

# On embeddings of $Out(F_n)$ , the outer automorphism group of the free group of rank $n$ , into the group $Out(F_m)$ for $m > n$

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Let  $F_n$  be the free group of finite rank  $n$ ,  $Aut(F_n)$ ,  $Out(F_n)$ , and  $Inn(F_n)$  be the groups of all, outer, and inner automorphisms of  $F_n$  respectively. Clearly, the group  $Aut(F_n)$  can be embedded into the group  $Aut(F_{n+1})$ . In the article of Magnus and Tretkoff [1] it is mentioned, that the group  $Out(F_n)$  can be embedded into the group  $Out(F_{n+1})$  for  $n \geq 2$ . This assertion was disproved by D. G. Khramtsov [2].

In this article we prove the following theorem.

**Theorem.** *For each  $n \geq 1$  the group  $Out(F_n)$  can be embedded into the group  $Out(F_m)$  where  $m = 1 + (n - 1)k^n$ , and  $k$  is an arbitrary natural number co-prime to  $n - 1$ .*

*Proof.* For  $n = 1$  the theorem is obvious. So, furthermore we assume that  $n \geq 2$ . Therefore  $Inn(F_n) \simeq F_n$ , and we identify the element  $x \in F_n$  with the automorphism  $\hat{x} \in Inn(F_n)$ , acting by the rule:  $g \mapsto x^{-1}gx$ ,  $g \in F_n$ .

Let  $k$  be a fixed natural number co-prime to  $n - 1$ , and let  $H$  be the verbal subgroup generated by the commutator subgroup  $F_n'$  and by all words of the form  $w^k$ . Then  $F_n/H \simeq \underbrace{Z_k \times \cdots \times Z_k}_n$ , and the rank of  $H$  is  $1 + (n - 1)k^n$ . The group  $Aut(F_n)$  can

be embedded into the group  $Aut(H)$  using the map  $\alpha \mapsto \alpha|_H$ , where  $\alpha \in Aut(F_n)$ . Indeed, if  $\alpha \in Aut(F_n)$  and the restriction of  $\alpha$  on  $H$  is the identical automorphism, then  $\alpha(x^{k^n}) = x^{k^n}$  for each  $x \in F_n$ . Since the root extraction in the group  $F_n$  is unique,  $\alpha(x) = x$ . This embedding induces an embedding of the group  $Aut(F_n)/H$  into the group  $Aut(H)/H \simeq Out(H) \simeq Out(F_m)$ . We have  $(Aut(F_n)/H)/(F_n/H) \simeq Out(F_n)$ . Further we will prove that the corresponding extension of the group  $F_n/H$  by the group  $Out(F_n)$  is splittable. Then the group  $Out(F_n)$  is embeddable into  $Aut(F_n)/H$ , and hence into  $Out(F_m)$  also.

Further we will use the following composition rule for automorphisms: if  $\phi, \psi \in Aut(F_n)$ , then  $\phi\psi(x) = \psi(\phi(x))$  for  $x \in F_n$ . Denote  $[x, y] = xyx^{-1}y^{-1}$ ,  $x^y = y^{-1}xy$ . Let  $X = \{x_1, \dots, x_n\}$  be a base of  $F_n$ . Nielsen proved (see for example [3]), that the group  $Aut(F_n)$  is generated by the set of automorphisms  $\{E_{xy} \mid x, y \in X^{\pm 1}, y \neq x, x^{-1}\}$ , of the form  $E_{xy} : x \mapsto xy, z \mapsto z$  for  $z \in X^{\pm 1} \setminus \{x, x^{-1}\}$ , together with the automorphism  $\tau$ , which inverts  $x_1$  and fixes the remaining generators from  $X$ . Gersten showed in [4] that  $Aut(F_n)$  has the presentation with this generator set and the following set of defining relations:

- (R1)  $E_{xy}^{-1} = E_{xy^{-1}}$ ,
- (R2)  $[E_{xy}, E_{zt}] = 1$  for  $z \neq x, y, y^{-1}, t \neq x, x^{-1}$ ,
- (R3)  $[E_{xy}, E_{yz}] = E_{xz}$  for  $z \neq x, x^{-1}$ ,
- (R4)  $B_{yx} = B_{y^{-1}x^{-1}}$ , where  $B_{yx} = E_{xy}E_{y^{-1}x}E_{x^{-1}y}^{-1}$ ,

