Z} \subset X$ and X contains one additional knot per interval $(i, i+1)$, $i \in \mathbb{Z}$, an explicit construction of such B is carried out in [5].

Remark. After presenting Theorem 3.3 at the conference, we were approached by a few people to claim that condition (b) should be equivalent to the 'symmetric' condition originating from (b) when the roles of $\mathbb{Z}$ and X are interchanged. This claim is false since (3.12) does not imply that A^*A is bounded below, too.

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References

1. Benedetto, J. J., Irregular sampling and frames, in *Wavelets: A Tutorial in Theory and Applications*, C.K. Chui (ed.), Academic Press, New York, 1992, 445–508.
2. de Boor, C., DeVore, R. and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$, *Trans. Amer. Math. Soc.* **341** (1994), 787–806.
3. Chui, C. K., Jetter, K., Stöckler, J. and J. D. Ward, Wavelets for analyzing scattered data: An unbounded operator approach, *Appl. Comp. Harmonic Analysis* **3** (1996), 254–267.
4. Chui, C. K. and X. L. Shi, On stability bounds of perturbed multivariate trigonometric systems, *Appl. Comp. Harmonic Analysis* **3** (1996), 283–287.
5. Dahmen, W. and C. A. Micchelli, Banded matices with banded inverses, II: Locally finite decomposition of spline spaces, *Constr. Approximation* **9** (1993), 263–281.
6. Favier, S. J. and R. A. Zalik, On the stability of frames and Riesz bases, *Appl. Comp. Harmonic Analysis* **2** (1995), 160–173.
7. Feichtinger, H. G. and K. Gröchenig, Iterative reconstruction of multivariate band-limited functions from irregular sampling values, *SIAM J. Math. Anal.* **23** (1992), 244–261.
8. Feichtinger, H. G. and K. Gröchenig, Irregular sampling theorems and series expansions of band-limited functions, *J. Math. Anal. Appl.* **167** (1992), 530–556.
9. Feichtinger, H. G. and K. Gröchenig, Theory and practice of irregular sampling, in *Wavelets: Mathematics and Applications*, J. J. Benedetto and M. W. Frazier (eds.), CRC Press, Boca Raton, 1994, 305–363.
10. Feichtinger, H. G., Gröchenig, K. and Th. Strohmer, Efficient numerical methods in non-uniform sampling theory, *Numer. Math.* **69** (1995), 423–440.

11. Gel'fand, I. M. and N. Ya. Vilenkin, *Generalized Functions*, vol. 4, Academic Press, New York, 1964.
12. Golomb, M. and H. F. Weinberger, Optimal approximation and error bounds, in *On Numerical Approximation*, R. E. Langer (ed.), The University of Wisconsin Press, Madison, 1959, 117–190.
13. Gröchenig, K., Irregular sampling, Toeplitz matrices, and the approximation of entire functions of exponential type, ms., 20 p.
14. Gröchenig, K. and H. Razafinjatovo, On Landau's necessary density conditions for sampling and interpolation of band-limited functions, ms., 11 p.
15. Jaffard, S., A density criterion for frames of complex exponentials, *Michigan Math. J.* **38** (1991), 339–348.
16. Jakimovski, A., Spline interpolation of data of power growth, in *Recent Advances in Fourier Analysis and its Applications*, J. S. Byrnes and J. F. Byrnes (eds.), Kluwer Academic Publishers, Dordrecht, 1990, 73–75.
17. Jetter, K., Multivariate approximation from the cardinal interpolation point of view, in *Approximation Theory VII*, E. W. Cheney, C. K. Chui and L. L. Schumaker (eds.), Academic Press, New York, 1993, 131–161.
18. Jetter, K. and J. Stöckler, A generalization of de Boor's stability result and symmetric preconditioning, *Advances in Comp. Math.* **3** (1995), 353–367.
19. Jetter, K. and J. Stöckler, Riesz bases of splines and regularized splines with multiple knots, *J. Approx. Theory*, to appear.
20. Jetter, K. and D.-X. Zhou, Order of linear approximation from shift-invariant spaces, *Constr. Approximation* **11** (1995), 423–438.
21. Jetter, K. and D.-X. Zhou, Approximation order of linear approximation schemes from shift-invariant spaces, ms., 19 p., January 1996.
22. Kadec, M. I., The exact value of the Paley-Wiener constant, *Soviet Math. Dokl.* **5** (1964), 559–561.
23. Landau, H. J., Necessary density conditions for sampling and interpolation of certain entire functions, *Acta Math.* **117** (1967), 37–52.
24. Madych, W. R. and S. A. Nelson, Multivariate interpolation: a variational theory, unpublished manuscript, 1983.
25. Schaback, R., Multivariate interpolation and approximation by translates of a basis function, in *Approximation and Interpolation*, Approximation Theory VIII, vol. 1, C. K. Chui and L. L. Schumaker (eds.), World Scientific, Singapore, 1995, 491–514.
26. Schoenberg, I. J., Metric spaces and completely monotone functions, *Annals of Math.* **39** (1938), 811–841.
27. Wu, Z.-M. and R. Schaback, Local error estimates for radial basis function interpolation of scattered data, *IMA J. Numer. Anal.* **13** (1993), 13 – 27.
28. Young, R., *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.

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