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$$(\mathcal{R5}) B_{yx}^4 = 1,$$

$$(\mathcal{R6}) \tau^2 = 1,$$

$$(\mathcal{R7}) \tau E_{xy} \tau = E_{\tau(x)\tau(y)}.$$

Write shortly the presentation of  $Aut(F_n)$  as

$$Aut(F_n) = \langle \tau, E_{xy} \mid r_1(E_{xy}, \tau) = \cdots = r_l(E_{xy}, \tau) = 1 \rangle,$$

where  $r_1, \dots, r_l$  are words, corresponding to the relations  $(\mathcal{R1}) - (\mathcal{R7})$ . Then

$$\begin{aligned} Aut(F_n)/H &= \langle \tau, E_{xy}, x_1, \dots, x_n \mid r_1(E_{xy}, \tau) = \cdots = r_l(E_{xy}, \tau) = 1, \\ &\quad x_1 = r_{l+1}(E_{xy}), \dots, x_n = r_{l+n}(E_{xy}) \rangle / H, \end{aligned} \quad (1)$$

where the words  $r_{l+1}, \dots, r_{l+n}$  express automorphisms  $\hat{x}_1, \dots, \hat{x}_n$  in generators  $E_{xy}$ :

$$r_{l+i}(E_{xy}) = E_{x_1 x_i} E_{x_1^{-1} x_i} \cdots E_{x_{i-1} x_i} E_{x_{i-1}^{-1} x_i} E_{x_{i+1} x_i} E_{x_{i+1}^{-1} x_i} \cdots E_{x_n x_i} E_{x_n^{-1} x_i}.$$

We have  $Out(F_n) = \langle \tau, E_{xy} \mid r_1(E_{xy}, \tau) = \cdots = r_{l+n}(E_{xy}, \tau) = 1 \rangle$ .

For each  $E_{xy}$  find such a word  $w_{xy}$  from  $F_n$  (call this word by “multiplying”), that using notations  $E'_{xy} = E_{xy} w_{xy}$  we have the following inclusions:

$$\begin{aligned} r_i(E'_{xy}, \tau) &\in F'_n \quad (i = 1, \dots, l). \\ r_i(E'_{xy}, \tau) &\in H \quad (i = l+1, \dots, l+n). \end{aligned} \quad (2)$$

Then  $Aut(F_n)/H = \langle \tau, E'_{xy}, x_1, \dots, x_n \mid r_i(E'_{xy}, \tau) = 1 \quad (i = 1, \dots, l+n), x_j^{E'_{xy}} = E'_{xy}(x_j), x_j^\tau = \tau(x_j) \quad (j = 1, \dots, n), h = 1 \quad (h \in H) \rangle \simeq (F_n/H) \rtimes Out(F_n)$ , and the splitting will be proved.

The theorem will be proved if the system of equations for multiplying words, coming from (2), is decidable. Let us make computations for  $n \geq 4$ . For  $n = 2$  and 3 computations are similar, moreover some cases are excluded, and the system will be simpler.

There is a natural epimorphism from  $Aut(F_n)$  to  $GL_n(\mathbb{Z}) \simeq Aut(F_n/F'_n)$ . Matrices which are images of automorphisms  $E_{xy}$  and  $B_{xy}$  denote by  $\mathbf{E}_{xy}$  and  $\mathbf{B}_{xy}$ . For  $w \in F_n$  and  $x \in X^{\pm 1}$  denote by  $l_x(w)$  the exponent by  $x$  in the abelianization of the word  $w$ , by  $l(w)$  the vector  $(l_{x_1}(w), \dots, l_{x_n}(w))$ , and by  $\mathbf{0}$  the vector  $l(1) = (0, \dots, 0)$ . Note that  $l_{x_i^{-1}}(w) = -l_{x_i}(w)$ ,  $l(w^{-1}) = -l(w)$ . For  $g, h \in F_n$  write  $g = h \pmod{F'_n}$  if  $gh^{-1} \in F'_n$ .

*Consider relation  $(\mathcal{R1})$ .* From the condition (2) for multiplying words we have  $w_{xy}^{-1} E_{xy}^{-1} = E_{xy^{-1}} w_{xy^{-1}} \pmod{F'_n}$ . Then it is necessary that the equality  $(w_{xy}^{-1})^{E_{xy}^{-1}} = w_{xy^{-1}} \pmod{F'_n}$  is satisfied. This equality is equivalent to the vector equation  $-l(w_{xy}) \cdot \mathbf{E}_{xy}^{-1} = l(w_{xy^{-1}})$ . From this we get the following equations for multiplying words:

$$\begin{aligned} -l_v(w_{xy}) &= l_v(w_{xy^{-1}}) \text{ for } v \neq y, y^{-1}, v \in X^{\pm 1}, \\ -l_y(w_{xy}) + l_x(w_{xy}) &= l_y(w_{xy^{-1}}). \end{aligned}$$

*Consider relation  $(\mathcal{R2})$ .* By  $(\mathcal{R1})$  it may be assumed that  $y, t \in X$ . Then the relation is equivalent to the system of relations (a)–(f):

$$(a) [E_{xy}, E_{zt}] = 1, \text{ where } t \neq y, y^{-1}; z \neq x^{-1},$$

- (b)  $[E_{xy}, E_{x^{-1}y}] = 1$ ,
- (c)  $[E_{xy}, E_{x^{-1}y^{-1}}] = 1$ ,
- (d)  $[E_{xy}, E_{x^{-1}t}] = 1$ , where  $t \neq y, y^{-1}$ ,
- (e)  $[E_{xy}, E_{zy}] = 1$ , where  $z \neq x^{-1}$ ,
- (f)  $[E_{xy}, E_{zy^{-1}}] = 1$ , where  $z \neq x^{-1}$ .

Relations (c) and (f) can be deduced from relations (b) and (e) using relations  $(\mathcal{R}1)$ .

Consider the case (a):  $E_{xy}E_{zt}E_{xy}^{-1}E_{zt}^{-1} = 1$ , where  $t \neq y, y^{-1}$ ;  $z \neq x^{-1}$ . By (2) we have  $E_{xy}w_{xy}E_{zt}w_{zt}w_{xy}^{-1}E_{xy}^{-1}w_{zt}^{-1}E_{zt}^{-1} = 1 \pmod{F'_n}$ . Then it is necessary that the following inclusion is satisfied:

$$w_{xy}^{E_{zt}E_{xy}^{-1}E_{zt}^{-1}}(w_{zt}w_{xy}^{-1})^{E_{xy}E_{zt}^{-1}}(w_{zt}^{-1})^{E_{zt}^{-1}} \in F'_n.$$

This inclusion is equivalent to the vector equality

$$l(w_{xy}) \cdot \mathbf{E}_{\mathbf{xy}}^{-1} + (l(w_{zt}) - l(w_{xy})) \cdot \mathbf{E}_{\mathbf{xy}}^{-1}\mathbf{E}_{\mathbf{zt}}^{-1} + (-l(w_{zt})) \cdot \mathbf{E}_{\mathbf{zt}}^{-1} = \mathbf{0}.$$

Herefrom we get the following equations for multiplying words:

$$\begin{aligned} l_y(w_{xy}) - l_x(w_{xy}) + l_y(w_{zt}) - l_y(w_{xy}) - l_x(w_{zt}) + l_x(w_{xy}) - l_y(w_{zt}) &= 0, \\ l_t(w_{xy}) + l_t(w_{zt}) - l_t(w_{xy}) - l_z(w_{zt}) + l_z(w_{xy}) - l_t(w_{zt}) + l_z(w_{zt}) &= 0. \end{aligned}$$

All other cases can be considered in a similar way and they lead to the following equations:

- (a)  $-l_x(w_{zt}) = 0$ ,  $l_z(w_{xy}) = 0$  for  $t \neq x, x^{-1}, y, y^{-1}$ ;  $z \neq x, x^{-1}, y, y^{-1}$
- (b)  $-l_x(w_{x^{-1}y}) - l_x(w_{xy}) = 0$ ,
- (d)  $-l_x(w_{x^{-1}t}) = 0$ ,  $-l_x(w_{xy}) = 0$
- (e)  $-l_x(w_{zy}) + l_z(w_{xy}) = 0$  for  $z \neq x, x^{-1}$ .

Now consider only relation  $(\mathcal{R}5)$ . For all other relations of the given presentation of  $\text{Aut}(F_n)$  write only equations for multiplying words.

Relation  $(\mathcal{R}5)$ :  $B_{yx}^4 = 1$ , where  $B_{yx} = E_{xy}E_{y^{-1}x}E_{x^{-1}y}^{-1}$ .

An easy computation gives that  $l(w) \cdot \mathbf{B}_{\mathbf{yx}} = l'(w)$ , where  $l'_x(w) = l_y(w)$ ,  $l'_y(w) = -l_x(w)$ ,  $l'_z(w) = l_z(w)$  for  $z \neq x, x^{-1}, y, y^{-1}$ :

	$l_x(w)$	$l_y(w)$
$\mathbf{B}_{\mathbf{yx}}$	$-l_y(w)$	$l_x(w)$
$\mathbf{B}_{\mathbf{yx}}^2$	$-l_x(w)$	$-l_y(w)$
$\mathbf{B}_{\mathbf{yx}}^3$	$l_y(w)$	$-l_x(w)$
$\mathbf{B}_{\mathbf{yx}}^4$	$l_x(w)$	$l_y(w)$

By  $(\mathcal{R}2)$  it follows that  $(E_{xy}w_{xy}E_{y^{-1}x}w_{y^{-1}x}w_{x^{-1}y}^{-1}E_{x^{-1}y}^{-1})^4 = 1 \pmod{F'_n}$ .

Then it is necessary that the following inclusion is satisfied:

$$w_{xy}^{E_{y^{-1}x}E_{x^{-1}y}^{-1}B_{yx}^3}(w_{y^{-1}x}w_{x^{-1}y}^{-1})^{E_{x^{-1}y}^{-1}B_{yx}^3} \dots w_{xy}^{E_{y^{-1}x}E_{x^{-1}y}^{-1}}(w_{y^{-1}x}w_{x^{-1}y}^{-1})^{E_{x^{-1}y}^{-1}} \in F'_n.$$

This inclusion is equivalent to the vector equation

$$l(w_{xy}) \cdot \mathbf{E}_{\mathbf{y^{-1}x}}\mathbf{E}_{\mathbf{x^{-1}y}}^{-1}\mathbf{B}_{\mathbf{yx}}^3 + (l(w_{y^{-1}x}) - l(w_{x^{-1}y})) \cdot \mathbf{E}_{\mathbf{x^{-1}y}}^{-1}\mathbf{B}_{\mathbf{yx}}^3 + \dots +$$

$$+l(w_{xy}) \cdot \mathbf{E}_{\mathbf{y}^{-1}\mathbf{x}} \mathbf{E}_{\mathbf{x}^{-1}\mathbf{y}}^{-1} + (l(w_{y^{-1}x}) - l(w_{x^{-1}y})) \cdot \mathbf{E}_{\mathbf{x}^{-1}\mathbf{y}}^{-1} = \mathbf{0}.$$

Taking into consideration actions of powers of the matrix  $\mathbf{B}_{\mathbf{y}\mathbf{x}}$ , we get the following equation:

$$4(l_z(w_{xy}) + l_z(w_{y^{-1}x}) - l_z(w_{x^{-1}y})) = 0 \text{ for } z \neq x, x^{-1}, y, y^{-1}.$$

Thus, from the conditions  $r_i(E'_{xy}, \tau) \in F'_n$  for  $i = 1, \dots, l$  we obtain the following system of equations:

1.  $l_y(w_{xy}) = -l_y(w_{xy^{-1}}) + l_x(w_{xy}),$   
 $l_v(w_{xy}) = -l_v(w_{xy^{-1}})$  for  $v \neq y, y^{-1},$
2.  $l_x(w_{yz}) = 0$  for  $z \neq x, x^{-1},$
3.  $-l_x(w_{yz}) - l_y(w_{xz}) = 0,$   
 $l_y(w_{xy}) + l_x(w_{yz}) + l_x(w_{xz}) - l_z(w_{xz}) = 0$  for  $x \neq z, z^{-1},$
4.  $-l_y(w_{yx}) + l_x(w_{y^{-1}x}) - l_x(w_{x^{-1}y}) - l_y(w_{x^{-1}y}) + l_x(w_{yx}) - l_y(w_{yx}) = 0,$   
 $l_x(w_{xy}) + l_y(w_{y^{-1}x}) - l_y(w_{x^{-1}y}) + l_x(w_{y^{-1}x}) + l_x(w_{yx}) - l_y(w_{xy}),$   
 $l_v(w_{y^{-1}x}) + l_v(w_{yx})$  for  $v \neq x, x^{-1}, y, y^{-1},,$
5.  $l_z(w_{xy}) + l_z(w_{y^{-1}x}) - l_z(w_{x^{-1}y}) = 0$  for  $z \neq x, x^{-1}, y, y^{-1}.$
6. -
7.  $l_{x_1}(w_{xy}) = -l_{x_1}(w_{xy})$  for  $x_1 \neq x, x^{-1}, y, y^{-1},$   
 $l_{x_1}(w_{x_1y}) = -l_{x_1}(w_{x_1^{-1}y}), l_v(w_{x_1y}) = l_v(w_{x_1^{-1}y})$  for  $v \neq x_1, x_1^{-1},$   
 $l_v(w_{xx_1}) = -l_v(w_{xx_1})$  for  $v \neq x_1, x_1^{-1}.$

Further we will write equations, which can be deduced from conditions  $r_{l+i}(E'_{xy}) \in H,$   $i = 1, \dots, n.$  Recall that

$$r_{l+i}(E_{xy}) = E_{x_1x_i} E_{x_1^{-1}x_i} \dots E_{x_{i-1}x_i} E_{x_{i-1}^{-1}x_i} E_{x_{i+1}x_i} E_{x_{i+1}^{-1}x_i} \dots E_{x_nx_i} E_{x_n^{-1}x_i}.$$

So, we have

$$E_{x_1x_i} w_{x_1x_i} E_{x_1^{-1}x_i} w_{x_1^{-1}x_i} \dots E_{x_{i-1}x_i} w_{x_{i-1}x_i} E_{x_{i-1}^{-1}x_i} w_{x_{i-1}^{-1}x_i} \cdot \\ \cdot E_{x_{i+1}x_i} w_{x_{i+1}x_i} E_{x_{i+1}^{-1}x_i} w_{x_{i+1}^{-1}x_i} \dots E_{x_nx_i} w_{x_nx_i} E_{x_n^{-1}x_i} w_{x_n^{-1}x_i} = 1 \pmod{H}.$$

Then  $w_{x_1x_i}^{E_{x_1^{-1}x_i} r_{l+i}(E_{xy}) \dots E_{x_n^{-1}x_i}} \dots w_{x_n^{-1}x_i} \in x_i^{-1}H.$  Since the product of matrices  $\mathbf{E}_{\mathbf{x}_k\mathbf{x}_i}$  and  $\mathbf{E}_{\mathbf{x}_k^{-1}\mathbf{x}_i}$  is the identity matrix, the corresponding vector equation is

$$\sum_{\substack{j=1 \\ j \neq i}}^n (l(w_{x_jx_i}) \cdot \mathbf{E}_{\mathbf{x}_j^{-1}\mathbf{x}_i} + l(w_{x_j^{-1}x_i})) = e_i,$$

where  $e_i$  is a vector whose components are multiple of  $k$ , except the  $i$ -th component where  $i \equiv -1 \pmod{k}.$  Herefrom we get the following equations:

$$8. \sum_{\substack{j=1 \\ j \neq i}}^n (l_{x_i}(w_{x_jx_i}) - l_{x_1}(w_{x_jx_i}) + l_{x_i}(w_{x_j^{-1}x_i})) \equiv -1 \pmod{k}, i = 1, \dots, n.$$

Simplifying the system of equation 1 – 8, we get the final system:

$$l_z(w_{xy}) = 0 \text{ for all } z \neq y, y^{-1},$$

$$\begin{aligned}
l_y(w_{xy}) &= -l_y(w_{xy^{-1}}), \\
l_y(w_{xy}) &= l_z(w_{xz}), \\
l_x(w_{yx}) + l_x(w_{y^{-1}x}) &= l_y(w_{xy}) + l_y(w_{x^{-1}y}) \text{ for } x, y, z \in X^{\pm 1}, \\
\sum_{\substack{j=1 \\ j \neq i}}^n (l_{x_i}(w_{x_j x_i}) + l_{x_i}(w_{x_j^{-1} x_i})) &\equiv -1 \pmod{k}.
\end{aligned}$$

Since the congruence  $a \cdot (n - 1) \equiv -1 \pmod{k}$  is decidable if  $n - 1$  and  $k$  are co-prime, the final system is solvable also.

**Conjecture.** The group  $Out(F_n)$  can be embedded into the group  $Out(F_m)$  for  $m = 2n$ .

#### REFERENCE

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